Normal forms and linearity in nonflat domains outline of a talk

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February 4, 2015

1 Introduction

- Dana Scott and Juri Ershov [late 60's–70's]: Scott–Ershov domains with Scott-continuous functions provide an appropriate framework for higher-type computability and semantics of programming languages.
- Gordon Plotkin [Plotkin 1977]: There are inherently nonsequential functionals in Scott's model:
$$\mathrm{pcond}(q,x,y) = \begin{cases} x & \text{if } q = \mathtt{tt}, \\ y & \text{if } q = \mathtt{ff}, \\ x \cap y & \text{if } q = \bot. \end{cases}$$
- Gerard Berry [Berry1978]: If a functional is sequential, it has to be stable (that is, preserve consistent infima).
- Guo-Qiang Zhang [Zhang 1989–1992]: In order to represent stable domains by information systems, we have to require linearity (here, "atomicity"): if a formal neighborhood entails a token, it must do so with a single witness.
- Stability and atomicity are quite relevant to classical [Girard *et al.* 1989] and intuitionistic linear logic [Bucciarelli *et al.* 2009-10].
- Helmut Schwichtenberg and the Munich group [Schwichtenberg, Huber, B., Ranzi 2006–] working with *nonflat* base types, have shown among other things density, preservation of values, adequacy, and definability, sometimes within atomic systems alone, sometimes without.
- Why nonflat? (a) Trivially good reasons: injectivity of constructors and nonoverlapping of their ranges. (b) Deeper good reasons: more degrees of freedom in the model allow for stronger results.
- Why *not* nonflat? (a) Trivially good reasons: combinatorial chaos. (b) Deeper good reasons: *flat* base types are atomic but *nonflat* aren't. *But*: function spaces preserve atomicity!

2 Domain representations and types

- Information system $A = (\mathrm{Tok}, \mathrm{Con}, \vdash)$

$$\begin{aligned}
&\{a\} \in \mathrm{Con}, \\
&U \subseteq V \wedge V \in \mathrm{Con} \to U \in \mathrm{Con}, \\
&U \in \mathrm{Con} \wedge a \in U \to U \vdash a, \\
&U \vdash V \wedge V \vdash c \to U \vdash c, \\
&U \in \mathrm{Con} \wedge U \vdash b \to U \cup \{b\} \in \mathrm{Con}.
\end{aligned}$$

- Coherent information system

$$\forall_{a,a' \in U} \{a, a'\} \in \mathrm{Con} \to U \in \mathrm{Con}. \tag{1}$$

Write $a \asymp b$ for $\{a, b\} \in \mathrm{Con}$, and even $U \asymp V$ for $U \cup V \in \mathrm{Con}$.

- Function space $A \to B$

$$\begin{aligned}
&\langle U, b\rangle \in \mathrm{Tok} := U \in \mathrm{Con}_A \wedge b \in \mathrm{Tok}_B, \\
&\langle U, b\rangle \asymp \langle U', b'\rangle := U \asymp_A U' \to b \asymp_B b', \\
&W \vdash \langle U, b\rangle := WU \vdash_B b,
\end{aligned}$$

where

$$b \in WU := \exists_{U' \in \mathrm{Con}_A} \left(\langle U', b\rangle \in W \wedge U \vdash_A U'\right).$$

Fact 1. *The function space of two coherent information systems is itself a coherent information system.*

- Atomic information system

$$U \vdash b \to \exists_{a \in U} \{a\} \vdash b; \tag{2}$$

Fact 2. *The function space of two atomic information systems is itself an atomic information system.*

- Ideal $x \in \mathrm{Ide}$

$$\forall_{U \subseteq^f x} (U \in \mathrm{Con} \wedge \forall_{b \in \mathrm{Tok}} (U \vdash b \to b \in x)).$$

Coherent domains (with countable bases) are algebraic bounded complete cpo's, where every set of compacts has a least upper bound exactly when each of its *pairs* has a least upper bound.

Fact 3. *Let* $(\mathrm{Tok}, \mathrm{Con}, \vdash)$ *be a coherent information system. Then* $(\mathrm{Ide}, \subseteq, \varnothing)$ *is a coherent domain with compacts given by* $\{\overline{U} \mid U \in \mathrm{Con}\}$. *Conversely, every coherent domain can be represented by a coherent information system.*

- Approximable mapping $r \subseteq \mathrm{Con}_A \times \mathrm{Con}_B$

$$\begin{aligned}
&\langle \varnothing, \varnothing\rangle \in r, \\
&\langle U, V_1\rangle, \langle U, V_2\rangle \in r \to \langle U, V_1 \cup V_2\rangle \in r, \\
&U \vdash_\rho U' \wedge \langle U', V'\rangle \in r \wedge V' \vdash_\sigma V \to \langle U, V\rangle \in r.
\end{aligned}$$

Fact 4. *There is a bijective correspondence between the approximable mappings from ρ to σ and the ideals of the function space $\rho \to \sigma$; domains (with Scott continuous functions) and information systems (with approximable mappings) are categorically equivalent [Scott 1982]. Moreover, the equivalence is preserved if we restrict ourselves to the coherent case [B 2013].*

- Base types ι

$$\begin{aligned}
\mathbb{B} &= \{\texttt{tt}, \texttt{ff}\}, \\
\mathbb{N} &= \{\texttt{0}, \texttt{S0}, \texttt{SS0}, \ldots\}, \\
\mathbb{D} &= \{\texttt{0}, \texttt{1}, \texttt{S0}, \ldots, \texttt{B01}, \ldots, \texttt{BS0B01}, \ldots\},
\end{aligned}$$

and higher types $\rho \to \sigma$.

- Partiality at base types ι is not a distinguished *token* but a distinguished *nullary constructor* $*_\iota$: the base types are already *nonflat*:

$$\begin{aligned}
\mathbb{B} &= \{*, \texttt{tt}, \texttt{ff}\}, \\
\mathbb{N} &= \{*, \texttt{0}, \texttt{S}*, \texttt{S0}, \texttt{SS}*, \texttt{SS0}, \ldots\}, \\
\mathbb{D} &= \{*, \texttt{0}, \texttt{1}, \texttt{S}*, \texttt{S0}, \ldots, \texttt{B}*\texttt{1}, \ldots, \texttt{BS}*\texttt{B01}, \ldots\}.
\end{aligned}$$

- The information system induced by $\mathbb{D}$:

$$\begin{aligned}
&*, \texttt{0}, \texttt{1} \in \mathrm{Tok}, \\
&a \in \mathrm{Tok} \to \mathrm{S}a \in \mathrm{Tok}, \\
&a, b \in \mathrm{Tok} \to \mathrm{B}ab \in \mathrm{Tok}, \\[4pt]
&a \asymp * \wedge * \asymp a, \\
&a \asymp a' \to \mathrm{S}a \asymp \mathrm{S}a', \\
&a \asymp a' \wedge b \asymp b' \to \mathrm{B}ab \asymp \mathrm{B}a'b', \\[4pt]
&U \vdash *, \\
&U \vdash a \to \mathrm{S}U \vdash \mathrm{S}a, \text{ for } U \neq \varnothing, \\
&U \vdash a \wedge V \vdash b \to \mathrm{B}UV \vdash \mathrm{B}ab, \text{ for } U, V \neq \varnothing, \\
&U \vdash b \to U \cup \{*\} \vdash b,
\end{aligned}$$

where $\mathrm{B}UV := \{Bab \mid a \in U, b \in V\}$.

Fact 5. *Let ι be an algebra given by constructors. The triple $(\mathrm{Tok}_\iota, \mathrm{Con}_\iota, \vdash_\iota)$ is a coherent information system.*

- Our technical motivation draws from the following.

Inconvenience 6. *The systems $\mathbb{B}$ and $\mathbb{N}$ are atomic but $\mathbb{D}$ is not: $\{\texttt{B0}*, \texttt{B}*\texttt{1}\} \vdash \texttt{B01}$ but $\{\texttt{B0}*\} \nvdash \texttt{B01}$ and $\{\texttt{B}*\texttt{1}\} \nvdash \texttt{B01}$.*

Inconvenience 7. *At base types antisymmetry holds for tokens, but neither for neighborhoods (e.g., $\{\texttt{B0}*, \texttt{B}*\texttt{1}\} \sim \{\texttt{B01}\}$ and $\{\texttt{S0}, \texttt{S}*\} \sim \{\texttt{S0}\}$) nor, consequently, at higher types.*

3 Neighborhood mappings

- Let ρ, σ be types. A mapping $f : \mathrm{Con}_\rho \to \mathrm{Con}_\sigma$ is *compatible*, *monotone*, and *consistent* if

$$
\begin{aligned}
&U_1 \sim_\rho U_2 \to f(U_1) \sim_\sigma f(U_2),\\
&U_1 \vdash_\rho U_2 \to f(U_1) \vdash_\sigma f(U_2),\\
&U_1 \asymp_\rho U_2 \to f(U_1) \asymp_\sigma f(U_2),
\end{aligned}
$$

  respectively.

  Lemma 8. *Let $f : \mathrm{Con}_\rho \to \mathrm{Con}_\sigma$ be a neighborhood mapping.*

  1. *It is monotone if and only if it is compatible with equientailment and $f(U_1 \cup U_2) \vdash_\sigma f(U_1) \cup f(U_2)$ for every $U_1, U_2 \in \mathrm{Con}_\rho$ with $U_1 \asymp_\rho U_2$.*
  2. *If it is monotone, then it is also consistent.*

- The *idealization* $\hat{f}$ of a neighborhood mapping $f : \mathrm{Con}_\rho \to \mathrm{Con}_\sigma$ is the token set

$$
\hat{f} := \left\{ \langle U, b \rangle \in \mathrm{Tok}_{\rho\to\sigma} \mid \underset{U_1,\dots,U_m \in \mathrm{Con}_\rho}{\exists} \left(U \vdash_\rho \bigcup_{j=1}^{m} U_j \wedge \bigcup_{j=1}^{m} f(U_j) \vdash_\sigma b \right) \right\}.
$$

  Proposition 9. *Let ρ, σ be types, and f be a neighborhood mapping at type $\rho \to \sigma$. Then $\hat{f}$ is an ideal if and only if f is consistent.*

- Not all ideals are induced by neighborhood mappings: e.g., at type $\mathbb{N} \to \mathbb{N}$ take $\left\{ \langle \mathtt{0}, \mathtt{S}^n * \rangle \mid n = 0, 1, \dots \right\}$. Neighborhood mappings are those approximable maps r for which $r(U)$ is covered by a finite collection $V_1, \dots, V_m \in \mathrm{Con}_\sigma$ for every $U \in \mathrm{Con}_\rho$.

4 Normal forms at base types

- Let ρ be a type. A neighborhood-mapping $f : \mathrm{Con}_\rho \to \mathrm{Con}_\rho$ is a *normal form mapping (at type ρ)* if it preserves information and identifies equivalent neighborhoods, that is,

$$
\begin{aligned}
&f(U) \sim_\rho U,\\
&U_1 \sim_\rho U_2 \to f(U_1) = f(U_2).
\end{aligned}
$$

  Every normal form mapping is monotone (so by Lemma 8 also compatible and consistent).

- *Deductive closure*. Define

$$
\overline{U} := \left\{ b \in \mathrm{Tok} \mid U \vdash b \right\}.
$$

  The mapping $U \mapsto \overline{U}$ is a normal form mapping at base types.

- *Supremum*. For $a, b \in \mathrm{Tok}_{\mathbb{D}}$, define $\sup(a,b)$ by

$$\begin{aligned}
&\sup(a,*) = \sup(*,a) = a,\\
&\sup(\mathtt{S}a, \mathtt{S}a') = \mathtt{S}\sup(a,a'),\\
&\sup(\mathtt{B}ab, \mathtt{B}a'b') = \mathtt{B}\sup(a,a')\sup(b,b').
\end{aligned}$$

For a neighborhood $U \in \mathrm{Con}_{\mathbb{D}}$ define $\sup(U) \in \mathrm{Tok}$ by

$$\begin{aligned}
&\sup(\varnothing) := *,\\
&\sup(\{a_1,\dots,a_m\}) := \sup(\cdots\sup(a_1,a_2)\cdots,a_m).
\end{aligned}$$

The neighborhood mapping $U \mapsto \{\sup(U)\}$ is a normal form mapping at base types.

- *Path reduced neighborhood*. Define the paths in $\mathbb{D}$, $\mathrm{Tok}^p_{\mathbb{D}}$, by

$$\begin{aligned}
&*, \mathtt{0}, \mathtt{1} \in \mathrm{Tok}^p_{\mathbb{D}},\\
&a \in \mathrm{Tok}^p_{\mathbb{D}} \to \mathtt{S}a \in \mathrm{Tok}^p_{\mathbb{D}},\\
&a, b \in \mathrm{Tok}^p_{\mathbb{D}} \to \mathtt{B}a*, \mathtt{B}*b \in \mathrm{Tok}^p_{\mathbb{D}}.
\end{aligned}$$

Lemma 10. *Let ι be a base type.*

1. *Comparability: If $a \in \mathrm{Tok}^p_\iota$ and $b_1, b_2 \in \mathrm{Tok}_\iota$, then*

$$a \vdash_\iota b_1 \wedge a \vdash_\iota b_2 \to b_1 \vdash_\iota b_2 \vee b_2 \vdash_\iota b_1.$$

2. *Downward closure: If $a \in \mathrm{Tok}^p_\iota$ and $b \in \mathrm{Tok}_\iota$, then*

$$a \vdash_\iota b \to b \in \mathrm{Tok}^p_\iota.$$

3. *Atomicity: If $U \in \mathrm{Con}_\iota \backslash \varnothing$, and $b \in \mathrm{Tok}^p_\iota$, then*

$$U \vdash_\iota b \to \underset{a \in U}{\exists} \{a\} \vdash_\iota b.$$

A path reduced neighborhood is an inhabited neighborhood whose every token is maximal and a path.

Proposition 11 (Path normal form)**.** *There exists a normal form mapping* $\mathrm{nf}^p : \mathrm{Con}_\iota \to \mathrm{Con}_\iota$, *such that* $\mathrm{nf}^p(U)$ *is path reduced for every* $U \in \mathrm{Con}_\iota$.

5 Normal forms at higher types

- Some notation. Let $W = \{\langle U_1, b_1\rangle, \dots, \langle U_m, b_m\rangle\} \in \mathrm{Con}_{\rho\to\sigma}$. Let

$$\begin{aligned}
L(W) &:= \bigcup_{i=1}^{m} U_i = \{a \in U_i \mid i = 1,\dots,m\},\\
R(W) &:= \{b_i \mid i = 1,\dots,m\}.
\end{aligned}$$

These finite sets are not necessarily consistent! Also, write

$$\langle U, V\rangle := \{\langle U, b\rangle \mid b \in V\}.$$

- An *eigen-neighborhood* of W is a neighborhood $H = \langle U, V \rangle$, where $U \in \mathrm{Con}_{L(W)}$ (a subset of $L(W)$ which is consistent) and furthermore

$$U = \overline{U} \cap L(W) \wedge V = \overline{WU} \cap R(W).$$

  Write $H \in \mathrm{Eig}_W$. The *eigenform* of W is given by the neighborhood mapping

$$\mathrm{eig}(W) := \bigcup_{U \in \mathrm{Con}_{L(W)}} \left\langle \overline{U} \cap L(W), \overline{WU} \cap R(W) \right\rangle,$$

  that is, it is the union $\bigcup \mathrm{Eig}_W$ of its eigen-neighborhoods.

  Proposition 12 (Eigenform)**.** *Let ρ and σ be types, and $W, W_1, W_2 \in \mathrm{Con}_{\rho\to\sigma}$.*

  1. *The eigenform mapping is information preserving, that is, $W \sim_{\rho\to\sigma} \mathrm{eig}(W)$, and idempotent, that is $\mathrm{eig}(\mathrm{eig}(W)) = \mathrm{eig}(W)$.*
  2. *It is*

$$W_1 \vdash_{\rho\to\sigma} W_2 \leftrightarrow \forall_{H_2 \in \mathrm{Eig}_{W_2}} \exists_{H_1 \in \mathrm{Eig}_{W_1}} H_1 \vdash_{\rho\to\sigma} H_2,$$

$$W_1 \asymp_{\rho\to\sigma} W_2 \leftrightarrow \forall_{H_1 \in \mathrm{Eig}_{W_1}} \forall_{H_2 \in \mathrm{Eig}_{W_2}} H_1 \asymp_{\rho\to\sigma} H_2.$$

  (At base types we let $\mathrm{eig}(U) := U$ by convention.)

- The mapping eig is *not* a normal form mapping!

- Write Eig^0_W for the inhabited eigen-neighborhoods of W. Call $W \in \mathrm{Con}_{\rho\to\sigma}$ *eigen-maximal* if $W = \mathrm{eig}(W)$, and each $H \in \mathrm{Eig}_W$ is either empty or maximal, that is, if $H \in \mathrm{Eig}^0_W$, then for all $H' \in \mathrm{Eig}_W$ with $H' \vdash_{\rho\to\sigma} H$, it is $H' \sim_{\rho\to\sigma} H$.

  An eigen-maximal neighborhood is "flat", in the sense that the inclusion diagram of its eigen-neighborhoods forms a flat tree.

  Lemma 13. *Let ρ, σ be types. There exists a neighborhood mapping* emax *such that for every $W \in \mathrm{Con}_{\rho\to\sigma}$ the neighborhood* $\mathrm{emax}(W)$ *is eigen-maximal and* $W \sim_{\rho\to\sigma} \mathrm{emax}(W)$.

- The mapping emax is (again) *not* a normal form mapping!

- Write Fin_ρ for all (not necessarily consistent) finite token sets at type ρ. If $f : \mathrm{Con}_\rho \to \mathrm{Con}_\rho$ and $g : \mathrm{Con}_\sigma \to \mathrm{Con}_\sigma$, define their *eigenproduct* $\langle f, g \rangle : \mathrm{Con}_{\rho\to\sigma} \to \mathrm{Fin}_{\rho\to\sigma}$ by

$$\langle f, g \rangle(W) := \bigcup_{H \in \mathrm{Eig}^0_W} \langle f(L(H)), g(R(H)) \rangle.$$

  Proposition 14. *Let f and g be normal form mappings at types ρ and σ respectively. Then their eigenproduct is a normal form mapping at type $\rho \to \sigma$, when restricted to eigen-maximal neighborhoods.*

- As a corollary we obtain the following.

  Theorem 15 (Inductive normal forms)**.** *Let f and g be normal form mappings at types ρ and σ respectively. Then the mapping $\langle f, g \rangle \circ \mathrm{emax}$ is a normal form mapping at type $\rho \to \sigma$.*

6 Linearity

- There are two ways to work atomically in our setting, the implicit and the explicit way. Both are facilitated by the use of normal forms.

- *Implicit atomicity*. Call a type *implicitly atomic* when every neighborhood has an equivalent one which is atomic.

  All base types are implicitly atomic, since there are normal forms for every neighborhood which are atomic, like the closure and the supremum.

  Theorem 16. *Let ρ be an arbitrary type. There exists a neighborhood mapping* $\mathrm{at}_\rho : \mathrm{Con}_\rho \to \mathrm{Con}_\rho$*, such that* $\mathrm{at}_\rho(U)$ *is atomic and equivalent to U for all $U \in$* Con_ρ.

  Witness. $\mathrm{at}_{\rho\to\sigma}(W) := \langle \mathrm{id}, \mathrm{at}_\sigma \rangle (W)$. □

- *Explicit atomicity*. Fact 2 bluntly suggests the following strategy: render your base type information systems in an atomic manner and you're done. The problem is that in restricting ourselves to atomic base types, we want to obtain essentially the same ideals.

- Write $\rho \cong \sigma$ if the ideals of ρ and the ideals of σ are in a bijective correspondence.

  Theorem 17. *Let ι be a finitary base type. There exists an atomic-coherent information system η, such that $\eta \cong \iota$.*

  Proofsketch. Given a finitary base type ι, define the *path subsystem of* ι, ι^p, by letting

$$
\begin{aligned}
\mathrm{Tok}_{\iota^p} &:= \mathrm{Tok}_\iota^p, \\
\mathrm{Con}_{\iota^p} &:= \mathrm{Con}_\iota \cap \mathscr{P}_f(\mathrm{Tok}_{\iota^p}), \\
\vdash_{\iota^p} &:= \vdash_\iota \cap (\mathrm{Con}_{\iota^p} \times \mathrm{Tok}_{\iota^p}).
\end{aligned}
$$

  The triple ι^p is a coherent information system and it is $\iota^p \cong \iota$.

  To see that it is atomic, let $U \in \mathrm{Con}_{\iota^p}$ and $b \in \mathrm{Tok}_{\iota^p}$ be such that $U \vdash_{\iota^p} b$. Since b is a path, by Proposition 10.3 there is an $a \in U$ with $\{a\} \vdash_\iota b$. But a is itself a path, so $\{a\} \vdash_{\iota^p} b$. □

7 Outlook

- Further applications of neighborhood mappings: study of Fin, finite density, definability etc.

- What is "atomicity" in formal topological parlance? What are "eigen-neighborhoods"?

- What are the consequences of working on the basis of Theorem 17? For example, do we obtain naturally a model for linear logic?