Searchability of Cototal natural numbers

Nils Köpp (jww Chuangjie Xu)

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Math. Institut LMU München

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Introduction

- given some X decide for $p : X \to \mathbb{B}$ if

$$\exists_x\, px = 0 \vee \forall_x\, px = 1$$

- X finite $\rightsquigarrow$ ✓
- Are there infinite X?
  If $X = \mathbb{N}$ then LPO.
- But for the one-point compactification $X = \mathbb{N}_\infty$ ✓ ?
- Martin Escardo proved this, modelling N_∞ as decreasing binary sequences ($n = 1 \ldots 10 \ldots$ and $\infty = \lambda_n 1$)
- $N_\infty \equiv {}^{co}\ T_{\mathbb{N}}$ ($\infty = \mathrm{SSS} \ldots$)

Notation

- We work in the framework presented by Prof. Schwichtenberg yesterday
- $\mathbb{N} := \mu_\xi\,(\xi, \xi \to \xi) = 0^{\mathbb{N}} \,|\, \mathsf{S}^{\mathbb{N}\to\mathbb{N}}$,
  $\mathbb{B} = \mu_\xi\,(\xi, \xi) = 0^{\mathbb{B}} \,|\, 1^{\mathbb{B}} = \text{False} \,|\, \text{True}$
- We use $\widehat{n}, \widehat{m}$ for variables of type $\mathbb{N}$
- n, m for "total variables" of type $\mathbb{N}$, meaning that $\forall, \exists$ are relativized. (e.g. $\forall_n(A(n)) \equiv \forall_{\widehat{n}}(T_{\mathbb{N}}\widehat{n} \to A(\widehat{n})))$
- α, β for "cototal variables". Everything relativized to ${}^{co}T_{\mathbb{N}}$.
- p for variables of type $\mathbb{N} \to \mathbb{B}$.
- $=$ (without subscript or dots) is *Leibniz equality*

Extensionality and $\mathbb{N}$ I

- Totality

$$T_{\mathbb{N}}0 \qquad \forall_{\widehat{n}}\,(T_{\mathbb{N}}\widehat{n} \to T_{\mathbb{N}}\,(\mathsf{S}\widehat{n}))$$
$$\forall_n\,(P0 \to \forall_m\,(Pn \to P\,(\mathsf{S}n)) \to Pn)$$

- Similarity

$$0 \doteq_{T_{\mathbb{N}}} 0 \qquad \forall_{\widehat{n},\widehat{m}}\,(\widehat{n} \doteq_{T_{\mathbb{N}}} \widehat{m} \to \mathsf{S}\widehat{n} \doteq_{T_{\mathbb{N}}} \mathsf{S}\widehat{m})$$
$$\forall_{\widehat{n},\widehat{m}}\big(\widehat{n} \doteq_{T_{\mathbb{N}}} \widehat{m} \to P00 \to \forall_{\widehat{n},\widehat{m}}\,(\widehat{n} \doteq_{T_{\mathbb{N}}} \widehat{m} \to P\widehat{n}\widehat{m} \to P(\mathsf{S}\widehat{n}\mathsf{S}\widehat{m}))$$
$$\to P\widehat{n}\widehat{m}\big)$$

- Closure axioms for ${}^{co}T_{\mathbb{N}}$ and $\dot{\approx}_{{}^{co}T_{\mathbb{N}}}$.

$$\forall_\alpha\,(\alpha = 0 \vee \exists_{\alpha_0}\,\alpha = \mathsf{S}\alpha_0)$$
$$\forall_{\alpha,\beta}\big(\alpha \approx \beta \to (\alpha = 0 \wedge \beta = 0) \vee$$
$$\exists_{\alpha_0,\beta_0}\,(\alpha = \mathsf{S}\alpha_0 \wedge \beta = \mathsf{S}\beta_0 \wedge \alpha_0 \approx \beta_0)\,\big)$$

Extensionality and $\mathbb{N}$ II

- Greatest fixed point (coinduction) for ${}^{co}T_{\mathbb{N}}$

$$\forall_{\widehat{n}}\big(P\widehat{n} \to$$
$$\forall_{\widehat{n}}\,(P\widehat{n} \to \widehat{n} = 0 \lor \exists_{\widehat{n}_0}\,(\widehat{n} = \mathsf{S}\widehat{n}_0 \land (P\widehat{n}_0 \lor \widehat{n}_0 \in {}^{co}T_{\mathbb{N}})))$$
$$\to \widehat{n} \in {}^{co}T_{\mathbb{N}}\big)$$

- GFP for $\approx$

  If $P\widehat{n}\widehat{m}$ and (the costep)

$$\forall_{\widehat{n},\widehat{m}}\big(P\widehat{n}\widehat{m} \to$$
$$(\widehat{n} = 0 \land \widehat{m} = 0) \lor$$
$$\exists_{\widehat{n}_0,\widehat{m}_0}\,(\widehat{n} = \mathsf{S}\widehat{n}_0 \land \widehat{m} = \mathsf{S}\widehat{m}_0 \land (P\widehat{n}_0\widehat{m}_0 \lor \widehat{n}_0 \approx \widehat{m}_0))\big)$$

  then $\widehat{n} \approx \widehat{m}$.

Extensionality and $\mathbb{N}$ III

- $p \in Ext_{T_{\mathbb{N}} \to T_{\mathbb{B}}} \quad \Leftrightarrow$

$$\begin{aligned} & \forall_{\widehat{n},\widehat{m}} (\widehat{n} \doteq \widehat{m} \to p\widehat{n} \doteq_{T_{\mathbb{B}}} p\widehat{m}) \\ \Leftrightarrow \quad & \forall_n pn \in T_{\mathbb{B}} \wedge \forall_{n,m}(n \doteq m \to pn = pm) \end{aligned}$$

- $p \in Ext_{{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}} \quad \Leftrightarrow$

$$\begin{aligned} & \forall_{\widehat{n},\widehat{m}} (\widehat{n} \approx \widehat{m} \to p\widehat{n} \doteq p\widehat{m}) \\ \Leftrightarrow \quad & \forall_\alpha p\alpha \in T_{\mathbb{B}} \wedge \forall_{\alpha,\beta} (\alpha \approx \beta \to p\alpha = p\beta) \end{aligned}$$

- $Ext_{{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}} \subset Ext_{T_{\mathbb{N}} \to T_{\mathbb{B}}}$
- Note that

$$\widehat{n} \approx \widehat{m} \to \widehat{n}, \widehat{m} \in {}^{co}T_{\mathbb{N}}$$

Some arithmetic on ${}^{co}T_{\mathbb{N}}$

Definition

- *Minimum* is a constant with defining equations

$$min(0,\widehat{n}) = min(\widehat{n},0) := 0 \quad min(\mathsf{S}\widehat{n},\mathsf{S}\widehat{m}) := \mathsf{S}\,(min(\widehat{n},\widehat{m}))$$

- $\leq$ is a constant with

$$(0 \leq \widehat{n}) := \mathsf{True} \quad (\mathsf{S}\widehat{n} \leq 0) := \mathsf{False} \quad (\mathsf{S}\widehat{n} \leq \mathsf{S}\widehat{m}) := (\widehat{n} \leq \widehat{m})$$

- $<$ is an inductive predicate with clauses

$$\forall_{\widehat{n}}\,(0 < \mathsf{S}\widehat{n}) \quad \forall_{\widehat{n},\widehat{m}}\,(\widehat{n} < \widehat{m} \to \mathsf{S}\widehat{n} < \mathsf{S}\widehat{m})$$

- ∞ is

$$\infty := {}^{co}\mathcal{R}_{\mathbb{U}\Rightarrow\mathbb{N}}\mathsf{Dummy}\,([u]\mathsf{InR}(\,\mathsf{InR}u\,)) = \mathsf{SSS}\ldots$$

Things we can prove

Lemma

- *$<$ is an order relation. It is not decidable, but*

$$\forall_{n,\alpha,\beta}\,(n < \alpha \to n < \beta \vee \beta < \alpha)$$

- *Furthermore*

$$\begin{aligned}
&\forall_{n,\widehat{m}}\,(T_{\mathbb{N}}(min(n,\widehat{m})) \wedge T_{\mathbb{N}}(min(\widehat{m},n))) \\
&\forall_{n,\widehat{m}}\,(min(n,\widehat{m}) < Sn) \\
&\forall_{\widehat{n},\widehat{m}}\,(\widehat{n} < \widehat{m} \to T_{\mathbb{N}}\widehat{n}) \\
&\forall_{\alpha}\,\alpha \not< \alpha \\
&\forall_{\widehat{n}}\,\infty \not< \widehat{n}
\end{aligned}$$

Things we can't prove

Lemma

$$\forall_{\alpha,\beta}\ \alpha < \beta \vee \beta < \alpha \quad \Leftrightarrow \quad \forall_\alpha\, T_{\mathbb{N}}\alpha \vee \neg T_{\mathbb{N}}\alpha \quad \Leftrightarrow \quad \mathsf{LPO}$$

$$\forall_\alpha\, (\alpha \not\approx \infty \to T_{\mathbb{N}}\alpha) \quad \Leftrightarrow \quad \mathsf{MP}$$

$$\forall_\alpha\ \alpha \approx \infty \vee \alpha \not\approx \infty \quad \Leftrightarrow \quad \mathsf{WLPO}$$

Definition (Selection function)

A *selction function* for $\mathfrak{F} \subseteq (X \to \mathbb{B})$ is a functional $\epsilon : (X \to \mathbb{B}) \to X$ such that for all $p \in \mathfrak{F}$

$$p(\epsilon(p)) = 1 \to \forall_{x \in X}\, p(x) = 1 \tag{1}$$

$(X, \mathfrak{F})$ is *searchable* if there exists a selection function.

Lemma

For a selection function ϵ *and* $p \in \mathfrak{F}$

$$\exists_x\, p(x) = 0 \Leftrightarrow p(\epsilon(p)) = 0 \tag{2}$$

Definition

Define $\eta : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$.

$$\eta p := \begin{cases} 0 & \text{if p(0)=0} \\ \mathsf{S}(\eta(p \circ \mathsf{S})) & \text{else} \end{cases} \tag{3}$$

- η first checks if $p0 = 0$. If yes then return 0.
- if not check if $p1 = 0 \ldots$

Lemma

$$\forall_p \, (p \in Ext_{\,T_{\mathbb{N}} \to T_{\mathbb{B}}} \to \eta p \in {}^{co}T_{\mathbb{N}}) \tag{4}$$

Proof.

Easy coinduction on $A(\widehat{n}) := \exists_{p_0 \in Ext_{\,T_{\mathbb{N}} \to T_{\mathbb{B}}}} (\eta p_0 = \widehat{n})$. □

Theorem

η is a selection function for $Ext_{{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}$, i.e. for all $p \in Ext_{{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}$:

$$p(\eta(p)) = 1 \to \forall_\alpha p(\alpha) = 1 \tag{5}$$

Proof.

We first prove some intermediate statements:

① $\forall_\alpha (\forall_n \alpha \not\approx n \to \alpha \approx \infty)$

② $\forall_{p \in Ext_{{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}} (\forall_n\ pn = 1 \to p\infty = 1 \to \forall_\alpha\ p\alpha = 1)$

③ $\forall_{n, p \in Ext_{{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}} (\eta p \approx \infty \to pn = 1)$

④ $\forall_{n,p} (\eta p \approx n \to pn = 0)$

Proof.

① $\forall_\alpha (\forall_n\, \alpha \not\approx n \to \alpha \approx \infty)$

By coinduction on $P\widehat{n}\widehat{m} := \widehat{n} \in {}^{co}T_{\mathbb{N}} \wedge \forall_k\, \widehat{n} \not\approx k \wedge \widehat{m} = \infty$.

Need to prove the costep, i.e. for $\widehat{n}, \widehat{m}$ with $P\widehat{n}\widehat{m}$

$$(\widehat{n} = 0 \wedge \widehat{m} = 0) \vee$$
$$\exists_{\widehat{n}_0, \widehat{m}_0} \big(\widehat{n} = S\widehat{n}_0 \wedge \widehat{m} = S\widehat{m}_0 \wedge (P\widehat{n}_0\widehat{m}_0 \vee \widehat{n}_0 \approx \widehat{m}_0)\big)$$

Since $\widehat{n} \in {}^{co}T_{\mathbb{N}}$: $\widehat{n} = 0$ (which is impossible) or $\widehat{n} = S\widehat{n}_0$ for some $\widehat{n}_0 \in {}^{co}T_{\mathbb{N}}$. Since $\widehat{m} = \infty = S\infty$ we are in the right side of the disjunction with $\widehat{m}_0 = \infty$.

② $\forall_{p \in Ext\, {}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}} (\forall_n\, pn = 1 \to p\infty = 1 \to \forall_\alpha\, p\alpha = 1)$

③ $\forall_{n, p \in Ext\, {}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}} (\eta p \approx \infty \to pn = 1)$

④ $\forall_{n,p} (\eta p \approx n \to pn = 0)$

Proof.

① $\forall_\alpha\,(\forall_n\,\alpha \not\approx n \to \alpha \approx \infty)$

② $\forall_{p \in Ext\,{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}\,(\forall_n\, pn = 1 \to p\infty = 1 \to \forall_\alpha\, p\alpha = 1)$

Assume $p\alpha \neq 1$. Then by the extensionality of p: $\forall_n\,\alpha \not\approx n$ and $\alpha \not\approx \infty$. This is impossible by ①.

③ $\forall_{n, p \in Ext\,{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}\,(\eta p \approx \infty \to pn = 1)$

④ $\forall_{n,p}\,(\eta p \approx n \to pn = 0)$

Proof.

① $\forall_\alpha\,(\forall_n\, \alpha \not\approx n \to \alpha \approx \infty)$
② $\forall_{p\in Ext\, {}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}\,(\forall_n\, pn = 1 \to p\infty = 1 \to \forall_\alpha\, p\alpha = 1)$
③ $\forall_{n,p\in Ext\, {}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}\,(\eta p \approx \infty \to pn = 1)$
By induction on n: $p0 = 1$, since otherwise $\eta p = 0 \not\approx \infty$.
Assume $\forall_p\,(\eta p \approx \infty \to pn = 1)$ and $\eta p \approx \infty$. By Definition $\eta\,(p \circ \mathsf{S}) \approx \infty$. So by the induction hypothesis $p(\mathsf{S}n) = 1$.
④ $\forall_{n,p}\,(\eta p \approx n \to pn = 0)$

Proof.

① $\forall_\alpha \, (\forall_n \, \alpha \not\approx n \to \alpha \approx \infty)$

② $\forall_{p \in Ext \, {}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}} \, (\forall_n \, pn = 1 \to p\infty = 1 \to \forall_\alpha \, p\alpha = 1)$

③ $\forall_{n, p \in Ext \, {}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}} \, (\eta p \approx \infty \to pn = 1)$

④ $\forall_{n,p} \, (\eta p \approx n \to pn = 0)$

By induction. If $\eta p \approx 0 \Leftrightarrow p0 = 0$. Assume $\forall_p (\eta p \approx n \to pn = 0)$ and $\eta p \approx \mathsf{S}n$. Apply $p \circ \mathsf{S}$ to the Induction hypothesis to get

$$\eta(p \circ \mathsf{S}) \approx n \to p(\mathsf{S}n) = 0$$

and $\mathsf{S}(\eta(p \circ \mathsf{S})) = \eta p \approx \mathsf{S}n \Rightarrow \eta(p \circ \mathsf{S}) \approx n$.

Proof.

① $\forall_\alpha (\forall_n \alpha \not\approx n \to \alpha \approx \infty)$
② $\forall_{p \in Ext\, {}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}} (\forall_n pn = 1 \to p\infty = 1 \to \forall_\alpha p\alpha = 1)$
③ $\forall_{n, p \in Ext\, {}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}} (\eta p \approx \infty \to pn = 1)$
④ $\forall_{n,p} (\eta p \approx n \to pn = 0)$

Finally assume $p(\eta p) = 1$. We have

- $\forall_n \eta p \not\approx n$. If $\eta p \approx n$ then $p(\eta p) = 0$ by ④.
- $\eta p \approx \infty$. By ①.
- $\forall_n pn = 1$. By ③.
- $p\infty = 1$. By $\eta p \approx \infty$, extensionality and the assumption $p(\eta p) = 1$.
- $\forall_\alpha p\alpha = 1$. By ②.

□

Corollary

$Ext_{\,^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}$ is searchable, i.e.

$$\forall_{p \in Ext_{\,^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}} \left(\exists_{\alpha_0}\, p\alpha_0 = 0 \lor \forall_{\alpha}\, p\alpha = 1 \right) \tag{6}$$

The realizer is given by

$$\lambda_p \left[\text{if } (p(\eta p) = 0) \; (InL(\eta p)) \text{ else } InR \right]$$

Definition

A map $p \in Ext_{\,{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}$ is

- *strongly extensional* iff

$$\forall_\alpha \left(p(\alpha) \neq p(\infty) \to \alpha \in T_{\mathbb{N}}\right) \tag{7}$$

- *continuous* iff

$$\exists_n \forall_\alpha \left(n < \alpha \to p(\alpha) = p(\infty)\right) \tag{8}$$

Corollary

If $p \in Ext_{\,{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}$ is strongly extensional then

$$\exists_n (pn \neq p\infty) \vee \forall_n (pn = p\infty)$$

Theorem

Any $p \in Ext_{T_{\mathbb{N}} \to T_{\mathbb{B}}}$ that is eventually constant extends to a $\widetilde{p} \in Ext_{{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}$ and $\widetilde{p}$ is strongly extensional and continuous.

Proof.

Assume $n_0 \in T_{\mathbb{N}}$ with $\forall_m \, (n_0 \leq m \to pm = pn_0)$ and define

$$\widetilde{p} := \lambda_{\widehat{n}} \, [\text{if } (\widehat{n} < n_0) \; (p\widehat{n}) \text{ else } (pn_0)]$$

(i) We need to show $\widetilde{p} \in Ext_{{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}$, so let $\alpha \approx \beta$. We have $min(n_0, \alpha) < \mathsf{S}n_0$, so

$$\alpha < \mathsf{S}n_0 \vee min(n_0, \alpha) < \alpha$$

- (L) $\alpha < \mathsf{S}n_0 \Rightarrow \alpha = \beta \in T_{\mathbb{N}}$ and $\widetilde{p}\alpha = p\alpha = p\beta = \widetilde{p}\beta$.
- (R) $min(n_0, \alpha) < \alpha \Rightarrow n_0 < \alpha, \beta$ and $\widetilde{p}\alpha = pn_0 = \widetilde{p}\beta$.

Proof (Cont.)

(ii) Clearly continuous with n_0.

(iii) Assume $\widetilde{p}\alpha \neq \widetilde{p}\infty$. Again $\alpha < \mathrm{n}_0 \vee min(n_0, \alpha) < \alpha$.

L $\alpha < \mathrm{n}_0 \Rightarrow \alpha < \infty \Rightarrow \alpha \in T_{\mathbb{N}}$

R $min(n_0, \alpha) < \alpha \Rightarrow n_0 < \alpha \Rightarrow \widetilde{p}\alpha = pn_0 = \widetilde{p}\infty \lightning$. $\square$

Extending decreasing functions?

Definition

For $p \in Ext_{T_{\mathbb{N}} \to T_{\mathbb{B}}}$ define $\widehat{p} \in Ext_{T_{\mathbb{N}} \to T_{\mathbb{B}}}$ by

$$\widehat{p}0 := p0 \quad \widehat{p}(\mathsf{S}n) := (\widehat{p}n)(p(\mathsf{S}n)) \tag{9}$$

Lemma

For $p \in Ext_{T_{\mathbb{N}} \to T_{\mathbb{B}}}$ $\widehat{p}$ is decreasing and

$$\forall_n \big(\widehat{p}n = 0 \to \exists_{m \leq n}\, pn = 0\big) \wedge \big(\forall_n\, \widehat{p}n = 1 \to \forall_n\, pn = 1\big)$$

($\triangle$) Any decreasing $p \in Ext_{T_{\mathbb{N}} \to T_{\mathbb{B}}}$ extends to a strongly extensional continuous $\widetilde{p} \in Ext_{co}{}_{T_{\mathbb{N}} \to T_{\mathbb{B}}}$
($\square$) Any decreasing $p \in Ext_{T_{\mathbb{N}} \to T_{\mathbb{B}}}$ with $\neg\forall_n pn = 1$ extends to a strongly extensional continuous $\widetilde{p} \in Ext_{co}{}_{T_{\mathbb{N}} \to T_{\mathbb{B}}}$

Theorem

(i) $(\triangle) \Leftrightarrow$ *LPO* $\left(\forall_{p \in Ext_{T_{\mathbb{N}} \to T_{\mathbb{B}}}} (\exists_n\, pn = 0 \vee \forall_n\, pn = 1)\right)$
(ii) $(\square) \Leftrightarrow$ *MP*

Proof.

(i) $\Rightarrow$. Assume $p \in Ext_{T_{\mathbb{N}} \to T_{\mathbb{B}}}$. By $(\triangle)$ $\widehat{p}$ extends to some $q \in Ext_{{}^{co}T_{\mathbb{N}} \to T_{\mathbb{B}}}$. We have (with the selection function):

$$\exists_{\alpha_0}\, q\alpha_0 = 0 \vee \forall_\alpha\, q\alpha = 1$$

(L) $q\alpha_0 = 0$. We distinguish cases on $q\infty \in T_{\mathbb{B}}$.
(1) $q\infty = 1$. By strong extensionality $T_{\mathbb{N}}\alpha_0$ and $\widehat{p}\alpha_0 = 0 \Rightarrow \exists_{m \leq \alpha_0} pm = 0$.
(2) $q\infty = 0$. By continuity exists n_0 such that $\forall_{\alpha > n_0}\, q\alpha = q\infty$. I.p. $\widehat{p}Sn_0 = qSn_0 = 0 \Rightarrow \exists_{m \leq Sn_0}\, pm = 0$.
(R) $\forall_n\, \widehat{p}n = 1 \Rightarrow \forall_n\, pn = 1$. $\square$

References

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Thank you!