Implicit atomicity and finite density for non-flat domains

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DRAFT OF 13 DEC 2013, 3:26 P.M.

Abstract

The Kleene-Kreisel density theorem states that total ideals are dense in the finitely generated partial ideals of a given type. We investigate the status of this statement for non-flat domains, through their representation as non-flat coherent Scott information systems, in an internal, bottom-up approach. We prove that such information systems are implicitly atomic, in the sense that, at each type, a neighborhood has an equivalent one whose closure is atomic, and use this fact to provide finite witnesses for density and separation.

1 Introduction

Adhering to the paradigm of functional programming, we view data types as countably based Scott domains in the tradition that started with Dana Scott's and Yuri Ershov's independent work in the late sixties and early seventies. More particularly, we view these domains through their representations as information systems, which were again introduced by Scott in [9]. For us, programs are typed terms with denotations lying in corresponding function spaces over information systems.

The systems we use turn out to be *coherent*, in the sense that the consistency of information reduces to a series of independent binary tests. A crucial choice in our setting is to work with *non-flat* rather than flat domains, where varying degrees of partiality are allowed. Among other things, this is a sufficient imposition to have in order to obtain injectivity and disjoint ranges for the constructors of the base types.[1] More intuitively, non-flatness yields richer domains, a fact that may sometimes facilitate arguments that wouldn't carry through in the flat setting. The "finite density" argument of this paper is one good example.

A bottom-up approach to the problem of density

Branching out of the general domain-theoretic setting for computability, the theory of coherent information systems as we study it aims at a theory where *partiality* is the norm rather than a freak of nature: an ideal corresponds to an algorithm, and as such it may in general have arguments to which it can not reply; in particular, these arguments are allowed to be partial too.

[1] We assume acquaintance with coherent Scott information systems as well as with algebraic coherent Scott information systems, that is, with systems induced by given constructors and the function spaces over them; finally, we assume acquaintance with the resulting type system, and with the Scott topology naturally associated with each type. For details one may consult [8].

On the other hand, an algorithm may indeed happen to be “total”. We define the *total ideals* at type ρ, and write G_ρ, by the following: at base type α, an ideal x is total if it contains a total token (that is, a token that is constructed without use of the partiality pseudoconstructor $*$); at type $\rho \to \sigma$, f is total when, fed a total input, it yields a total output, that is, when

$$\underset{x\in\rho}{\forall}\left(x \in G_\rho \to f(x) \in G_\sigma\right) ,$$

where $b \in f(x)$ if and only if $\langle U, b\rangle \in f$, for some $U \subseteq x$.

The *density property* for a type, the latter being understood as a space governed by the Scott topology, is the property that every open set in the space nurtures total points, in other words, that total points are dense in the space, despite our fundamental premise that we work with spaces of generally *partial* points. More formally, the property “ρ is dense” is stated in our setting as

$$\underset{U\in\mathsf{Con}_\rho}{\forall}\ \underset{x\in\rho}{\exists}\left(x \in G_\rho \wedge U \subseteq x\right) .$$

Density was first stated by Georg Kreisel already in [5], and since Ulrich Berger’s [1], the standard way to deal with it is to show (by mutual induction) that both density and “separation” hold at every type; intuitively, a function space $\rho \to \sigma$ is considered to feature the *separation property*, if any two open sets W_1 and W_2 of conflicting information can be separated by a point x of type ρ, meaning, that x draws conflicting values $W_1(x)$ and $W_2(x)$ when tested on the neighborhoods[2].

In search of finite witnesses

It turns out that this generic domain-theoretic method, when made concrete within our setting in a top-down manner, can prove rather too abstract, and even cumbersome. In an information system ideals are approximated by very tangible finite lists of tokens, the neighborhoods, and one would expect that constructions of the like of total or separating witnesses, should also be made as tangible as it gets. In spite of such expectations, in all adaptions of the density argument to the information system setting so far, both the separating and the totalizing witnesses are given on the level of ideals.[3]

We attempted to improve this situation in [4], where we showed that one may first prove “finite separation” at every type—where the witnesses are not total ideals anymore but simply neighborhoods—and then use this as a lemma to prove density at every type, a bonus advantage being that one can avoid the mutual inductive argument.[4] Here we go one step further, and attempt to witness density as well in a finite way.

Let us now roughly sketch our bottom-up intuition. We want to capture totality, as we know it from the generally infinite level of ideals, within the finite level of neighborhoods. In particular, we understand the definition of totality on the infinite level as follows: a set at type $\rho \to \sigma$ is total when (a) it is an ideal, that is, consistent and deductively closed, (b) it admits all totals of type ρ as arguments—a property we think

[2]Closer to the spirit of Ulrich Berger, and assuming that the type of booleans is around, one may give a separator as a functional $\mathcal{X}$ of type $(\rho \to \sigma) \to \mathbb{B}$, such that if W_1 and W_2 are inconsistent, then for example $\mathcal{X}(W_1) = \mathtt{tt}$ and $\mathcal{X}(W_2) = \mathtt{ff}$.

[3]The theory of non-flat coherent Scott information systems as a model for higher-type computability, aiming at an implementation on a proof assistant has been one of the main strands of research within the Munich logic group. For several takes on the density argument see [6, 7, 2, 3], as well as [4].

[4]See [4, §2.4].

of as "omniception"[5]—, and (c) it responds to every total argument with a total value at type σ. We bring the notion down to the finite level by simply disposing of half of the demand (a), namely, that the set of tokens be deductively closed: we prove that every neighborhood extends to a "total neighborhood" ("finite density"), and that the closure of every total neighborhood is a total ideal, thus establishing the standard density theorem for our setting. As we will see, the argument will interestingly be based on a traditional mutual induction between an appropriate notion of separation on the one hand, and a notion of "finite" totality on the other.

Eigen-neighborhoods and implicit atomicity

A central tool in our approach is the notion of an "eigen-neighborhood" of a given neighborhood. At type $\rho \to \sigma$, two tokens $\langle U_1, b_1 \rangle$ and $\langle U_2, b_2 \rangle$ may be trivially consistent, in the sense that U_1 and U_2 are inconsistent, or they can be consistent both on the left and on the right, that is, $U_1 \asymp_\rho U_2$ and $b_1 \asymp_\sigma b_2$; in an eigen-neighborhood, all pairs of tokens are non-trivially consistent. Moreover, an eigen-neighborhood H of a neighborhood W also features left deductive closure (with respect to W), meaning that there is no pair in W whose left part is entailed by the (consistent) left parts of H and is not already contained in H.

The reason for considering such special sublists of neighborhoods is intimately related to our view on the density argument. A total neighborhood W should provide support for an arbitrary total ideal of the left type; this support (together with its corresponding right parts) is exactly an eigen-neighborhood of W.

But besides density, eigen-neighborhoods prove to be of wider importance in the theory of coherent information systems, since they act as generalized tokens which portray an atomic behavior regarding entailment. Indeed, we show that our coherent information systems are "implicitly atomic", in the sense that every neighborhood has an equivalent one whose closure is atomic.[6] As we will see, acknowledgment of implicit atomicity in our systems plays a crucial role in simplifying the density argument.

2 Neighborhoods in lists

Whenever we regard the arguments of a higher-type neighborhood we face a list which is not necessarily consistent. Let $\Gamma \in \mathsf{Lst}_\rho$ be such a list,[7] and denote its *consistent* sublists by Con_Γ; for example, it is $a \in \mathsf{Con}_\Gamma$ (seen as a neighborhood), for every $a \in \Gamma$, as well as $\varnothing \in \mathsf{Con}_\Gamma$ for every $\Gamma \in \mathsf{Lst}_\rho$. Clearly, if $\Gamma \in \mathsf{Con}_\rho$ already, then $\mathsf{Con}_\Gamma = \mathcal{P}(\Gamma)$, while in general it is $\mathsf{Con}_\Gamma \subseteq \mathcal{P}(\Gamma)$.

Call $M \in \mathsf{Con}_\Gamma$ a *maximal neighborhood* in Γ, and write $M \in \operatorname{Max}\Gamma$, if

$$\forall_{a \in \Gamma} (a \asymp_\rho M \to a \in M) \ .$$

An easy observation is that for all $V \in \mathsf{Con}_\rho$, it is $U \vdash_\rho V$ for some $U \in \mathsf{Con}_\Gamma$ if and only if $M \vdash_\rho V$ for some $M \in \operatorname{Max}\Gamma$ (leftwards let $U := M$ and rightwards use transitivity of entailment at type ρ).

[5] A rather pompous but arguably grammatically smoother synonym for *admission* or *acceptance of all*.

[6] Recall that a neighborhood U has an *atomic closure* when, for an arbitrary token b, if $U \vdash b$ then there is a token $a \in U$, such that $\{a\} \vdash b$.

[7] In the following we just write Lst_ρ for possibly inconsistent lists of tokens of type ρ.

Moreover, the consistency of a list is characterized easily through its left and right maximal neighborhoods: at base types, clearly, a list is consistent if and only if it is its own sole maximal neighborhood, while for higher types we have the following.

Proposition 1. *A list $\Theta \in \mathsf{Lst}_{\rho\to\sigma}$ is a neighborhood if and only if for each left maximal $M \in \mathrm{Max}\,(\arg\Theta)$ there is a right maximal $N \in \mathrm{Max}\,(\mathrm{val}\,\Theta)$ with $\Theta M \subseteq N$.*

Proof. Write Γ and Δ for $\arg\Theta$ and $\mathrm{val}\,\Theta$ respectively.

From left to right, let $\Theta \in \mathsf{Con}_{\rho\to\sigma}$ and $M \in \mathrm{Max}\,\Gamma$. If $U, U' \in M$, it will be $U \asymp_\rho U'$, and then $WU \asymp_\sigma WU'$; so there must be a maximal $N \in \mathrm{Max}\,\Delta$ with $WU \subseteq N$ for every $U \in \mathrm{Max}\,$.

From right to left, let $\Theta \in \mathsf{Lst}_{\rho\to\sigma}$ with the property that for each left maximal $M \in \mathrm{Max}\,\Gamma$ there is a right maximal $N \in \mathrm{Max}\,\Delta$ such that $\Theta M \subseteq N$; let $\langle U, b\rangle, \langle U', b'\rangle \in \Theta$ with $U \asymp_\rho U'$; there will be a maximal $M \in \mathrm{Max}\,\Gamma$ with $U, U' \in \Gamma$; by hypothesis, there will be an $N \in \mathrm{Max}\,\Delta$ such that

$$b + b' \subseteq WU + WU' \subseteq \Theta W \subseteq N \ ,$$

so Θ is consistent. □

We need to be a bit careful with the way we use the lists of arguments in given neighborhoods: do we mean them as lists of neighborhoods, that is, lists of type $N\rho$, or as lists of the neighborhoods' tokens, that is, lists of type ρ? For our purposes, it turns out that, given a list in $N\rho$, we can work with its underlying "flat" list in ρ, and then draw safe conclusions about it in $N\rho$ again.

Define a *flattening* mapping $\mathrm{fl} : \mathsf{Lst}_{N\rho} \to \mathsf{Lst}_\rho$ in the usual way:

$$\mathrm{fl}(\Gamma) := \sum_{U\in\Gamma} \sum_{a\in U} a \ ;$$

in set-theoretical notation we may as well write $\mathrm{fl}(\Gamma) = \cup\Gamma$.

Proposition 2. *Let $\Gamma \in \mathsf{Lst}_{N\rho}$. Then*

$$\bigvee_{M\in\mathrm{Max}\,\Gamma} \ \exists!_{M_f\in\mathrm{Max}\,\mathrm{fl}(\Gamma)} \ M_f \vdash_{N\rho} M \quad\wedge\quad \bigvee_{M_f\in\mathrm{Max}\,\mathrm{fl}(\Gamma)} \ \exists!_{M\in\mathrm{Max}\,\Gamma} \ M_f \vdash_{N\rho} M \ .$$

Proof. Let $\Gamma \in \mathsf{Lst}_{N\rho}$. For the first conjunct, let $M \in \mathrm{Max}\,\Gamma$; it is consistent, so $U \asymp_{N\rho} U'$, for all $U \in M$, which means that $a \asymp_\rho a'$, for all $a \in U$, $a' \in U'$; then there must exist a maximal neighborhood M_f in $\mathrm{fl}(\Gamma)$, which will contain all $a \in U$, for any $U \in M$, so $M_f \vdash_{N\rho} M$. Suppose that M'_f is yet another maximal neighborhood in $\mathrm{fl}(\Gamma)$, with $M'_f \vdash_{N\rho} M$; then, since entailment preserves consistency, for all $a \in \mathrm{fl}(\Gamma)$ it is

$$[a] \asymp_{N\rho} M \to a \asymp_\rho M_f \wedge a \asymp_\rho M'_f \ ,$$

which yields $M_f = M'_f$ due to their maximality with respect to consistency.

For the second conjunct, let $M_f \in \mathrm{Max}\,\mathrm{fl}(\Gamma)$; since the situation is finite, we may argue indirectly; let $M_1, \ldots, M_T$ be all maximals in Γ, and suppose that $M_f \nvdash_{N\rho} M_t$ for any $t = 1, \ldots, T$; by the first conjunct, there is an $M_f^t \in \mathrm{Max}\,\mathrm{fl}(\Gamma)$, with $M_f^t \vdash M_t$, for every t, and since M_t's together cover Γ, their corresponding M_f^t's together must cover $\mathrm{fl}(\Gamma)$; on the other hand, the supposition yields $M_f \not\asymp_\rho M_f^t$, for all t, which would mean that there are $a \in M_f \smallsetminus \mathrm{fl}(\Gamma)$, a contradiction. Assume now that $M_f \vdash_{N\rho} M$ and $M_f \vdash_{N\rho} M'$, for $M, M' \in \mathrm{Max}\,\Gamma$; then $M \asymp_{N\rho} M'$, so $M = M'$ by maximality. □

3 Implicit atomicity

Let W be a neighborhood at a type ρ. We define the set $\mathrm{Eig}\, W$ of its *eigen-neighborhoods* as follows. At a base type α, the only eigen-neighborhoods of a neighborhood U are $\varnothing$ and U itself. At a higher type $\rho \to \sigma$, a sublist $H \subseteq W$ is an eigen-neighborhood of W, if it is *left-consistent*, that is,

$$\underset{U_1, U_2 \in \arg H}{\forall} U_1 \asymp_\rho U_2$$

(so consequently $b_1 \asymp_\sigma b_2$ for the corresponding arguments as well), and *closed under left entailment (relatively to W)*, that is,

$$\underset{U \in \arg W}{\forall} (\arg H \vdash_{N\rho} U \to U \in \arg H) \ ,$$

where $N\rho$ is the corresponding information system of the neighborhoods of ρ[8].

Every $U \in \mathsf{Con}_\rho$ generates an eigen-neighborhood $H_{U,W}$ of W, by

$$\langle U_0, b_0 \rangle \in H_{U,W} :\Leftrightarrow \langle U_0, b_0 \rangle \in W \wedge U \vdash_\rho U_0 \ .$$

Observe then that

$$WU \vdash_\sigma b \leftrightarrow \mathrm{val}\, H_{U,W} \vdash_\sigma b \ .$$

Moreover, it is clear that the set of eigen-neighborhoods is finite and that $W \sim_{\rho\to\sigma} \sum_{H \in \mathrm{Eig}\, W} H$.

The eigen-neighborhoods behave as generalized tokens to some extent, enough to reveal a hidden atomicity that underlies the otherwise non-atomic algebraic entailment. The following anticipates Theorem 5.

Proposition 3. *Let $W_1, W_2 \in \mathsf{Con}_{\rho\to\sigma}$. The following hold:*

$$W_1 \asymp_{\rho\to\sigma} W_2 \leftrightarrow \underset{H_1 \in \mathrm{Eig}\, W_1}{\forall} \ \underset{H_2 \in \mathrm{Eig}\, W_2}{\forall} (\arg H_1 \asymp_{N\rho} \arg H_2 \to \mathrm{val}\, H_1 \asymp_\sigma \mathrm{val}\, H_2) \ ,$$

$$W_1 \vdash_{\rho\to\sigma} W_2 \leftrightarrow \underset{H_2 \in \mathrm{Eig}\, W_2}{\forall} \ \underset{H_1 \in \mathrm{Eig}\, W_1}{\exists} (\arg H_2 \vdash_{N\rho} \arg H_1 \wedge \mathrm{val}\, H_1 \vdash_\sigma \mathrm{val}\, H_2) \ .$$

Proof. See [4]. □

Write $\langle U, V \rangle$ for $\sum_{b \in V} \langle U, b \rangle$ and $U \sim_\rho U'$ for $U \vdash_\rho U' \wedge U' \vdash_\rho U$[9]. With the use of eigen-neighborhoods we can achieve manageable *conservative extensions* of a neighborhood.

Proposition 4 (Conservative extension)**.** *Let $W \in \mathsf{Con}_{\rho\to\sigma}$, and $H_1, \dots, H_m \in \mathrm{Eig}\, W$. For any choice of $U_1, \dots, U_m \in \mathsf{Con}_\rho$ and $V_1, \dots, V_m \in \mathsf{Con}_\sigma$ with the property that $U_i \vdash_\rho \arg H_i$ and $\mathrm{val}\, H_i \vdash_\sigma V_i$, for $i = 1, \dots, m$, it is*

$$W \sim_{\rho\to\sigma} W + \sum_{i=1}^{m} \langle U_i, V_i \rangle \ .$$

[8] Recall that $N\rho = (\mathsf{Con}_\rho, \mathsf{Con}_{N\rho}, \vdash_{N\rho})$, where $\sum_{j=1}^{l} U_j \in \mathsf{Con}_{N\rho}$ if and only if $\cup_{j=1}^{l} U_j \in \mathsf{Con}_\rho$ and $\sum_{j=1}^{l} U_j \vdash_{N\rho} U$ if and only if $\cup_{j=1}^{l} U_j \vdash_\rho U$. This is an information system which is coherent if ρ is coherent and moreover the two have isomorphic domains of ideals, see [4, Chapter 3].

[9] Equientailment is clearly an equivalence on neighborhoods.

Proof. For the consistency of the extension $W + \sum_{i=1}^{m} \langle U_i, V_i \rangle$, let $i, j = 1, \dots, m$; then

$$U_i \asymp_\rho U_j \Rightarrow \arg H_i \asymp_\rho \arg H_j \Rightarrow \operatorname{val} H_i \asymp_\sigma \operatorname{val} H_j \Rightarrow V_i \asymp_\sigma V_j \ ,$$

by propagation of consistency and consistency of W; this suffices.

For the equientailment, let $i = 1, \dots, m$; it is

$$H_i = \langle \arg H_i, \operatorname{val} H_i \rangle \vdash_{\rho\to\sigma} \langle U_i, V_i \rangle \ ,$$

so $W \vdash_{\rho\to\sigma} W + \sum_{i=1}^{m} \langle U_i, V_i \rangle$. The converse is trivial. □

For every $H \in \operatorname{Eig} W$ there is exactly one $U^H \in \mathsf{Con}_\rho$ (up to equientailment) and exactly one $V^H \in \mathsf{Con}_\sigma$ (up to equientailment), so that $H \sim_{\rho\to\sigma} \left\langle U^H, V^H \right\rangle$; just set

$$U^H := \operatorname{fl}(\arg H) \ \text{ and } \ V^H := \operatorname{val} H \ .$$

Say that W is in *eigenform*, if for every $H \in \operatorname{Eig} W$ it is $\left\langle U^H, V^H \right\rangle \subseteq W$. Furthermore call W *monotone*, if for all $\langle U_1, V_1 \rangle, \langle U_2, V_2 \rangle \subseteq W$ it is

$$U_1 \vdash_\rho U_2 \to V_1 \vdash_\sigma V_2 \ .$$

We will also need the elementary fact that every base-type neighborhood has an *eigentoken*, that is, an equientailing token. The quickest way to see this is through the fact that the eigentoken is the "supremum" of the neighborhood: the *supremum* $\sup(a, b)$ of two *consistent* tokens is defined inductively as follows:

$$\begin{gathered} \sup(a, *) = \sup(*, a) = a \ , \\ \sup(Ca_1 \cdots a_r, Cb_1 \cdots b_r) = C \sup(a_1, b_1) \cdots \sup(a_r, b_r) \ , \end{gathered}$$

for every r-ary constructor C of the base type. If $U = \left\{a_1, \dots, a_l\right\}$ is a consistent list, then we set

$$\sup(U) := \sup(a_l, \cdots \sup(a_2, a_1) \cdots) \ .$$

It is direct to see that this is well-defined and also that $\sup(U) \sim U$.

Theorem 5 (Implicit atomicity)**.** *Every type is* implicitly atomic*, that is, for every neighborhood there exists an equientailing one, whose closure is atomic.*

Proof. We want to show that

$$\underset{U \in \mathsf{Con}_\rho}{\forall} \ \underset{U^E \in \mathsf{Con}_\rho}{\exists} \left(U^E \sim_\rho U \wedge \underset{b \in T_\rho}{\forall} \left(U^E \vdash_\rho b \to \underset{a \in U^E}{\exists} \left\{a\right\} \vdash_\rho b \right) \right) \ .$$

At a base type α, given a neighborhood U, set $U^E := \left\{\sup(U)\right\}$. At a higher type $\rho \to \sigma$, assume that σ is implicitly atomic, and let $W \in \mathsf{Con}_{\rho\to\sigma}$. We will use an equivalent neighborhood of W in monotone eigenform. Set

$$W^E := \sum_{H \in \operatorname{Eig} W} \left\langle \arg H, (\operatorname{val} H)^E \right\rangle \ .$$

It is easy to see that this list is finite and consistent; that it is monotone follows from the monotonicity of application; that it is in eigenform is obvious by construction.

We show the equientailment. Let $\langle U, b\rangle \in W$, and consider the eigen-neighborhood $H_{U,W}$ of W that is generated by U. It is $U \vdash_\rho \arg H_{U,W}$ and $b \in \operatorname{val} H_{U,W}$, so $(\operatorname{val} H_{U,W})^E \vdash_\sigma b$ by the induction hypothesis, and $W^E \vdash_{\rho\to\sigma} W$. For the other direction, let $\langle \arg H, b\rangle \in W^E$, for some eigen-neighborhood H of W; by the construction of W^E, it is $b \in (\operatorname{val} H)^E$; then by the induction hypothesis it is $\operatorname{val} H \vdash_\sigma b$, so we have found an eigen-neighborhood of W (namely H itself) that entails $\langle \arg H, b\rangle$.

Now we show the atomic closure of W^E. Let $\langle U, b\rangle \in T_{\rho\to\sigma}$ be an arbitrary token, and assume that $W^E \vdash_{\rho\to\sigma} \langle U, b\rangle$; starting by the definition of entailment we get

$$W^E U \vdash_\sigma b \Leftrightarrow \operatorname{val} H_{U,W^E} \vdash_\sigma \Leftrightarrow (\operatorname{val} H_{U,W})^E \vdash_\sigma b\ .$$

By the induction hypothesis, there exists a $b_0 \in (\operatorname{val} H_{U,W})^E$, such that $\{b_0\} \vdash_\sigma b$, so we have found a token $\langle \arg H_{U,W}, b_0\rangle \in W^E$, for which $\langle \arg H_{U,W}, b_0\rangle \vdash_{\rho\to\sigma} \langle U, b\rangle$. □

Note that in general there may be many equientailing atomic neighborhoods for a given neighborhood. In the following, the reader is invited to assume that the neighborhoods we consider have an atomic closure, in other words, that they are *atomic*, and ponder on the implications for the arguments at hand, though we will restrict ourselves to the use of Proposition 3 every time we invoke implicit atomicity.

4 Separating lists and total neighborhoods

At type ρ consider a not necessarily consistent, but finite list $\Gamma \in \mathsf{Lst}_\rho$, and assume it is nonempty to avoid trivialities. Call it *omniceptive (for totals)* if every total ideal $x \in G_\rho$ shares an *inhabited* neighborhood with Γ[10], that is, if

$$\underset{U\in\mathsf{Con}^0_\rho}{\exists} (U \in \mathsf{Con}_\Gamma \wedge U \subseteq x)\ .$$

Notice that, by its deductive closure, if x shares a neighborhood U with a list Γ, then it also shares every $U' \in \mathsf{Con}_\Gamma$ with $U \vdash_\rho U'$. At type $\mathbb{N}$, the list $\Delta_1 := 0 + S0$, with inhabited neighborhoods $\{\{0\}, \{S0\}\}$, is not omniceptive, since it does not share a neighborhood with, say, the total ideal $\{SS0, SS*, S*, *\}$, but the list $\Delta_2 := 0 + S*$ is. An important case of omniception is when the shared neighborhood is maximal in the list: a list is *omniceptive by maximals*, or traditionally *separating*, when for every total ideal $x \in G_\rho$,

$$\underset{M\in\mathsf{Con}_\rho}{\exists} (M \in \operatorname{Max}\Gamma \wedge M \subseteq x)\ .$$

The importance of separation is that the shared maximal will be inconsistent to all other maximals, making it easy to pair this neighborhood with a value of our choice without endangering consistency.

Now consider a neighborhood $U \in \mathsf{Con}_\rho$ in monotone eigenform, whose (flattened) list of arguments is omniceptive. At base types this is a trivial notion due to the absence of argument lists.[11] At a higher type $\rho \to \sigma$, if W is a neighborhood with $\arg W$ being omniceptive, then it would suffice for the corresponding values to be “total” in order for the closure of W to be a total ideal.

[10] Write Con^0 for the collection of inhabited neighborhoods.

[11] Intuitively, one may think of a base type α as being morally equivalent to the type $\mathbb{U} \to \alpha$ ($\mathbb{U}$ being the *unit type* induced by an algebra with one nullary constructor) so then any $U \in \mathsf{Con}_{\mathbb{U}\to\alpha}$ would have an omniceptive argument list.

Formally, at type ρ, define $U \in G\mathsf{Con}_\rho$, and say that U is a *total neighborhood*, by the following; at type α,

$$U \in G\mathsf{Con}_\alpha := \underset{a \in GT_\alpha}{\exists} U \vdash_\alpha a \; ;$$

at type $\rho \to \sigma$,

$$W \in G\mathsf{Con}_{\rho\to\sigma} := \underset{x \in G_\rho}{\forall} \; \underset{H \in \operatorname{Eig} W}{\exists} (\arg H \subseteq x \wedge \operatorname{val} H \in G\mathsf{Con}_\sigma) \; .$$

Lemma 6. *Let ρ and σ be arbitrary types. For all $W \in \mathsf{Con}_{\rho\to\sigma}$, $x \in G_\rho$, and $V \in \mathsf{Con}_\sigma$, if $W(x) \vdash_\sigma V$, then $\overline{V} \subseteq \overline{W}(x)$.*

Proof. Assume that $W(x) \vdash_\sigma V$. We want to show that $\overline{V} \subseteq \overline{W}(x)$, so let $b \in T_\sigma$ be such a token, that $V \vdash_\sigma b$; we want now to show that $b \in \overline{W}(x)$, that is, we need to find a $U \in \mathsf{Con}_\rho$, such that $\langle U, b\rangle \in \overline{W}$ and $U \subseteq x$. By the assumptions and the transitivity of entailment we get $W(x) \vdash_\sigma b$, so there exists an eigen-neighborhood H of W, such that $\arg H \subseteq x$ and $\operatorname{val} H \vdash_\sigma b$; it is $H \subseteq W \subseteq \overline{W}$ by transitivity, and $\langle \arg H, b\rangle \in \overline{W}$ by closure of W, so we may choose $U := \arg H$. ☐

Lemma 7 (Extension lemma)**.** *An ideal that includes a total ideal is itself total.*

Proof. At a base type α, let $x \in G_\alpha$ and $y \in \mathsf{Ide}_\alpha$ be two ideals with $x \subseteq y$. Then there is a total token $a \in T_\alpha$, such that $a \in x$, so also $a \in y$. At a higher type $\rho \to \sigma$, let $f \in G_{\rho\to\sigma}$, $g \in \mathsf{Ide}_{\rho\to\sigma}$, and assume that $f \subseteq g$. We want to show that g is also total, so consider an arbitrary $x \in G_\rho$. By the totality of f we have that $f(x) \in G_\sigma$, and since it is straightforward to see that $f(x) \subseteq g(x)$, we get $g(x) \in G_\sigma$ by the induction hypothesis at σ. ☐

Proposition 8. *At an arbitrary type, a neighborhood is total if and only if its closure is a total ideal.*

Proof. Let ρ be a type. We want to show that

$$\underset{U \in \mathsf{Con}_\rho}{\forall} \left(U \in G\mathsf{Con}_\rho \leftrightarrow \overline{U} \in G_\rho\right) \; .$$

At a base type α, it is immediate that U entails a total token if and only if its closure contains it.

At a type $\rho \to \sigma$, let $W \in \mathsf{Con}_{\rho\to\sigma}$. Assume that $W \in G\mathsf{Con}_{\rho\to\sigma}$. We want to show that $\overline{W} \in G_{\rho\to\sigma}$, that is, that for every $x \in G_\rho$, it is also $\overline{W}(x) \in G_\sigma$. So let x be a total at type ρ; by the assumption, there is an eigen-neighborhood H of W, such that $\arg H \subseteq x$ and $\operatorname{val} H \in G\mathsf{Con}_\sigma$; it follows that

$$W(x) \vdash_\sigma \operatorname{val} H \; , \tag{$\star$}$$

where $\operatorname{val} H \in G\mathsf{Con}_\sigma$. By the induction hypothesis at σ, it is $\overline{\operatorname{val} H} \in G_\sigma$, so by Lemma 6 the formula ($\star$) gives $\overline{\operatorname{val} H} \subseteq \overline{W}(x)$; it is then $\overline{W}(x) \in G_\sigma$ by Lemma 7.

For the other direction, assume that $\overline{W} \in G_{\rho\to\sigma}$. To show that $W \in G\mathsf{Con}_{\rho\to\sigma}$, we have to come up with an eigen-neighborhood H for each $x \in G_\rho$, which will satisfy both $\arg H \subseteq x$ and $\operatorname{val} H \in G\mathsf{Con}_\sigma$. So let x be a total at ρ. The assumption yields that $\overline{W}(x) \in G_\sigma$. By the definition of application, this means that $\sum_{A_{W,x}(b)} b \in G_\sigma$, where

$$A_{W,x}(b) := \underset{U \in \mathsf{Con}_\rho}{\exists} \left(\langle U, b\rangle \in \overline{W} \wedge U \subseteq x\right) \; ;$$

such U's indeed exist, since the ideal is inhabited (being total). But $\langle U, b\rangle \in \overline{W}$ means $W \vdash_{\rho\to\sigma} \langle U, b\rangle$, which, by implicit atomicity, means that there is an eigen-neighborhood H of W, such that $U \vdash_\rho \arg H$ and $\operatorname{val} H \vdash_\sigma b$; for this eigen-neighborhood, by the deductive closure of x, we get that $\arg H \subseteq x$, so $A_{W,x}(b)$ implies

$$B_{W,x}(b) := \underset{H\in\operatorname{Eig} W}{\exists} (\arg H \subseteq x \wedge \operatorname{val} H \vdash_\sigma b) \ ,$$

while it's not hard to see that also $B_{W,x}(b)$ implies $A_{W,x}(b)$. So the assumption now reads $\sum_{B_{W,x}(b)} b = \overline{\operatorname{val} H} \in G_\sigma$. By the induction hypothesis at σ, it is $\operatorname{val} H \in G\mathsf{Con}_\sigma$, so we're done. □

Corollary 9. *A total functional maps total neighborhoods to total functionals.*

Proof. Let $\rho \to \sigma$ be some higher type, and let $f \in G_{\rho\to\sigma}$ and $U \in G\mathsf{Con}_\rho$. We want to show that $f(U) \in G_\sigma$. By Proposition 8, since U is a total neighborhood, its closure will be a total ideal, so $f(\overline{U}) \in G_\sigma$; this straightforwardly implies that $f(U) \in G_\sigma$. □

We can also prove a "finite" analogue to the Extension Lemma 7.

Proposition 10 (Finite extension lemma)**.** *Extension preserves finite totality, that is, a neighborhood that entails a total neighborhood is itself total.*

Proof. We want to prove the following.

$$\underset{U,U'\in\mathsf{Con}_\rho}{\forall} (U' \vdash_\rho U \wedge U \in G\mathsf{Con}_\rho \to U' \in G\mathsf{Con}_\rho) \ .$$

At a base type α, let $U, U' \in \mathsf{Con}_\alpha$, such that $U' \vdash_\alpha U$ and $U \in G\mathsf{Con}_\alpha$. Let a be the total token that U entails; by transitivity of entailment, it is $U' \vdash_\alpha a$ as well.

At a higher type $\rho \to \sigma$, let $W, W' \in \mathsf{Con}_{\rho\to\sigma}$, such that $W' \vdash_{\rho\to\sigma} W$ and $W \in G\mathsf{Con}_{\rho\to\sigma}$. We want to show that $W' \in G\mathsf{Con}_{\rho\to\sigma}$, so let $x \in G_\rho$ be an arbitrary total. By the totality of W, there is an eigen-neighborhood H of W, such that $\arg H \subseteq x$ and $\operatorname{val} H \in G\mathsf{Con}_\sigma$; by implicit atomicity, there is an eigen-neighborhood H' of W', such that $\arg H \vdash_\rho \arg H'$ and $\operatorname{val} H' \vdash_\sigma \operatorname{val} H$, since x is closed, it is $\arg H' \subseteq x$, and by the induction hypothesis at σ it follows that $\operatorname{val} H' \in G\mathsf{Con}_\sigma$, so H' is indeed an eigen-neighborhood as we need it. □

5 Finite density

We will use the following conventions. Say that a (not necessarily consistent) list Γ_1 *extends* the list Γ_2, if for each $U_2 \in \mathsf{Con}_{\Gamma_2}$ there is a $U_1 \in \mathsf{Con}_{\Gamma_1}$, such that $U_1 \vdash_\rho U_2$. For example, the list Δ_1 from before extends Δ_2, while the list $\Delta_3 := 0 + S0 + SS{*}$ extends both of them, and the (already consistent) $U_1 := SS0$ extends $U_2 := SS{*}{+}S{*}$. It is clear that if Γ_1 extends Γ_2, then every maximal neighborhood of Γ_1 extends a unique maximal neighborhood of Γ_2. Say that an eigen-neighborhood H of W is *maximal* in W, and write $H \in M\operatorname{Eig} W$, if $\arg H \in \operatorname{Max} \operatorname{fl}(\arg W)$.

We will also need the *size* of a token of an algebra α, defined as the number of its proper constructors: for example, in our pet algebra we would have $\|0\| = 1$, $\|B{*}BS{*}1\| = 4$, and so on.

Theorem 11 (Finite density). *Assume that the type system is built upon finitary algebras, each of which features a distinguished nullary token. At an arbitrary type, every list can be extended to a separating list and every neighborhood can be extended to a total neighborhood.*

Proof by mutual induction. We concentrate first on separation. At a base type α, let $\Gamma \in \mathsf{Lst}_\alpha$. Set s_Γ to be the maximum size of Γ's tokens, that is, $s_\Gamma := \max_{a\in\Gamma} \|a\|$. Then set $\Gamma^S := \left\{a \in T_\alpha \mid \|a\| \leq s_\Gamma\right\}$. This is a finite list that obviously does the job.[12]

At a higher type $\rho \to \sigma$, assume that neighborhoods at ρ can be extended to total neighborhoods and that lists at σ can be extended to separating lists. Let $\Theta \in \mathsf{Lst}_{\rho\to\sigma}$, and write Γ for $\mathrm{fl}(\arg\Theta)$. Set

$$\Theta^S := \Theta + \sum_{U\in\mathrm{Max}\,\Gamma} \left\langle U^G, (\Theta U^G)^S \right\rangle \ .$$

It is obvious by definition that the list Θ^S is finite and extends Θ. To show that it is separating consider a total ideal $f \in G_{\rho\to\sigma}$. By Corollary 9, it is $f(U^G) \in G_\sigma$; by induction hypothesis, since $(\Theta U^G)^S$ is separating, it will share one of its maximal neighborhoods with $f(U^G)$, that is, there will exist a maximal $N \in \mathsf{Con}_\sigma$, such that

$$N \in \mathrm{Max}\,(\Theta U^G)^S \wedge N \subseteq f(U^G) \ ;$$

then $\left\langle U^G, N\right\rangle$ is a neighborhood that f and Θ^S share, which extends a maximal of Θ, so is itself maximal in Θ^S by the finite density at ρ.

Now we turn to finite density. At a base type α, let $U \in \mathsf{Con}_\alpha$. By implicit atomicity, this neighborhood has an eigentoken a_U. If $a_U = *$, then set $a_U^G := 0_\alpha$, where 0_α is the distinguished nullary token of α. If $a_U = Ca_1\cdots a_r$, for a constructor C of arity $r \geq 0$, then set $a_U^G := Ca_1^G\cdots a_r^G$. For the total extension of U, finally, set $U^G := \left\{a_U^G\right\}$.

At a higher type $\rho \to \sigma$, assume that lists at ρ can be extended to separating lists and that neighborhoods at σ can be extended to total neighborhoods. Let $W \in \mathsf{Con}_{\rho\to\sigma}$. Write Γ for $\mathrm{fl}(\arg W)$; this is a list at type ρ which can be extended to a separating list Γ^S; consider the neighborhood

$$\tilde{W} := W + \sum_{U\in\mathrm{Max}\,_{\Gamma^S}} \langle U, WU\rangle \ ;$$

since Γ^S extends Γ, it follows that for every maximal neighborhood U in the former there exists a unique maximal eigen-neighborhood H of W such that $U \vdash_\rho \arg H$ while $WU = \mathrm{val}\,H$ by construction; so by Proposition 4, $\tilde{W}$ is a conservative extension of W. Set

$$W^G := \tilde{W} + \sum_{H\in M\mathrm{Eig}\,\tilde{W}} \left\langle \arg H, (\mathrm{val}\,H)^G \right\rangle \ .$$

The list W^G is obviously finite and extends the neighborhood $\tilde{W}$ (and consequently W as well).

Consistency follows by Proposition 1, since every maximal neighborhood U of $\mathrm{fl}(\arg W^G)$ features by construction as an argument in $W^G \smallsetminus \tilde{W}$, in the form $\arg H$

[12] Note that this is a rather crude construction, yielding a brute-force method for providing a separating extension for a given list.

for some maximal eigen-neighborhood H of $\tilde{W}$, and is paired with $(\mathrm{val}\, H)^G$, which is a maximal neighborhood in $\mathrm{val}\, W^G$ by the induction hypothesis at σ.

It remains to show that it is a total neighborhood. Let x be a total ideal at type ρ. Since $\arg W^G = \arg \tilde{W}$ is separating by construction, there will be a maximal eigen-neighborhood H^x of W^G, such that $\arg H^x \subseteq x$. By the construction of W^G, it is $W(x) = W \cdot \arg H^x = \mathrm{val}\, H^x \in G\mathsf{Con}_\sigma$. □

Corollary 12 (Density)**.** *Every type is dense.*

Proof. Let ρ be a type, and $U \in \mathsf{Con}_\rho$. By Theorem 11 there exists a total neighborhood U^G that extends U. By Proposition 8, U^G has a total closure, so the ideal that we seek is $\overline{U^G}$. □

6 Discussion

It is very simple, but quite interesting and important to notice that the construction in our Theorem 11 would not succeed if we worked with flat domains. The reason is that separation, as we defined it, would fail at non-trivial base types, like the natural numbers: suppose that we are given the singleton list $\mathit{\Gamma} := S0$; in our setting, a separating list extending $\mathit{\Gamma}$ is $0+S0+SS*$ (actually, by the high-complexity version of our proof, it would be $* + 0 + S* + S0 + SS*$); but in a flat setting, there is no finite part of the token carrier $\{*, 0, S0, SS0, \ldots\}$, that could ever support all totals.

This remark was pointed out to the author by Davide Rinaldi, who has independently arrived at a similar construction as ours, motivated by formal topological considerations and working with structures that we believe are very similar in nature to the collections of eigen-neighborhoods in a given information system. A further collaborative pursuit should help clarify the connections of our essentially domain-theoretical approach to the viewpoint of formal topology, which among other things could help pinpoint the notion of atomicity in a formal topological setting—an issue which is yet unresolved to the best knowledge of the author.

Outlook

There are two or three points in the above exposition that could allow elaboration and even improvement. Firstly, we avoided using the feature of implicit atomicity, so to speak, explicitly, and we confined ourselves to the use of the more moderate Proposition 3. A heavy, explicit use of Theorem 5 should render a tighter exposition.

Secondly, our construction of the separating witness in Theorem 11 craves for a brave trimming. An investigation—unavoidably of combinatorial nature—on low-complexity witnesses of separation may also lead to recognition of new normal forms for higher-type neighborhoods, and is certainly a subject for further study.

Thirdly, though we have succeeded in providing finite witnesses for density, it would arguably make for a conceptually neater exposition to have a characterization of finite totality exclusively by finite means. This is also something to look into.

Now, apart from the present topic of density, further work in this direction would include using implicit atomicity in the sense of our Theorem 5, to tackle other known problems in higher-type computability theory, like definability, which up to now are established for our setting, if at all, only partially, or as results of non-intrinsic and maybe redundantly general and powerful domain-theoretic machinery.

Acknowledgments

Thanks to Davide Rinaldi for the very interesting discussions on the topic. Also thanks to Helmut Schwichtenberg for his crucial feedback on several earlier attempts.

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