Atomicity in non-atomic information systems

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Algebras

- We consider **finitary algebras** ι, given by a finite collection of **constructors**
$$\mathsf{C} : \underbrace{\iota \to \cdots \to \iota}_{r} \to \iota \ ,$$
with arity $r \geq 0$. We demand that each ι has a nullary constructor (to ensure inhabitedness) and a nullary pseudoconstructor $*_\iota : \iota$ for **partiality**.
- Natural numbers $\mathbb{N}$ are given by the constructors $*_\mathbb{N} : \mathbb{N}$ for partiality, $0 : \mathbb{N}$ for zero, and $\mathsf{S} : \mathbb{N} \to \mathbb{N}$ for successor.
- Derivations $\mathbb{D}$ are given by $*_\mathbb{D} : \mathbb{D}$ for partiality, $0 : \mathbb{D}$ for axioms, $\mathsf{S} : \mathbb{D} \to \mathbb{D}$ for single premise rules and $\mathsf{B} : \mathbb{D} \to \mathbb{D} \to \mathbb{D}$ for double premise rules.

Coherence

- All algebras induce **coherent information systems**, where consistency is binary (write "$\asymp$"): for a finite collection U of tokens, it is
$$\forall_{a,b \in U} a \asymp b \to U \in \mathsf{Con} .$$
- Coherent information systems correspond to coherent domains, coherent precusl's, and coherent Scott-Ershov formal topologies [B. 2013].
- Can we have a binary entailment too?

Atomicity

- The algebra $\mathbb{N}$ is simple enough to allow for a binary entailment: $\{\mathsf{SS0}, \mathsf{S}*\} \vdash \mathsf{SS}*$ reduces to $\mathsf{SS0} \vdash \mathsf{SS}*$.
- **Comparability Lemma**. Let ι be an algebra given by **at most unary constructors**. For tokens a, b, if $a \asymp_\iota b$, then either $\{a\} \vdash_\iota b$ or $\{b\} \vdash_\iota a$.
- So if $\{a_1, \ldots, a_l\} \vdash_{\mathbb{N}} b$, then there is an index $j = 1, \ldots, l$, for which a_j is maximal in the set—indeed, equivalent to it—and we have $\{a_j\} \vdash b$.
- **Atomicity** in general means

$$U \vdash b \to \exists_{a \in T} (a \in U \wedge \{a\} \vdash b) \ ,$$

although in the case above we have something stronger:

$$U \vdash b \to \exists!_{a \in T} (a \in U \wedge \{a\} \sim U \wedge \{a\} \vdash b) \ .$$

Non-atomicity

- Information systems where atomicity holds in general have been studied in [Schwichtenberg 2006] and from another viewpoint in [Bucciarelli–Carraro–Ehrhard–Salibra 2009]. They behave quite good (they are closed under exponentiation for a start), but unfortunately not good enough.
- The case of $\mathbb{D}$ is not atomic: in the entailment

  $$\{\mathsf{B}0*, \mathsf{B}*0\} \vdash \mathsf{B}00 \ ,$$

  no element on the left is redundant (Coquand, MAP2006).
- So the hope of basing the theory on **atomic** information systems alone falters; we need to work with entailments of arbitrary arities.
- Now how hopeless is this really?...

Atomicity at the base of entailment

- The non-atomic entailment

$$\{\mathsf{B}0{*}, \mathsf{B}{*}0\} \vdash \mathsf{B}00$$

holds because every argument in the right is **atomically** entailed by a corresponding argument in the neighborhood tokens.

- This suggests the following understanding: it is

$$\mathsf{B}\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \vdash \mathsf{B}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

because, row-wise,

$$\begin{bmatrix} 0 & * \end{bmatrix} \vdash^{A} 0 \;\; \text{and} \;\; \begin{bmatrix} * & 0 \end{bmatrix} \vdash^{A} 0$$

—we write $\vdash^{A}$ for atomic entailment.

Matrices over atomic systems

- A **(coherently consistent) matrix** over a given algebra ι is an array of tokens

$$\begin{bmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rl} \end{bmatrix},$$

  where every row is consistent.
- Consistency and *atomic* entailment for matrices are defined in terms of consistency and *atomic* entailment of their respective rows. In this way they form an atomic information system, the **matrix system** $M(\iota)$ of ι.

General entailment

- The **application** of an r-ary constructor C to an $r \times l$ matrix is defined by

$$\mathsf{C}\begin{bmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rl} \end{bmatrix} := [\mathsf{C}a_{11}\cdots a_{r1} \quad \cdots \quad \mathsf{C}a_{1l}\cdots a_{rl}] \ .$$

- If

$$\begin{aligned} \begin{bmatrix} a_{11} & \cdots & a_{1l} \end{bmatrix} &\vdash a_1 \ , \\ &\vdots \\ \begin{bmatrix} a_{r1} & \cdots & a_{rl} \end{bmatrix} &\vdash a_r \ , \end{aligned}$$

then define

$$\mathsf{C}\begin{bmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rl} \end{bmatrix} \vdash \mathsf{C}\begin{bmatrix} a_1 \\ \vdots \\ a_r \end{bmatrix} \ .$$

Matrix representation of a neighborhood

- The idea of applying a constructor to a matrix naturally extends to applying a whole **constructor context** to a matrix. Then we can make sense of the following:

$$
\begin{aligned}
\begin{bmatrix} \mathsf{BSB0}{*}{*} & \mathsf{BSB}{*}{*}0 & \mathsf{BSB}{*}0{*} \end{bmatrix} &\sim \mathsf{B}(\bullet,\bullet)\begin{bmatrix} \mathsf{SB0}{*} & \mathsf{SB}{*}{*} & \mathsf{SB}{*}0 \\ * & 0 & * \end{bmatrix} \\
&\sim \mathsf{B}(\mathsf{S}(\bullet),\bullet)\begin{bmatrix} \mathsf{B0}{*} & \mathsf{B}{*}{*} & \mathsf{B}{*}0 \\ * & 0 & * \end{bmatrix} \\
&\sim \mathsf{B}(\mathsf{S}(\mathsf{B}(\bullet,\bullet)),\bullet)\begin{bmatrix} 0 & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix} \quad .
\end{aligned}
$$

- **Matrix representation**. For every matrix A over ι there exist a unique constructor context K (in an appropriate normal form) and a unique nullary matrix M, so that $A \sim \mathsf{K}(M)$.

Characterizations of entailment

- The matrix representation reduces the question of **entailment** between neighborhoods to the question of **equality** between normal contexts and **atomic entailment** between nullary matrices (needs some work to see):

$$\text{“}\vdash_\iota\text{” is characterized by “}=_{K_{nf}(\iota)}\text{” and “}\vdash^A_{M_0(\iota)}\text{”}$$

- Moreover, every nullary matrix is equivalent to a nullary **vector**:

$$\begin{bmatrix} 0 & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix} \sim \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ,$$

- **Eigentokens**. Every neighborhood U in ι is equivalent to a token $e(U)$, its **eigentoken** (only for finitary algebras!). In this way we obtain a characterization of entailment between neighborhoods by a (trivially atomic) entailment between tokens.

Higher-type entailment

- Let $U_1, \dots, U_l, U \in \mathsf{Con}_\rho$, $b_1, \dots, b_l, b \in T_\sigma$, and define (list-list **application**)

$$\{\langle U_1, b_1\rangle, \dots, \langle U_l, b_l\rangle\} U := \{b_i \mid U \vdash_\rho U_i, i = 1, \dots, l\} \ ;$$

  if $WU \vdash_\sigma b$, then define $W \vdash_{\rho\to\sigma} \langle U, b\rangle$.
- Atomicity in higher types, in case we had it, would have entailment unfold as follows:

$$\begin{aligned} &\{\langle U_1, b_1\rangle, \dots, \langle U_l, b_l\rangle\} \vdash^A \langle U, b\rangle \\ &\Rightarrow \exists_{j=1}^{l} \{\langle U_j, b_j\rangle\} \vdash \langle U, b\rangle \\ &\Rightarrow \exists_{j=1}^{l} \{\langle U_j, b_j\rangle\} U \vdash b \\ &\Rightarrow \exists_{j=1}^{l} (U \vdash U_j \wedge \{b_j\} \vdash b) \ . \end{aligned}$$

- Based on non-atomic algebras, the higher function spaces are a fortiori non-atomic.

Two remarks

- Two higher-type tokens $\langle U_1, b_1\rangle$ and $\langle U_2, b_2\rangle$ may be consistent either, so to speak, trivially, when $U_1 \not\asymp U_2$, or essentially, when $U_1 \asymp U_2$ and $b_1 \asymp b_2$.
- Let $W = \{\langle U_j, b_j\rangle \mid j = 1, \ldots, l\} \in \mathsf{Con}_{\rho\to\sigma}$, and $U \in \mathsf{Con}_\rho$. If for some j and k it happens that $U_j \vdash U_k$ and $U \vdash U_j$, then, by transitivity of entailment, it is $U \vdash U_k$ as well, and so $b_j, b_k \in WU$ both.
- These suggest considering sub-neighborhoods with left consistency and left closure.

Eigen-neighborhoods

- Write $\arg W$ for the list of left-hand sides of the pairs in W, and $\operatorname{val} W$ for the corresponding list of the right-hand sides.
- An **eigen-neighborhood** of W is a sublist $E \subseteq W$ which is **left-consistent**, that is,

$$\forall_{U,U' \in \arg E} U \asymp U'$$

(therefore, also **right consistent**) as well as **closed under entailment with respect to** W, that is,

$$\forall_{U \in \arg W} (\arg E \vdash U \to U \in \arg E) \ .$$

Write $E(W)$ for the collection of eigen-neighborhoods of W.

Characterization of higher-type entailment

- **Eigen-neighborhoods**. Let $W \in \mathsf{Con}_{\rho\to\sigma}$. It is $W \sim \{E \mid E \in E(W)\}$. Moreover, if $\langle U, b\rangle \in T_{\rho\to\sigma}$, then

$$W \vdash \langle U, b\rangle \to \underset{E\in E(W)}{\exists} (U \vdash \arg E \wedge \operatorname{val} E \vdash b) \enspace .$$

- **Eigenform**. Any neighborhood W is equivalent to the neighborhood

$$\{\langle \arg \mathrm{E}, \operatorname{val} E\rangle \mid E \in E(W)\} \enspace ,$$

where we write $\langle U, V\rangle$ for $\{\langle U, b\rangle \mid b \in V\}$.

- A suggestive application of the eigenform is that it gives a clear-cut way to get **conservative extensions** of a neighborhood: Let $W \in \mathsf{Con}_{\rho\to\sigma}$, and $E_1, \ldots, E_m \in E(W)$. For any choice of $U_1, \ldots, U_m \in \mathsf{Con}_\rho$ and $V_1, \ldots, V_m \in \mathsf{Con}_\sigma$ with the property that $U_i \vdash_\rho \arg E_i$ and $\operatorname{val} E_i \vdash_\sigma V_i$, for $i = 1, \ldots, m$, it is

$$W \sim_{\rho\to\sigma} W \cup \{\langle U_i, V_i\rangle \mid i = 1, \ldots, m\} \enspace .$$

Morals

- Atomicity is not enough to model arithmetic for partial computable functionals as we would wish, but clearly plays a fundamental role in the general theory that demands attention.
- At base types, that is, at systems induced by algebras, atomicity manifests itself through matrices over atomic systems and leads to the development of a theory with ramifying technicalities at times, but for the same reason very illuminating.
- At higher types atomicity appears in a generalized form, on an intermediate level between tokens and neighborhoods, namely on the level of **eigen-neighborhoods**, which play a crucial role in the operation of application.
- In both cases, atomicity is the key to pinpointing interesting notions of normal forms.

Outlook

- At base types, the next step is to hone the matrix theory by utilizing it to help implement the endless first steps of $\mathrm{TCF}+$ (see [Huber–B.–Schwichtenberg 2010]). The canonical proof assistant to this end would be MINLOG (`http://www.math.lmu.de/~minlog/`).
- At higher types, the first goal is to make systematic use of eigen-neighborhoods in revisiting old favorites like **definability** [Plotkin 1997] and **density** [Berger 1993]. The hope is to provide bottom-up proofs, native to the setting of coherent information systems, and compare them to the well-known top-down adaptations of similar or more general arguments.
- A mini side-goal already in the agenda is also to study eigen-neighborhoods formal-topologically. Knowing that in a formal topological setting tokens are unobservable, eigen-neighborhoods, being neighborhoods first of all, might give a way to see atomicity of information in structures with no tokens of information.

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