Ergodic Behaviour of Affine Recursions II

Invariant Measures and Ergodic Theorems

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Summary. This paper is concerned with the discrete–time Markov process $(X_n)_{n\geq 0}$ solving the stochastic difference equation $X_n = Y_n X_{n-1} + Z_n$ for $n \in \mathbf{N}$, where $(Y_n, Z_n)_{n\in\mathbf{N}}$ is a sequence of i.i.d. random variables independent of the initial variable X_0 and, in accordance with most applications, the state space is restricted to $\mathbf{R}_+$. In this part the emphasis is on the recurrent case, where existence and uniqueness of an invariant measure as well as mean and pointwise ergodic theorems can be established.

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Introduction

This is the continuation of work begun in ... and to be finished in .... It studies affine recursions on $\mathbf{R}_+$, i.e. sequences $(X_n)_{n\geq 0}$ defined by

$$X_n = Y_n X_{n-1} + Z_n \qquad \text{for } n \in \mathbf{N}.$$

Here, $(Y_n, Z_n)_{n\in\mathbf{N}}$ is a sequence of independent identically distributed $\mathbf{R}_+^2$–valued random variables which is independent of the initial variable $X_0 \geq 0$. Without loss of generality, the common distribution ν of $(X_n, Z_n), n \in \mathbf{N}$, will be assumed to belong to the class $\mathcal{N}$ defined in Section 0.

A central result of Part I then states that the lower and upper limit of $(X_n)_{n\geq 0}$ are constants $\underline{x}$ and $\overline{x}$, independent of the initial law $\mathcal{L}(X_o)$. Accordingly the sequence is called recurrent in the case $\underline{x} < \infty$ and transient in the case $\underline{x} = \infty$. The aim of Part II is a more specific study of the asymptotic behaviour in the recurrent case; its contents are summarized below.

Section 4. For information about limit points of the sequence $(X_n)_{n\geq 0}$ different from $\underline{x}$ and $\overline{x}$ existence and uniqueness of invariant measures for the corresponding transition kernel are essential. Here, in accordance with the topological structure of the state space, only locally finite measures are of interest. The support M of such an invariant measure μ satisfies a functional equation (4.1), by which lower and upper limit can be identified as minimum and maximum of M (4.3). Moreover, M inherits connectedness from the support of the joint law $\mathcal{L}(Y_n, Z_n)$ (4.4). An important property of μ itself consists in the fact that the measure $\mu([0,t])$ grows only polynomially for $t \to \infty$, unless the underlying affine maps are expansive almost surely (4.5).

Section 5. The main result of this section is basic for all that follows. To each recurrent sequence $(X_n)_{n\geq 0}$ it assigns an essentially unique locally finite invariant measure (5.6). While its existence is settled by usual compactness arguments (5.3), the uniqueness requires an elaborate localization. It is complicated by the fact that the hitting kernel of an interval $[0,t]$ with $\underline{x} < t < \infty$, though being still stochastic, need no longer be a Feller kernel. To determine the invariant measure explicitly, an integral equation for its Laplace transform is available (5.7), which, however, is more or less of theoretical interest. In the example

$$\text{(E)} \qquad X_n = Y_n X_{n-1} + 1 \qquad \text{with} \qquad \mathbf{P}(Y_n = \frac{1}{2}) = \frac{1}{2} = \mathbf{P}(Y_n = 2)$$

it leads to the functional equation

$$\psi(u) = e^{-u}\,\frac{1}{2}\,(\psi(\frac{u}{2}) + \psi(2u)) \qquad \text{for } u > 0,$$

which can hardly be solved.

Section 6. In view of the computational problems it is important to study the invariant measure μ at least qualitatively. The first assertion concerns its support M, which, whenever unbounded, can be shown to be an interval (6.1). More precisely, here as in some of the following results, $\mathbf{P}(Y_n = 0) = 0$ has to be assumed. It is not hard to show μ to be either absolutely continuous or singular with respect to Lebesgue measure (6.2). The first statement applies, unless the joint law $\mathcal{L}(Y_n, Z_n)$ is singular (6.3). It is more involved to prove that μ, apart from a trivial exception, is nonatomic (6.4). The section closes with a stability result that may be used for approximating the invariant measure (6.5).

Section 7. Here the main ergodic theorems for ratios are established. While the version for means follows easily from earlier results (7.1), for the pointwise version a localization as in Section 5 is necessary (7.4). It makes essential use of the fact that the hitting kernel from Section 5 enjoys at least some weaker Feller property. As a consequence of the ergodic theorems the support M and its complement can be identified with the conservative and dissipative part of the process (7.5). If specialized to the example (E) above, this means that the set of limit points of the sequence $(X_n)_{n\geq 0}$ equals the interval $[2, \infty]$ almost surely. Rather general results on irreducibility and aperiodicity conclude the section (7.6).

4. Excessive and invariant measures

Basic for the ergodic theorems in Sections 5 and 7 are some properties of measures $\mu \in \mathcal{M}(\mathbf{R}_+)$ that are excessive or invariant with respect to ν. Here the reference to ν, suppressed in general, actually refers to the corresponding kernel P. Moreover, it should be noted that for this section it is irrelevant whether the underlying distribution ν is recurrent or transient. The results concern mainly the support of μ, where in the first assertion $\overline{A}$ denotes the closure of a subset A of $\mathbf{R}_+$:

(4.1) Proposition. *With the mapping*

$$h: \ \mathbf{R}_+ \times \mathbf{R}_+^2 \ni (x; y, z) \to xy + z \in \mathbf{R}_+$$

the support M of a measure $\mu \in \mathcal{M}(\mathbf{R}_+)$ satisfies:

(a) $\quad M \supset \overline{h[M \times N]}$, *if μ is excessive,*

(b) $\quad M = \overline{h[M \times N]}$, *if μ is invariant.*

Proof. The measure μP is the image of $\mu \otimes \nu$ under the continuous mapping h. Therefore, due to a general result from topological measure theory, the support of μP is the closure of the image of

$$\text{supp } (\mu \otimes \nu) = M \times N$$

under h, which clearly proves (a) and (b). $\square$

Equally simple is the following auxiliary result:

(4.2) Lemma. *If $\mu \in \mathcal{M}(\mathbf{R}_+)$ is a nontrivial excessive measure with support M, then*

$$\frac{z}{1-y} \in M \qquad \textit{for } (y,z) \in N \textit{ with } y < 1.$$

Proof. With an arbitrary $x_0 \in M \neq \emptyset$ and (y,z) as above, (4.1a) yields recursively

$$x_n := y\, x_{n-1} + z \in M \qquad \text{for } n \in \mathbf{N}.$$

Since M is closed, this implies

$$\frac{z}{1-y} = \lim_{n\to\infty} (y^n x_0 + y^{n-1} z + \ldots + z) = \lim_{n\to\infty} x_n \in M. \quad \square$$

An excessive measure μ obviously preserves this property, if P is replaced by some power P^n. Thus it is clear that for $(y_m, z_m) \in N$ with $y_m < 1$ the support M contains not only the fixed points $z_m \,/\, (1-y_m)$ of the associated affine maps $g_m : x \to y_m x + z_m$ but as well those of the composition $g_1 \circ \ldots \circ g_n$ for any $n \in \mathbf{N}$.

The next result is closely related to (1.3):

(4.3) Theorem. *If $\mu \in \mathcal{M}(\mathbf{R}_+)$ is a nontrivial invariant measure with support M, then, with the notation of* (1.3),

(a) $\quad \inf M \;= \inf\{\frac{z}{1-y} : (y,z) \in N_c\},$

(b) $\quad \sup M = \sup\{\frac{z}{1-y} : (y,z) \in N_c\} \qquad$ *if* $N_e = \emptyset$,

$\qquad\qquad = \infty \qquad\qquad\qquad\qquad\qquad\qquad$ *otherwise.*

Proof. 1. If $\inf M$ and $\sup M$ are denoted by $\underline{m}$ and $\overline{m}$, respectively, it is immediate from (4.2) that

$$\underline{m} \leq \inf\{\ldots\} \qquad \text{and} \qquad \overline{m} \geq \sup\{\ldots\}.$$

Moreover, $\overline{m} = \infty$ holds in the case $N_e \neq \emptyset$. Indeed, (4.1a) implies

(1) $\quad M \cap \,]0,\infty[\neq \emptyset,$

and $0 \neq x_0 \in M$ combined with $(y_0, z_0) \in N_e$ yields, again by (4.1a),

$$M \ni y_0^n x_0 + y_0^{n-1} z_0 + \ldots + z_0 \to \infty.$$

2. To prove the inverse inequality for $\underline{m}$, abbreviate

$$\gamma := \inf\{\frac{z}{1-y} : (y,z) \in N_c\},$$

which so far may be infinite, and choose $\delta \in [0, \gamma[$. Then

$$y\,\delta + z \geq \delta \qquad \text{for all } (y, z) \in N,$$

because this inequality is trivial for $y \geq 1$. Therefore

$$\{x : yx + z < \delta\} \subset \{x : x < \delta\} \qquad \text{for all } (y, z) \in N,$$

while by the invariance of μ on the other hand

$$\textstyle\int_N \mu(\{x : yx + z < \delta\})\, d\nu = \mu(\{x : x < \delta\}) = \int_N \mu(\{x : x < \delta\})\, d\nu.$$

Since μ is locally finite, this implies

$$\{x : yx + z < \delta\} \overset{\mu}{=} \{x : x < \delta\} \qquad \text{for } \nu\text{-almost all } (y, z) \in \mathbf{R}_+^2.$$

If δ varies through $[0, \gamma[$, this yields

$$(yx + z) \wedge \gamma \overset{\mu}{=} x \wedge \gamma \qquad \text{for } \nu\text{-almost all } (y, z) \in \mathbf{R}_+^2.$$

Since both sides of this equation are continuous functions of x and (y, z), they agree therefore on the support of $\mu \otimes \nu$, hence

$$(yx + z) \wedge \gamma = x \wedge \gamma \qquad \text{for } x \in M \text{ and } (y, z) \in N.$$

By choosing $(y_0, z_0) \in N$ with $z_0 > 0$ thus

$$(y_0 x + z_0) \wedge \gamma = x \wedge \gamma \qquad \text{for all } x \in M.$$

This proves indeed the inequality $\underline{m} \geq \gamma$, because any $x_0 \in M \cap [0, \gamma[$ would lead to $y_0 x_0 + z_0 = x_0$, hence $y_0 < 1$, and thus to the contradiction

$$x_0 = \frac{z_0}{1 - y_0} \geq \gamma.$$

3. The corresponding proof for $\overline{m}$, under the additional hypothesis $N_e = \emptyset$, differs only at the beginning. Define here

$$\gamma := \sup\, \{\frac{z}{1 - y} : (y, z) \in N_c\},$$

which now may be assumed to be finite, and choose $\delta \in [\gamma, \infty[$. Then

$$y\,\delta + z \leq \delta \qquad \text{for all } (y, z) \in N,$$

because $N \setminus N_c$ may contain $(y, z) = (1, 0)$ only. Therefore

$$\{x : yx + z \leq \delta\} \supset \{x : x \leq \delta\} \qquad \text{for all } (y, z) \in N,$$

and the proof continues in complete analogy to part 2. $\square$

While it is easily seen that (b) extends to excessive measures, this fails for (a). Indeed, if ν is transient, the excessive measure $\varepsilon_0\,(\sum_{n \geq 0} P^n)$ is locally finite according to (2.2b) with inf $M = 0$, while the infimum on the right–hand side may be arbitrarily large.

A simple consequence of (4.3) is the following result:

(4.4) Proposition. *The support M of a nontrivial invariant measure $\mu \in \mathcal{M}(\mathbf{R}_+)$ is an interval whenever N is connected.*

Proof. With $0 \neq x_0 \in M$ (see (1) in the proof of (4.3)) the set

$$M_n := \{y^n x_0 + y^{n-1} z + \ldots + z : (y, z) \in N\}$$

is the continuous image of a connected set, hence

(1) $\quad]\underline{m}_n, \overline{m}_n[\subset M_n \subset [\underline{m}_n, \overline{m}_n] \qquad \text{for } n \in \mathbf{N}$

with appropriate bounds $\underline{m}_n$ and $\overline{m}_n$. Moreover, according to (4.1a) and (4.3)

(2) $\quad M_n \subset M \qquad \text{for } n \in \mathbf{N},$

(3) $\quad \underline{m}_n \to \inf M \qquad \text{and} \qquad \overline{m}_n \to \sup M,$

where in the case $N_e \neq \emptyset$ the choice $x_0 \neq 0$ is essential. Together, (1) – (3) prove the assertion. □

As will be seen in (6.1), the converse of (4.4) fails in a surprisingly general sense.

The final part of this section concerns properties not only of the support but of excessive measures themselves. The following technical result will be important:

(4.5) Proposition. *Let $(y_0, z_0) \in \mathbf{R}_+^2$ with $y_0 < 1$ be given. Then for $p > 0$ and $s > z_0 \,/\, (1 - y_0)$ there exist finite constants*

$$\alpha = \alpha(p) \qquad \textit{and} \qquad \gamma = \gamma(p, s)$$

such that under the hypothesis

$$\nu([0, y_0[\times [0, z_0[) \geq p$$

each excessive measure $\mu \in \mathcal{M}(\mathbf{R}_+)$ satisfies

$$\mu([0, t]) \leq \gamma \, \mu([0, s]) \, t^\alpha \qquad \textit{for } t \geq s.$$

Proof. If μ is excessive with respect to ν, then

$$\begin{aligned} \mu([0, t]) \, \nu([0, y_0[\times [0, z_0[) &\leq (\mu \otimes \nu)(\{(x; y, z) : yx + z \leq y_0 t + z_0\}) \\ &\leq \mu([0 \,,\, y_0 t + z_0]) \qquad \text{for all } t \geq 0. \end{aligned}$$

This yields the estimate

$$\begin{aligned} \mu([0, t]) &\leq p^{-1} \mu([0 \,,\, y_0 t + z_0]) \\ &\leq p^{-2} \mu([0 \,,\, y_0(y_0 t + z_0) + z_0]), \end{aligned}$$

hence by iteration

$$\begin{aligned}\mu([0,t]) &\leq p^{-k}\mu([0\,,\,y_0^k t + y_0^{k-1} z_0 + \ldots + z_0]) \\ &\leq p^{-k}\mu([0\,,\,y_0^k t + \frac{z_0}{1-y_0}]) \qquad \text{for } k \in \mathbf{N}.\end{aligned}$$

Therefore

$$\mu([0,t]) \leq p^{-k}\mu([0,s]) \qquad \text{whenever} \qquad y_0^k t + \frac{z_0}{1-y_0} \leq s.$$

This condition holds for $k \in \mathbf{N}$ satisfying

$$k \geq [(\log{(s - \frac{z_0}{1-y_0})} - \log t)\,/\,\log y_0] \geq 0,$$

with the convention $\log y_0 = -\infty$ for $y_0 = 0$. Since for $t \geq s$ this bound is nonnegative, finally

$$\mu([0,t]) \leq p^{-([\ldots]+1)}\,\mu([0,s]) \qquad \text{for } t \geq s,$$

and a simple computation provides the constants

$$\alpha := \log p\,/\,\log y_0 \qquad \text{and} \qquad \gamma := \frac{1}{p}\,(s - \frac{z_0}{1-y_0})^{-\alpha}. \quad \Box$$

The essential content of this result lies in the fact that the growth of an excessive measure is only polynomial whenever $\mathbf{P}(Y < 1) > 0$.

The last result of this section will be required for stability theorems:

(4.6) Lemma. *Let $\mathcal{N} \ni \nu_k \xrightarrow{\text{w}} \nu$ and $\mu_k \in \mathcal{M}(\mathbf{R}_+)$ be excessive with respect to ν_k. Then $\mu_k \xrightarrow{\text{v}} \mu \in \mathcal{M}(\mathbf{R}_+)$ implies that μ is excessive with respect to ν.*

Proof. Vague convergence is implied by weak convergence and compatible with forming product measures, hence

$$\mu_k \otimes \nu_k \xrightarrow{\text{v}} \mu \otimes \nu.$$

By monotone approximation this yields

$$\textstyle\int g\,d(\mu \otimes \nu) \leq \liminf_{k\to\infty} \int g\,d(\mu_k \otimes \nu_k) \qquad \text{for } 0 \leq g \in \mathcal{C}(\mathbf{R}_+ \times \mathbf{R}_+^2).$$

Therefore, with P_k denoting the kernel corresponding to ν_k,

$$\begin{aligned}\mu\,Pf &= \textstyle\int f(yx+z)\,\mu(dx)\nu(dy,dz) \\ &\leq \liminf_{k\to\infty} \textstyle\int f(yx+z)\,\mu_k(dx)\nu_k(dy,dz) \\ &= \liminf_{k\to\infty} \mu_k P_k f \\ &\leq \liminf_{k\to\infty} \mu_k f \\ &= \mu f \qquad \text{for } 0 \leq f \in \mathcal{K}(\mathbf{R}_+),\end{aligned}$$

which proves $\mu P \leq \mu$. $\square$

It should be mentioned that this result does not carry over to invariant measures, as can be shown by somewhat involved examples.

5. Existence and uniqueness of invariant measures

To derive ergodic theorems in the recurrent case, the following "localization" is essential:

(5.1) Definition. *Let ν be recurrent and $\underline{x} < t < \infty$. Then:*

(a) *${}^tP$ denotes the "hitting kernel" belonging to P and $[0,t]$, i.e.*

$$ {}^tP(x; B) := \mathbf{P}^x(X_T \in B) \qquad \text{for } x \in [0,t] \text{ and } B \in \mathcal{B}([0,t]), $$

where

$$ T := \inf\{n \in \mathbf{N} : X_n \in [0,t]\}; $$

(b) *$({}^tX_n)_{n\geq 0}$ denotes the "sojourn process" belonging to $(X_n)_{n\geq 0}$ and $[0,t]$, i.e.*

$$ {}^tX_n := X_{T_n} \qquad \text{for } n \geq 0, $$

where $T_0 < T_1 < \ldots$ are the random times when $(X_n)_{n\geq 0}$ is in $[0,t]$ and the notation ${}^tX_n^x$ is used in the case $X_0 = x$.

For easy reference a simple conclusion from probabilistic potential theory is stated explicitly:

(5.2) Lemma. *Let ν be recurrent and $\mu \in \mathcal{M}(\mathbf{R}_+)$ be excessive. If ${}^t\mu$ denotes the restriction of μ to $[0,t]$, $\underline{x} < t < \infty$, then*

(a) *${}^t\mu$ is invariant with respect to ${}^tP$,*

(b) *μ is invariant with respect to P.*

Proof. 1. If I_A for $A \in \mathcal{B}(\mathbf{R}_+)$ denotes the kernel

$$ I_A(x; \cdot) := 1_A(x)\, \varepsilon_x \qquad \text{for } x \in \mathbf{R}_+, $$

the crucial point is the inequality

(1) $\quad \mu(I_A \sum_{n\geq 0} (PI_{\mathbf{R}_+\setminus A})^n) \leq \mu$

(see e.g. IX, (62.4) and (31.6), in [8]).

2. Multiplied by PI_A from the right and specialized to $A = [0,t]$, (1) yields

$$ \begin{aligned} ({}^t\mu\, {}^tP)(B) &\leq (\mu P)(B) \\ &\leq \mu(B) \\ &= {}^t\mu(B) \qquad \text{for } B \in \mathcal{B}([0,t]). \end{aligned} $$

This proves (a), because ${}^t\mu$ is finite and ${}^tP$ is a stochastic kernel.

3. For $0 \le f \in \mathcal{K}(\mathbf{R}_+)$ choose now $t > \underline{x}$ with $\operatorname{supp} f \subset [0,t]$ and denote the restriction of f to $[0,t]$ by ${}^tf$. Then (a) implies

$$\begin{aligned} \mu f &= {}^t\mu\, {}^tf \\ &= ({}^t\mu\, {}^tP)\, {}^tf \\ &\le (\mu P) f \qquad \text{for } 0 \le f \in \mathcal{K}(\mathbf{R}_+), \end{aligned}$$

once more by (1), and this yields the inequality $\mu \le \mu P$ needed for (b). □

Now the existence of an invariant measure can be proved in the usual way (see e.g. [14]), under some simplification due to the monotonicity. More generally, the following version is required in Section 7:

(5.3) Proposition. *If ν is recurrent and $\underline{x} < t < \infty$, the measures*

$$\varrho_n(B) := \textstyle\sum_{0 \le m < n} \mathbf{P}(X_m \in B) \,/\, \sum_{0 \le m < n} \mathbf{P}(X_m \le t) \ \ \textit{for}\ \ B \in \mathcal{B}(\mathbf{R}_+),$$

defined for $n \ge n_0$ according to (2.2a)*, satisfy:*

(a) $\{\varrho_n : n \ge n_0\}$ *is a sequentially compact subset of* $\mathcal{M}(\mathbf{R}_+)$,

(b) *each limit point μ of the sequence $(\varrho_n)_{n \ge n_0}$ is a nontrivial invariant measure.*

Proof. (a) For arbitrary $s \ge 0$ choose $l \in \mathbf{N}$ such that

$$\vartheta := \mathbf{P}(X_l^s \le t) > 0,$$

which is possible in view of $t > \underline{x}$. With $\mu_m := \mathcal{L}(X_m)$ this implies by monotonicity

$$\begin{aligned} \mathbf{P}(X_{m+l} \le t) &\ge \textstyle\int_{0 \le x \le s} \mathbf{P}(X_l^x \le t)\, \mu_m(dx) \\ &\ge \textstyle\int_{0 \le x \le s} \mathbf{P}(X_l^s \le t)\, \mu_m(dx) \\ &= \vartheta\, \mathbf{P}(X_m \le s) \qquad \text{for } m \ge 0. \end{aligned}$$

With the norming constants

$$r_n := \textstyle\sum_{0 \le m < n} \mathbf{P}(X_m \le t)$$

this provides the estimate

$$\textstyle\sum_{0 \le m < n} \mathbf{P}(X_m \le s) \le \vartheta^{-1} \sum_{0 \le m < n} \mathbf{P}(X_{m+l} \le t) \le \vartheta^{-1} (r_n + l).$$

Since $r_n \to \infty$ by (2.2a), this yields

$$\limsup_{n \to \infty} \varrho_n([0,s]) \le \vartheta^{-1} < \infty,$$

i.e. the measures ϱ_n, $n \ge n_0$, are uniformly locally finite, and the assertion follows from general results of topological measure theory.

(b) The assumption $\varrho_{n_k} \xrightarrow{\text{v}} \mu$ yields $\mu \neq 0$, because

$$\mu([0,t]) \geq \limsup_{k\to\infty} \varrho_{n_k}([0,t]) = 1.$$

With $\mu_0 := \mathcal{L}(X_0)$, moreover,

$$\varrho_{n_k} f = r_{n_k}^{-1} \sum_{0\leq m<n_k} \mu_0 P^m f \qquad \text{for } 0 \leq f \in \mathcal{K}(\mathbf{R}_+).$$

By monotone approximation of $Pf \in \mathcal{C}(\mathbf{R}_+)$ this yields

$$\begin{aligned} \mu Pf &\leq \liminf_{k\to\infty} \varrho_{n_k} Pf \\ &= \liminf_{k\to\infty} r_{n_k}^{-1} \sum_{0<m\leq n_k} \mu_0 P^m f \\ &= \liminf_{k\to\infty} r_{n_k}^{-1} \sum_{0\leq m<n_k} \mu_0 P^m f \\ &= \mu f \qquad \text{for } 0 \leq f \in \mathcal{K}(\mathbf{R}_+), \end{aligned}$$

because $r_{n_k} \to \infty$ and $\mu_0 P^m f$ is bounded by $\max f$. Therefore μ is excessive and thus invariant by (5.2b). $\square$

The proof of the uniqueness of μ is much more involved. For the first step (2.6) is crucial:

(5.4) Lemma. *Let ν be recurrent and define*

$$C := \{(s,t) : 0 < s < t \text{ and } \underline{x} < t < \infty\},$$

$$\underline{Q}^x_{s,t} := \liminf_{n\to\infty} \sum_{0\leq m<n} 1_{[0,s]}(X^x_m) \,/\, \sum_{0\leq m<n} 1_{[0,t]}(X^x_m),$$

$$\overline{Q}^x_{s,t} := \limsup_{n\to\infty} \sum_{0\leq m<n} 1_{[0,s]}(X^x_m) \,/\, \sum_{0\leq m<n} 1_{[0,t]}(X^x_m)$$

for $(s,t) \in C$ and $x \in [0,t]$. Then the set

$$C^0 := \{(s,t)\in C\colon \underline{Q}^0_{\cdot,\cdot} \text{ and } \overline{Q}^0_{\cdot,\cdot} \text{ are continuous at } (s,t) \text{ almost surely}\},$$

with λ denoting Lebesgue measure, satisfies

(a) $\quad \lambda^2(C \setminus C^0) = 0,$

(b) $\quad \underline{Q}^x_{s,t} = \underline{Q}^0_{s,t}$ *a.s. and* $\overline{Q}^x_{s,t} = \overline{Q}^0_{s,t}$ *a.s. for* $x \in [0,t]$

whenever $(s,t) \in C^0$.

Proof. (a) For brevity let $Q^x_{s,t}$ stand for $\underline{Q}^x_{s,t}$ or $\overline{Q}^x_{s,t}$ in the sequel. Then $Q^x_{s,t}$ is well–defined, due to $X^x_0 \leq t$, and $(s,t;\omega) \to Q^x_{s,t}(\omega)$ is a measurable mapping from $C \times \Omega$ to [0,1]. Therefore

$$A := \{(s,t;\omega) \in C \times \Omega : \sup_{k\in\mathbf{N}} Q^0_{s-\frac{1}{k},t+\frac{1}{k}}(\omega) < \inf_{k\in\mathbf{N}} Q^0_{s+\frac{1}{k},t-\frac{1}{k}}(\omega)\}$$

is a measurable set such that

(1) $\quad Q^0_{\cdot,\cdot}(\omega)$ is discontinuous at (s,t) $\quad$ if and only if $\quad (s,t) \in A_\omega,$

because $Q^0_{\cdot,\cdot}(\omega)$ is nondecreasing in s and nonincreasing in t. Again by this monotonicity the section A_ω is countable along each line $s+t =$ const., hence

$$\lambda^2(A_\omega) = 0 \qquad \text{for all } \omega \in \Omega.$$

By Fubini this implies

$$\mathbf{P}(A_{s,t}) = 0 \qquad \text{for } \lambda^2\text{–almost all } (s,t) \in C,$$

and by (1) this verifies assertion (a).

(b) If r stands for s or t, monotonicity in x yields

$$(2) \qquad 1_{[0,r]}(X^x_m) \le 1_{[0,r]}(X^0_m) \qquad \text{for all } m \ge 0.$$

Now consider $\delta > 0$ small enough to satisfy

$$\delta < s \wedge (t-s) \wedge (t - \underline{x})$$

and choose $f \in \mathcal{K}(\mathbf{R}_+)$ such that

$$1_{[0,r-\delta]} \le f \le 1_{[0,r]}.$$

Then according to (2.6) the set

$$\{m \ge 0 : 1_{[0,r]}(X^x_m) < 1_{[0,r-\delta]}(X^0_m)\} \subset \{m \ge 0 : f(X^x_m) = 0, f(X^0_m) = 1\}$$

is almost surely finite, hence with probability 1

$$(3) \qquad 1_{[0,r]}(X^x_m) \ge 1_{[0,r-\delta]}(X^0_m) \qquad \text{for almost all } m \ge 0.$$

Since for $r = t$ the terms on the right–hand side of both (2) and (3) sum up to ∞ almost surely, this implies

$$Q^0_{s-\delta\,,\,t} \le Q^x_{s,t} \le Q^0_{s\,,\,t-\delta} \ \text{ a.s.}$$

Now assertion (b) follows for $\delta = 1/k \to 0$ from the hypothesis $(s,t) \in C^0$. $\square$

At the next step the pointwise ergodic theorem enters:

(5.5) Lemma. *Let ν be recurrent and $\mu \in \mathcal{M}(\mathbf{R}_+)$ be a nontrivial invariant measure. Then for $(s,t) \in C^0$, with the notation of* (5.4)*:*

$$\text{(a)} \qquad \mathbf{E}(\underline{Q}^x_{s,t}) = \mu([0,s]) \,/\, \mu([0,t]) = \mathbf{E}(\overline{Q}^x_{s,t}) \qquad \textit{for all } x \in [0,t],$$

$$\text{(b)} \qquad \frac{1}{n} \textstyle\sum_{0 \le m < n} \mathbf{P}({}^tX^x_m \le s) \to \mu([0,s]) \,/\, \mu([0,t])$$

for μ–almost all $x \in [0,t]$.

Proof. 1. Since $\mu([0,t])$ is finite and, by (4.3a) and (1.3a), strictly positive, the restriction ${}^t\mu$ of μ to $[0,t]$ may be assumed to be normalized, hence by (5.2a) to be a stationary distribution for ${}^tP$. Assume in the sequel that X_0

is distributed according to (the trivial extension of) ${}^t\mu$. Then $({}^tX_n)_{n\geq 0}$ is a stationary process, hence the classical ergodic theorem ensures that

$$(1)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{0\leq m<n} 1_{[0,s]}({}^tX_m) \quad \text{exists almost surely.}$$

Now the sequence of quotients defining $\underline{Q}^x_{s,t}$ and $\overline{Q}^x_{s,t}$ in (5.4) arises from the sequence of successive means of $(1_{[0,s]}({}^tX^x_n))_{n\geq 0}$ through "extension to the right by constancy" in an evident sense. Therefore it follows from (1) by Fubini that

$$(2)\qquad \frac{1}{n} \sum_{0\leq m<n} 1_{[0,s]}({}^tX^x_m) \to Q^x_{s,t} \ \text{ a.s.} \qquad \text{for } \mu\text{–almost all } x \in [0,t],$$

where $Q^x_{s,t}$ stands for $\underline{Q}^x_{s,t}$ or $\overline{Q}^x_{s,t}$.

2. Next, by stationarity and bounded convergence

$$\begin{aligned}
\mu([0,s]) &= (\lim_{n\to\infty}) \mathbf{E}\Big(\frac{1}{n} \sum_{0\leq m<n} 1_{[0,s]}({}^tX_m)\Big) \\
&= \mathbf{E}\Big(\lim_{n\to\infty} \frac{1}{n} \sum_{0\leq m<n} 1_{[0,s]}({}^tX_m)\Big) \\
&= \int_{[0,t]} \mathbf{E}\Big(\lim_{n\to\infty} \frac{1}{n} \sum_{0\leq m<n} 1_{[0,s]}({}^tX^x_m)\Big)\, \mu(dx) \\
&= \int_{[0,t]} \mathbf{E}(Q^x_{s,t})\, \mu(dx)
\end{aligned}$$

in view of (2). This verifies (a), because $\mathbf{E}(Q^x_{s,t})$ by (5.4b) is in fact independent of $x \in [0,t]$. Finally, (a) implies (b) by taking expectations in (2). □

Now one of the main results can be established:

(5.6) Theorem. *If ν is recurrent, then there exists a nontrivial invariant measure $\mu \in \mathcal{M}(\mathbf{R}_+)$ such that each excessive measure $\mu' \in \mathcal{M}(\mathbf{R}_+)$ is a multiple of μ.*

Proof. Choose μ according to (5.3) and let $\mu' \in \mathcal{M}(\mathbf{R}_+)$ be another nontrivial excessive, hence by (5.2b) invariant, measure. For any r choose, moreover, $t > r$ such that

$$(1)\qquad \gamma := \mu([0,t]) > 0 \qquad \text{and} \qquad \gamma' := \mu'([0,t]) > 0,$$

$$(2)\qquad \lambda((C \setminus C^0)_t) = 0,$$

which is possible according to (5.4a). If ${}^t\mu$ and ${}^t\mu'$ denote the restrictions of μ and μ' to $[0,t]$, the measures $\gamma^{-1}\,{}^t\mu$ and $(\gamma')^{-1}\,{}^t\mu'$ are contained in $\mathcal{M}_1(\mathbf{R}_+)$ by (1) and according to (5.5a) agree for sets $[0,s]$ with $(s,t) \in C^0$, hence on $\mathcal{B}([0,t])$, by (2). Therefore ${}^t\mu$ and ${}^t\mu'$ are linearly dependent, which for $t \to \infty$ extends to μ and μ'. □

Two comments on this result are in order:

— Uniqueness holds within the class of locally finite measures only, as is seen

already by a deterministic example: If $Y = 1/2 = Z$, then ν is recurrent with invariant measure $\mu = \varepsilon_1$. Since $x \to x/2 + 1/2$ is a bijection of the set $A := \mathbf{Q} \cap]1, \infty[$, however, the definition $\mu'(B) := |A \cap B|$ yields another (σ–finite) invariant measure.
— In the transient case nontrivial invariant measures $\mu \in \mathcal{M}(\mathbf{R}_+)$ may be absent, as follows from (4.3a) in the case $Y \geq 1$, or present, as can be shown in the case $Y = \gamma \in]0, 1[$ by a limiting procedure.

It is another question how to get the invariant measure from (5.6). It is not hard to translate the equation $\mu P = \mu$ into an integral equation for the function $F(t) := \mu([0, t])$, but in general it is impossible to solve it. Passing to Laplace transforms simplifies at least the integral equation:

(5.7) Proposition. *Let ν and μ be given according to* (5.6). *Then:*

(a) $\psi(u) := \int e^{-ux} \mu(dx) < \infty$ *for* $u > 0$,

(b) *up to a scalar, ψ is uniquely determined by the equation*

$$\psi(u) = \int \psi(uy) e^{-uz}\, \nu(dy, dz) \qquad \textit{for } u > 0.$$

Proof. While (a) is a simple consequence of (4.5), (b) follows from

$$\int e^{-ux}\, (\mu P)(dx) = \int \int e^{-u(xy+z)}\, \mu(dx)\, \nu(dy, dz)$$

and (5.6), because the Laplace transform determines μ. $\square$

If Y and Z are independent (as in the additive or multiplicative model), the integral equation in (5.7b) simplifies to

$$\psi(u) = \mathbf{E}(e^{-uZ})\, \mathbf{E}(\psi(uY)) \qquad \text{for } u > 0,$$

but is still rarely solvable.

6. Main properties of the invariant measure

The measure μ that (5.6) assigns to a recurrent distribution ν actually stands for a one–dimensional family. Nevertheless it will be briefly called "the invariant measure" in this and the following section. As pointed out at the end of the preceding section its quantitative determination is in general out of reach. Thus it is important to obtain at least a qualitative description.

A first property of the support M of μ entered already: (1.3) and (4.3) combine to the equations

$$\begin{aligned} \inf M &= \underline{x} = \inf\{\frac{z}{1-y} : (y, z) \in N_c\}, \\ \sup M &= \overline{x} = \sup\{\frac{z}{1-y} : (y, z) \in N_c\} \qquad \text{if } N_e = \emptyset, \\ &= \infty \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

It is a natural question to ask under which conditions, less restrictive than in (4.4), the whole interval $[\underline{x}, \overline{x}]$ is exhausted by M. A surprisingly general answer is given by the following result (for a special case see [2]):

(6.1) Theorem. *Let ν be recurrent with $\mathbf{P}(Y = 0) = 0$ and $\overline{x} = \infty$. Then the invariant measure μ has the support*

$$M = [\underline{x}, \infty[.$$

Proof. 1. By the hypothesis $\mathbf{P}(Y=0) = 0$ points $(0, z) \in N$ cannot be isolated, hence

$$\underline{x} = \inf\{\frac{z}{1-y} : (y, z) \in N \text{ with } 0 < y < 1\}.$$

Thus it suffices to prove that $x \in M$ if

$$x_0 = \frac{z_0}{1 - y_0} < x < \infty \qquad \text{with} \qquad (y_0, z_0) \in N \quad \text{and} \quad 0 < y_0 < 1.$$

To this end points (y, z) and the associated mappings $g : x \to yx + z$ will be identified throughout this proof. It will be accomplished by constructing two sequences $n_1, n_2, \ldots \in \mathbf{N}$ and $g_1, g_2, \ldots \in N$ such that

$$0 = n_0 \leq n_1 \leq \ldots,$$

$$g_i = (y_i, z_i) \qquad \text{with } y_i > 0,$$

$$0 \leq x - x_k \leq (1 - y_0)^k (x - x_0) \qquad \text{for } x_k = g_1 \circ \ldots \circ g_{n_k}(x_0).$$

Clearly, this implies $x \in M$ by (4.1a) and (4.2).

2. It suffices to construct n_{k+1} and $g_{n_k+1}, \ldots, g_{n_{k+1}}$ from n_k and $g_1, \ldots, g_{n_k}$ under the additional assumption $x_k < x$, because otherwise the definition $n_{k+1} = n_k$ works. Now the hypothesis $\overline{x} = \infty$ enters, providing

$$g_i' = (y_i', z_i') \in N \quad \text{with } y_i' > 0 \qquad \text{for } 1 \leq i \leq m$$

such that

$$\delta_k := y_1 \ldots y_{n_k}(g_1' \circ \ldots \circ g_m'(x_0) - x_0) > x - x_k.$$

Indeed, this follows from $\limsup_{n\to\infty} X_n^{x_0} = \infty$, because the relations $Y_n > 0$ and $(Y_n, Z_n) \in N$ hold almost surely.

3. Let now $l \in \mathbf{N}$ be defined by

$$(1) \qquad y_0^{l-1}\delta_k > x - x_k \geq y_0^l\,\delta_k.$$

Then the construction can be continued by $n_{k+1} = n_k + l + m$ and

$$g_i := \begin{cases} g_0 & \text{for } n_k < i \leq n_k + l, \\ g'_{i-(n_k+l)} & \text{for } n_k + l < i \leq n_{k+1}. \end{cases}$$

Indeed, since x_0 is a fixed point of $g_0 = (y_0, z_0)$, by affinity

$$\begin{aligned} x_{k+1} - x_k &= g_1 \circ \ldots \circ g_{n_k} \circ g_0^l \, (g_1' \circ \ldots \circ g_m' \, (x_0)) \\ &- g_1 \circ \ldots \circ g_{n_k} \circ g_0^l \, (x_0) \\ &= y_1 \ldots y_{n_k} y_0^l \, (g_1' \circ \ldots \circ g_m' \, (x_0) - x_0) \\ &= y_0^l \, \delta_k. \end{aligned}$$

By (1) this implies the required inequalities

$$x_{k+1} = x_k + y_0^l \, \delta_k \le x,$$

$$x - x_{k+1} = (x - x_k) - y_0 \, y_0^{l-1} \delta_k < (1 - y_0)(x - x_k). \quad \square$$

Clearly, the condition $\mathbf{P}(Y = 0) = 0$ is essential for this result, as is seen from the trivial case $Y = 0$, where μ is (a multiple of) the distribution of Z.

The argument used in the following proof can be traced back to Karlin [22]; it applies to locally finite measures as well:

(6.2) Theorem. *Let ν be recurrent with $\mathbf{P}(Y = 0) = 0$. Then the invariant measure μ is either absolutely continuous or singular with respect to Lebesgue measure.*

Proof. From the equation

$$(\mu P)(B) = \int_{y>0} \mu \, (\frac{B - z}{y}) \, d\nu \qquad \text{for } \; B \in \mathcal{B}(\mathbf{R}_+)$$

it is clear that $\mu \ll \lambda$ implies $\mu P \ll \lambda$ as well. If, therefore, μ is decomposed into the absolutely continuous part μ_c and the singular part μ_s, it follows from

$$\mu_c P + \mu_s P = \mu P = \mu = \mu_c + \mu_s$$

that $\mu_c P \le \mu_c$. Thus by (5.6) there exists a constant γ_c such that $\mu_c = \gamma_c \mu$, hence also a constant γ_s such that $\mu_s = \gamma_s \mu$. Now $\mu_c \wedge \mu_s = 0$ implies $\gamma_c \wedge \gamma_s = 0$ and verifies the assertion. $\square$

The remark following (6.1) shows that the condition $\mathbf{P}(Y = 0) = 0$ is essential for (6.2) as well.

It is another question how to decide on the alternative in (6.2). While some special cases are considered in Section 10, the only general result is provided by the following necessary conditions for singularity of μ:

(6.3) Proposition. *Let ν be recurrent and the invariant measure μ be singular with respect to Lebesgue measure. Then:*

(a) *ν is singular with respect to λ^2,*

(b) *ν_y and ν_z are singular with respect to λ, if $\nu = \nu_y \otimes \nu_z$.*

Proof. (a) Let $B \in \mathcal{B}(\mathbf{R}_+)$ satisfy

$$\lambda(B) = 0 \qquad \text{and} \qquad \mu(\mathbf{R}_+ \setminus B) = 0.$$

Then on the one hand

$$(1) \qquad \lambda^2(\{(y,z) : yx + z \in B\}) = \textstyle\int \lambda(B - yx)\, \lambda(dy) = 0 \qquad \text{for all } x \in \mathbf{R}_+,$$

while on the other hand

$$\begin{aligned} 0 &= \mu(\{x : x \notin B\}) \\ &= (\mu \otimes \nu)(\{(x; y, z) : yx + z \notin B\}) \\ &= \textstyle\int \nu(\{(y,z) : yx + z \notin B\})\, \mu(dx). \end{aligned}$$

In view of $\mu \neq 0$ this ensures the existence of $x_0 \in \mathbf{R}_+$ such that

$$(2) \qquad \nu(\{(y,z) : yx_0 + z \notin B\} = 0.$$

By (1) and (2), therefore, λ^2 and ν are supported by disjoint sets.

(b) With B as above it follows now that

$$\begin{aligned} 0 &= (\mu \otimes \nu_y \otimes \nu_z)(\{(x; y, z) : yx + z \notin B\}) \\ &= \textstyle\int \nu_y(\{y : yx + z \notin B\})\, \mu(dx)\, \nu_z(dz). \end{aligned}$$

Since $\operatorname{supp} \mu \neq \{0\}$ (see (1) in the proof of (4.3)), this ensures the existence of $x_0 > 0$ and $z_0 \geq 0$ such that

$$\lambda\,(\frac{B - z_0}{x_0}) = 0 \qquad \text{and} \qquad \nu_y\,(\mathbf{R}_+ \setminus \frac{B - z_0}{x_0}) = 0.$$

The corresponding argument for ν_z is even simpler. $\square$

It is an easy consequence of (1.4) that the independence of Y and Z is essential in (b). It applies, however, in the additive or multiplicative model. Moreover, it should be mentioned that (a) and (b) are by no means sufficient for singularity of μ (in this context see Section 10).

The following result is easily established for measures μ containing an atom with maximal mass (see [9]). Since this may fail for locally finite measures, the general proof gets more involved:

(6.4) Theorem. *Let ν be recurrent with $\mathbf{P}(Y = 0) = 0$ and $\underline{x} < \overline{x}$. Then the invariant measure μ is nonatomic.*

Proof. 1. It follows as in (6.2) that μ is either nonatomic or purely atomic. It suffices, therefore, to show that in the second case

$$(1) \qquad Z = \underline{x}\,(1 - Y),$$

thus in view of (1.4) obtaining a contradiction to the hypothesis $\underline{x} < \overline{x}$. To this end consider the countable set

$$R := \{x \in \mathbf{R}_+ : \mu(\{x\}) > 0\}.$$

By the invariance of μ it satisfies

$$0 = \mu(\mathbf{R}_+ \setminus R) = \textstyle\sum_{x \in R} \mu(\{x\})\, P(x; \mathbf{R}_+ \setminus R),$$

hence the definition

$$\widehat{P}(x, x') := P(x; \{x'\}) \qquad \text{for } \; x, x' \in R$$

yields a Markov chain on the state space R with invariant measure

$$\widehat{\mu}(x) := \mu(\{x\}) \qquad \text{for } \; x \in R.$$

2. For $\underline{x} < t < \infty$ define ${}^t\widehat{P}$ and ${}^t\widehat{\mu}$ in analogy to ${}^tP$ and ${}^t\mu$ in Section 5. Then ${}^t\widehat{P}$ is again a stochastic kernel with strictly positive and finite invariant measure ${}^t\widehat{\mu}$. By classical Markov chain theory this implies that all states $x \in R \cap [0, t]$ are (positive) recurrent with respect to ${}^t\widehat{P}$, hence also recurrent with respect to $\widehat{P}$. For $t \to \infty$ this extends to all states $x \in R$. The recurrent Markov kernel $\widehat{P}$ is in addition irreducible. Indeed, each restriction of $\widehat{\mu}$ to a single class yields again an invariant measure for $\widehat{P}$ and its trivial extension to $\mathbf{R}_+$ an invariant measure for P. Thus by the uniqueness property of μ there can exist only one class. Therefore, again by classical Markov chain theory, every σ–finite (not necessarily locally finite!) excessive measure for $\widehat{P}$ is in fact a multiple of $\widehat{\mu}$.

3. Now by the hypothesis $\mathbf{P}(Y = 0) = 0$ the sets

$$A(x, x') := \{(y, z) : yx + z = x'\}, \; x \in R,$$

are ν–almost disjoint for fixed x', hence

$$\textstyle\sum_{x \in R} \widehat{P}(x, x') = \sum_{x \in R} \nu(A(x, x')) \le 1 \qquad \text{for all } \; x' \in R,$$

i.e. the equidistribution on R is excessive for $\widehat{P}$. Therefore, according to part 2 of the proof,

$$\widehat{\mu}(x) = 1 \qquad \text{for all } \; x \in R \tag{2}$$

may be assumed in the sequel.

4. Since μ is locally finite, the support M of μ in view of (2) must consist of isolated points, hence R must contain $\underline{x} = \min M$. Now by (1.2a)

$$y\underline{x} + z \ge \underline{x} \quad \text{for } \nu\text{–almost all } \; (y, z),$$

which in view of $\mathbf{P}(Y = 0) = 0$ implies

$$yx + z > \underline{x} \quad \text{for } \nu\text{–almost all } \; (y, z) \qquad \text{whenever } \; x > \underline{x}.$$

By the invariance of $\widehat{\mu}$ this yields

$$\begin{aligned} 1 &= \textstyle\sum_{x \in R} \widehat{P}(x, \underline{x}) \\ &= \textstyle\sum_{x \in R} \nu(\{(y, z) : yx + z = \underline{x}\}) \\ &= \nu(\{(y, z) : y\underline{x} + z = \underline{x}\}), \end{aligned}$$

and (1) is established. $\square$

The remark following (6.1) shows again that the condition $\mathbf{P}(Y=0)=0$ is essential for (6.4).

The final result of this section is a strong stability statement, valid under an appropriate normalization:

(6.5) Theorem. *Let ν be recurrent with invariant measure μ and assume $\mathcal{N} \ni \nu_k \xrightarrow{w} \nu$. If $\mu_k \in \mathcal{M}(\mathbf{R}_+)$ is nontrivial and excessive with respect to ν_k, then*

$$(\mu_k([0,t]))^{-1}\,\mu_k \xrightarrow{v} (\mu([0,t]))^{-1}\,\mu \qquad \text{whenever } t > \underline{x} \text{ and } \mu(\{t\}) = 0.$$

Proof. By (1.3a) there exists $(y_0, z_0) \in N$ with $y_0 < 1$ satisfying the inequality $z_0\,/\,(1-y_0) < t$, where actually

$$\nu(\{(y,z) : y < y_0, z < z_0\}) > 0$$

may be assumed (slightly increasing y_0 and z_0 if necessary). By weak convergence therefore

$$p := \inf_{k\in\mathbf{N}} \nu_k(\{(y,z) : y < y_0, z < z_0\}) > 0$$

may be assumed next. Then (4.5) (with the roles of s and t interchanged) provides constants α and γ such that

(1) $\qquad \mu_k([0,s]) \le \gamma\,\mu_k([0,t])\,s^\alpha \qquad \text{for } s \ge t \text{ and all } k \in \mathbf{N}.$

Due to $\mu_k \neq 0$ this implies $\mu_k([0,t]) > 0$, hence

(2) $\qquad \mu_k([0,t]) = 1 \quad \text{for } k \in \mathbf{N} \qquad \text{and} \qquad \mu([0,t]) = 1$

may be assumed finally. By (1), moreover, the sequence $(\mu_k)_{k\in\mathbf{N}}$ is uniformly locally finite, hence (as in the proof of (5.3a)) each subsequence $(\mu'_k)_{k\in\mathbf{N}}$ contains a vaguely converging subsubsequence $(\mu''_k)_{k\in\mathbf{N}}$. By (4.6) and (5.6) its limit is of the form $\delta\mu$ assigning measure 0 to the set $\{t\}$. By (2) this implies

$$\delta = \delta\mu([0,t]) = \lim_{k\to\infty} \mu''_k([0,t]) = 1,$$

and the assertion follows. $\square$

This result offers an approximation method for the invariant measure μ associated with a recurrent distribution ν: choose

$$\nu_k := \frac{1}{k}\,\varepsilon_{(0,0)} + (1-\frac{1}{k})\,\nu \qquad \text{for } k \in \mathbf{N}$$

and observe that an invariant measure μ_k for ν_k can be determined more or less explicitly (see Section 8).

7. Ratio ergodic theorems

A first information on the fluctuation of a recurrent sequence $(X_n)_{n\geq 0}$ by means of its invariant measure is given by the following mean ergodic theorem for ratios, holding without any assumption on the initial law:

(7.1) Theorem. *If ν is recurrent with invariant measure μ, then*

$$\sum_{0\leq m<n} \mathbf{E}(f_1(X_m)) \,/\, \sum_{0\leq m<n} \mathbf{E}(f_2(X_m)) \to \mu f_1 \,/\, \mu f_2$$

for $f_i \in \mathcal{K}_\mu(\mathbf{R}_+)$ with $\mu f_2 \neq 0$.

Proof. Choose $t > \underline{x}$ with $\mu(\{t\}) = 0$ and consider the measures ϱ_n defined in (5.3). If $(\varrho_{n_k})_{k\in\mathbf{N}}$ is any vaguely converging subsequence, (5.3b) and (5.6) imply

(1) $\qquad \varrho_{n_k} f \to \delta\mu f \qquad$ for all $f \in \mathcal{K}(\mathbf{R}_+)$,

where the constant δ satisfies

$$\delta\mu([0,t]) = \lim_{k\to\infty} \varrho_{n_k}([0,t]) = 1,$$

hence is independent of the subsequence. Since (1) extends from $\mathcal{K}(\mathbf{R}_+)$ to $\mathcal{K}_\mu(\mathbf{R}_+)$ by monotone approximation, therefore

$$\varrho_n f_i \to \delta\mu f_i \qquad \text{for } i = 1,\ 2$$

by (5.3a). Finally, the constant δ disappears by taking quotients. $\square$

The proof of a pointwise analogue of (7.1) is much more involved. The first step consists in proving a counterpart of (5.4):

(7.2) Lemma. *Let ν be recurrent with invariant measure μ and for t in*

$$D :=]\underline{x}, \infty[$$

denote by ${}^t\mu$ the restriction of μ to $[0,t]$. Then the set

$$D^0 := \{t \in D : \mathbf{P}(\textstyle\bigcup_{n\geq 0} \{X_n^x = t\}) = 0 \text{ for } \mu\text{-almost all } x \in [0,t]\},$$

with λ denoting Lebesgue measure, satisfies

(a) $\qquad \lambda(D \setminus D^0) = 0,$

(b) $\qquad$ *${}^tP$ is a "${}^t\mu$-Feller kernel" whenever $t \in D^0$, i.e.*

$$ {}^tPf \in \mathcal{C}_{{}^t\mu}([0,t]) \qquad \text{for } f \in \mathcal{C}_{{}^t\mu}([0,t]).$$

Proof. 1. $X_n^x(\omega) - t$ is measurable in (t, x, ω) for all $n \in \mathbf{N}$, hence

$$A := \textstyle\bigcup_{n\in\mathbf{N}} \{(t,x,\omega) : 0 \leq x \leq t \text{ and } X_n^x(\omega) = t\}$$

is a measurable set with

$$\lambda(A_{x,\omega}) = 0 \qquad \text{for all } (x,\omega),$$

because the sections $A_{x,\omega}$ are countable. Therefore, by Fubini

$$(\mu \otimes \mathbf{P})(A_t) = 0 \qquad \text{for } \lambda\text{–almost all } t \in D.$$

This proves (a), because the last equation is equivalent to

$$\mathbf{P}(A_{t,x}) = 0 \qquad \text{for } \mu\text{–almost all } x \in [0,t].$$

2. To verify (b), fix $t \in D^0$ and $f \in \mathcal{C}_{^t\mu}([0,t])$. Then the set

$$B_1 := \{x \in [0,t] : \mathbf{P}(\bigcup_{n \geq 0} \{X_n^x = t\}) > 0\}$$

is a μ–null set, and with the notation

$$B := \{x \in [0,t] : f \text{ discontinuous at } x\}$$

this holds as well for the set

$$B_2 := \{x \in [0,t] : {}^tP(x; B) > 0\}.$$

Indeed, ${}^t\mu$ is by (5.2a) invariant with respect to ${}^tP$, hence

$$\int {}^tP(x;B)\, {}^t\mu(dx) = {}^t\mu(B) = 0.$$

Since $B' := B_1 \cup B_2$ is again a μ–null set, the proof will be completed by establishing

$$(1) \qquad ({}^tPf)(x_k) \to ({}^tPf)(x_0) \qquad \text{whenever} \qquad [0,t] \ni x_k \to x_0 \notin B'.$$

3. To this end denote by T_k the random time when $(X_n^{x_k})_{n \in \mathbf{N}}$ hits $[0,t]$ first. Since X_n^x depends continuously on x,

$$X_m^{x_0}(\omega) > t \quad \text{for } 0 < m < n \qquad \text{and} \qquad X_n^{x_0}(\omega) < t$$

entails

$$X_m^{x_k}(\omega) > t \quad \text{for } 0 < m < n \qquad \text{and} \qquad X_n^{x_k}(\omega) < t$$

for almost all $k \in \mathbf{N}$, hence

$$\{X_{T_0}^{x_0} \neq t\} \subset \{T_k = T_0 \text{ for almost all } k \in \mathbf{N}\}.$$

By assumption $x_0 \notin B_1$, hence $X_{T_0}^{x_0} \neq t$ with probability 1 and thus

$$T_k \to T_0 \text{ a.s.},$$

which, once more by the continuity of X_n^x in x, implies

$$(2) \qquad \mathbf{P}(X_{T_k}^{x_k} \to X_{T_0}^{x_0}) = 1.$$

By assumption $x_0 \notin B_2$ as well, hence

$$\mathbf{P}(X_{T_0}^{x_0} \in B) = {}^tP(x_0; B) = 0$$

and thus by the definition of B

(3) $\quad \mathbf{P}(f \text{ continuous at } X_{T_0}^{x_0}) = 1.$

Since f is bounded, (2) and (3) combine to

$$\mathbf{E}(f(X_{T_k}^{x_k})) \to \mathbf{E}(f(X_{T_0}^{x_0})),$$

and (1) is established. $\square$

At the next step (5.5) is essential:

(7.3) Lemma. *Let ν be recurrent with invariant measure μ. Suppose, with the notations of* (7.2) *and* (5.4),

$(*)$ $\quad t \in D^0 \quad$ *and* $\quad C_t^0$ *dense in* $[0,t]$

and, in accordance with (4.3), *${}^t\mu$ to be normalized. Then the process $({}^tX_n)_{n\geq 0}$ is stationary and ergodic, if X_0 is distributed according to (the trivial extension of) ${}^t\mu$.*

Proof. 1. The following fact will be used: the measures

$$\varrho_n^x := \frac{1}{n}\sum\nolimits_{0\leq m<n} \varepsilon_x({}^tP)^m \in \mathcal{M}_1([0,t])$$

satisfy

$$\varrho_n^x \overset{\text{w}}{\to} {}^t\mu \qquad \text{for } \mu\text{–almost all } x \in [0,t].$$

Indeed, this is a consequence of (5.5b), letting s vary through a countable dense subset of C_t^0. Thus

(1) $\quad \varrho_n^x\, g \to {}^t\mu\, g$ for $g \in \mathcal{C}_{{}^t\mu}([0,t])$ $\quad$ for μ–almost all $x \in [0,t]$.

2. The stationarity of $({}^tX_n)_{n\geq 0}$ is immediate from (5.2a). To prove the ergodicity it suffices to verify

$$\frac{1}{n}\sum\nolimits_{0\leq m<n} \mathbf{E}(f({}^tX_0,\ldots,{}^tX_k)\prod\nolimits_{1\leq i\leq l} f_i({}^tX_{m+k+i}))$$

$$\to \mathbf{E}(f({}^tX_0,\ldots,{}^tX_k))\,\mathbf{E}(\prod\nolimits_{1\leq i\leq l} f_i({}^tX_i))$$

for $k,l \in \mathbf{N}$ and $f \in \mathcal{C}(\prod_{0\leq i\leq k}[0,t]), f_i \in \mathcal{C}([0,t])$. With the notations

$$g(x) := \mathbf{E}(\prod\nolimits_{1\leq i\leq l} f_i({}^tX_i^x)) \qquad \text{and} \qquad h_n(x) := \varrho_n^x\, g$$

this is equivalent to

(2) $\quad \mathbf{E}(f({}^tX_0,\ldots,{}^tX_k)\,h_n({}^tX_k)) \to \mathbf{E}(f({}^tX_0,\ldots,{}^tX_k))\,{}^t\mu\, g.$

3. Now $g \in \mathcal{C}_{{}^t\mu}([0,t])$ holds even for $f_i \in \mathcal{C}_{{}^t\mu}([0,t])$. Indeed, this is true for $l=1$ by (7.2b) and extends to general l by induction, because by the Markov property

$$g(x) = \mathbf{E}(\prod\nolimits_{1\leq i<l} f_i({}^tX_i^x)\,({}^tPf_l)({}^tX_{l-1}^x))$$

and $\mathcal{C}_{{}^t\mu}([0,t])$ is closed with respect to multiplication. Therefore (1) applies and in view of $\mathcal{L}({}^tX_k) = {}^t\mu$ yields

$$h_n({}^tX_k) \to {}^t\mu\, g \ \text{ a.s.},$$

which by bounded convergence verifies (2). $\square$

Now the central result of this paper can be established, holding again without any assumption on the initial law:

(7.4) Theorem. *If ν is recurrent with invariant measure μ, then*

$$\textstyle\sum_{0\le m<n} f_1(X_m) \,/\, \sum_{0\le m<n} f_2(X_m) \to \mu f_1 \,/\, \mu f_2 \ \ \textit{a.s.}$$

for $f_i \in \mathcal{K}_\mu(\mathbf{R}_+)$ with $\mu f_2 \neq 0$.

Proof. 1. It suffices to prove the assertion under the assumption $X_0 = x_0$, because its general validity then follows by integration. According to (5.4a) and (7.2a) there exists t such that

(1) $\qquad t > x_0 \quad$ and $\quad \operatorname{supp} f_i \subset [0,t] \quad$ for $\ i = 1,\ 2,$

(2) $\qquad t$ satisfies condition $(*)$ in (7.3).

Therefore the classical ergodic theorem, combined with Fubini, yields

(3) $\qquad \dfrac{1}{n}\sum_{0\le m<n} 1_{[0,s]}({}^tX^x_m) \to \mu([0,s])/\mu([0,t])$ a.s.

for μ–almost all $x \in [0,t]$. But the sequence defining $\underline{Q}^x_{s,t}$ and $\overline{Q}^x_{s,t}$ in (5.4) is a trivial extension (as specified in the proof of (5.5)) of the sequence in (3). Thus (5.4b) applies and under the assumption $s \in C^0_t$ implies that (3) in fact holds for all $x \in [0,t]$, hence in particular for x_0.

2. Now assume f_i to be a linear combination of functions $1_{[0,s]}, s \in C^0_t$. Then it follows from part 1 that

$$\textstyle\sum_{0\le m<n} f_i(X_m) \,/\, \sum_{0\le m<n} 1_{[0,t]}(X_m) \to \dfrac{1}{\mu([0,t])} \int_{[0,t]} f_i \, d\mu \ \text{ a.s.}$$

Since C^0_t is dense in $[0,t]$, this convergence extends by monotone approximation first to the case $f_i \in \mathcal{K}(\mathbf{R}_+)$ and then to the general case. The assertion follows by taking quotients. $\square$

As a first consequence of this pointwise ergodic theorem it can be shown that a recurrent sequence $(X_n)_{n\ge0}$ is in fact recurrent and irreducible in a rather strong sense:

(7.5) Theorem. *Let ν be recurrent with invariant measure μ and define the random set*

$$L(\omega) := \{x \in \mathbf{R}_+ : x \textit{ is limit point of } (X_n(\omega))_{n\ge0}\}.$$

Then with probability 1

$$L(\omega) = \operatorname{supp} \mu.$$

Proof. 1. Let G_k, $k \in \mathbf{N}$, be a countable base of $\mathbf{R}_+$ consisting of bounded sets with a boundary of μ–measure 0 and let $t > \underline{x}$ satisfy $\mu(\{t\}) = 0$. Then by (7.4) with probability 1

$$\sum_{0 \le m < n} 1_{G_k}(X_m) \,/\, \sum_{0 \le m < n} 1_{[0,t]}(X_m) \to \mu(G_k) \,/\, \mu([0,t])$$

for all $k \in \mathbf{N}$. Therefore

$$\mathbf{P}(\sum_{n \ge 0} 1_{G_k}(X_n) = \infty \ \text{ whenever } \ \mu(G_k) > 0) = 1,$$

which implies the inclusion

$$L(\omega) \supset \operatorname{supp} \mu \ \text{ a.s.}$$

2. To prove the converse denote by $L_t(\omega)$ the analogue of $L(\omega)$ for the sequence $({}^tX_n)_{n \ge 0}$. Clearly

$$L(\omega) = \bigcup_{\underline{x} < t \in \mathbf{N}} L_t(\omega),$$

hence it suffices to verify

$$L_t(\omega) \subset \operatorname{supp} \mu \ \text{ a.s.} \qquad \text{for } \ t > \underline{x}.$$

This is obvious, if X_0 is distributed as in (7.3), because in this case with probability 1

$${}^tX_n \in \operatorname{supp} \mu \,\cap\, [0,t] \qquad \text{for all } \ n \ge 0.$$

Now an application of (2.6) to any function $f \in \mathcal{K}(\mathbf{R}_+)$ with $f(x) = x$ on $[0,t]$ shows that the distribution of X_0 is actually irrelevant. $\square$

Together, (2.2) and (7.5) imply that the two main characterizations of recurrence/transience from classical Markov chain theory carry over to affine recursions in the following form:
— If ν is recurrent, then for $x \in \operatorname{supp} \mu$ always

$$\mathbf{P}^x(X_n \in G \ \text{ infinitely often}) = 1,$$

hence

$$\mathbf{E}^x(|\{n \ge 0 : X_n \in G\}|) = \infty,$$

provided G is an open neighborhood of x.
— If ν is transient, then for $x \in \mathbf{R}_+$ always

$$\mathbf{E}^x(|\{n \ge 0 : X_n \in K\}|) < \infty,$$

hence

$$\mathbf{P}^x(X_n \in K \ \text{ infinitely often}) = 0,$$

provided K is a compact subset of $\mathbf{R}_+$.

The final result of this section shows that affine recursions in the recurrent case are not only irreducible but also aperiodic in a strong sense:

(7.6) Proposition. *Let ν be recurrent with invariant measure μ. Then for every $x_0 \in \mathbf{R}_+$ and each open subset G_0 of $\mathbf{R}_+$ with $\mu(G_0) > 0$ there exists $n_0 \in \mathbf{N}$ such that*

$$\mathbf{P}^{x_0}(X_n \in G_0) > 0 \qquad \textit{for all } n \geq n_0.$$

Proof. 0. If $X_n \to x$ a.s., then $\operatorname{supp}\mu = \{x\}$ (e.g. by (7.5)). Since the assertion is trivial in this case, $\underline{x} < \overline{x}$ will be assumed henceforth.

1. In a first step $m \geq 0$ and $s > \underline{x}$ can be found such that

$$(1) \qquad \mathbf{P}(X_m^x \in G_0) > 0 \qquad \text{for } \underline{x} \leq x < s.$$

Indeed, $\mathbf{P}(X_m^{\underline{x}} \in G_0) > 0$ for some m follows from (7.5) and extends to a neighborhood of $\underline{x}$, because P^m is again a Feller kernel and thus $P^m 1_{G_0}$ is lower semicontinuous.

2. In the next step $k \geq 0$ can be found such that

$$(2) \qquad \mathbf{P}(\underline{x} \leq X_k^{x_0} < s) > 0.$$

Indeed, $(X_n^{x_0})_{n\geq 0}$ hits $]\underline{x}, \infty[$ in view of $\underline{x} < \overline{x}$ and stays afterwards in $[\underline{x}, \infty[$ according to (1.2a), hence visits $[\underline{x}, s[$ infinitely often by (7.5) – all this with probability 1.

3. In a final step $l \geq 0$ can be found such that

$$(3) \qquad \mathbf{P}(\underline{x} \leq X_l^x < s) > 0 \quad \text{and} \quad \mathbf{P}(\underline{x} \leq X_{l+1}^x < s) > 0 \qquad \text{for } \underline{x} \leq x < s.$$

Indeed, fix x and choose $t > s$ and $l \geq 0$ such that

$$\mathbf{P}(X_1^s \leq t) > 0 \qquad \text{and} \qquad \mathbf{P}(X_l^t < s) > 0.$$

Then by (1.2a) and monotonicity

$$\mathbf{P}(\underline{x} \leq X_l^x < s) = \mathbf{P}(X_l^x < s) \geq \mathbf{P}(X_l^t < s) > 0.$$

With $\mu_1 := \mathcal{L}(X_1^x)$ moreover

$$\mathbf{P}(\underline{x} \leq X_{l+1}^x < s) \geq \int_{[\underline{x},t]} \mathbf{P}(\underline{x} \leq X_l^{x_1} < s)\, \mu_1(dx_1),$$

where as above

$$\mathbf{P}(\underline{x} \leq X_l^{x_1} < s) > 0 \qquad \text{for } x_1 \in [\underline{x}, t]$$

and

$$\mu_1([\underline{x}, t]) = \mathbf{P}(\underline{x} \leq X_1^x \leq t) \geq \mathbf{P}(X_1^s \leq t) > 0.$$

4. By combining k steps according to (2), i times l steps and j times $l+1$ steps according to (3), and m steps according to (1), the Markov property yields

$$\mathbf{P}^{x_0}(X_n \in G_0) > 0 \qquad \text{for } n = k + il + j(l+1) + m.$$

Since $i, j \geq 0$ are arbitrary, the requirement is met by

$$n_0 := k + (l-1)l + m. \quad \square$$

In view of (7.6) the following open problem can be posed: is it possible to strengthen (7.1) to a strong ratio limit theorem as valid for irreducible and aperiodic recurrent random walk?

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