Nonflatness and totality

Basil A. Karádais

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Abstract

We prove a strong version of the Kreisel density theorem by providing a witness generated by a compact element. This is achieved by interpreting finite types as domains over nonflat base types. Separation is obtained as a corollary, and the mismatch of the nonflat and flat-based versions of totality is discussed.

1 Introduction

In the area of denotational semantics of functional programming it is standard to view data types as countably based Scott domains in the tradition that started with Dana Scott's and Yuri Ershov's independent work in the late sixties and early seventies. More particularly, we may view these domains through their representations as Scott information systems, where programs are representatives of typed terms $x : \rho$ with denotations being ideals in appropriate information systems, that is, consistent and deductively closed sets of tokens $a \in x$; ideals are approximated by finite sets $U \subseteq x$, their so called formal neighborhoods.

A crucial choice in our setting is to work with *nonflat* rather than flat domains for the base types. These arise when we model base type partiality not as an extra pseudotoken $\perp$, but as an extra nullary pseudoconstructor $*$, which participates in the formation of further tokens and therefore leads to varying degrees of partiality. For example, instead of just a bottom for the natural numbers, we have the partial tokens $*$, $\mathtt{S}*$, $\mathtt{SS}*$ et cetera; tokens that do not involve the pseudoconstructor $*$, like $\mathtt{0}$, $\mathtt{S0}$, $\mathtt{SS0}$, and so on, are called *total tokens*. A basic advantage of this feature compared to flat base types is that we obtain injectivity and disjoint ranges for the constructors.

More generally, base-type nonflatness yields domains which are in a certain sense both richer, in that they contain more tokens, and tidier, in that they are finitely branching. Such domains seem to accommodate arguments that a flat setting cannot afford, and this paper intends to give one nontrivial example of this kind: an explicitly finitary approach to the *Kreisel density theorem*, a key result in the theory of higher-type computability[1]. Density was first stated and proved by Georg Kreisel in [25], and in different terms by Stephen Kleene in [24]. Building on work of Yuri Ershov [12, 13, 14], Ulrich Berger [2, 3] generalized and established density within domain theory, drawing as a corollary that it holds for the hierarchy of the partial continuous functionals over all finite types, and thus recovering the Kleene–Kreisel continuous functionals as equivalence classes of the abstractly total elements in the hierarchy.

[1] See section 6.

Helmut Schwichtenberg and collaborators have carried Berger's argument from the top (abstract domains with totality) down to the bottom (concrete Scott information systems induced by algebra constructors) numerous times in the past, starting with [42] and following up with [43, 19, 20, 44]. The present work builds on these latter approaches. We capture the concept of termination by a totality predicate G: at base types ι, an ideal x is total if it contains a total token; at type $\rho \to \sigma$, an ideal f is total when it preserves totality, that is, when

$$\forall_{x:\rho} (G_\rho(x) \to G_\sigma(fx)) \ ,$$

where $b \in fx$ for a token $b \in \mathrm{Tok}_\sigma$ if and only if $\langle U, b \rangle \in f$, for some formal neighborhood $U \in \mathrm{Con}_\rho$ with $U \subseteq x$. The *density property* for a type, the latter being understood as a space governed by the Scott topology, alleviates the omnipresent partiality by stating that every open set in the space nurtures total points, in other words, that total points are dense in the space. We formulate this here by saying that ρ is *dense* when

$$\forall_{U \in \mathrm{Con}_\rho} \exists_{x:\rho} \left(G_\rho(x) \wedge U \subseteq x \right) \ . \tag{D}$$

Namely, we are given a neighborhood U, comprising *finite* information, and we are supposed to come up with an ideal x as a witness, a set of tokens which is in principle *infinite*. It is reasonable to suspect that the element of infinity in x must be inessential as far as an actual process of "totalization" of U is concerned—whatever this process might be—and that there's nothing inherently infinitary about it. Indeed, the question that we claim to answer here in the positive is: can we devise a totalization process which will feature an explicitly finitary core, that is,

can we provide a witness for density which will be obviously finitary?

Our strategy can be summarized as follows: (a) define a notion of "total neighborhood"; (b) establish a "finite density theorem", that is, that every neighborhood extends to such a total neighborhood; (c) show that a total neighborhood extends to a total ideal in a straightforward way.

We begin in section 2 with a necessary preamble on domains over nonflat base types represented by information systems. In section 3 we pave the road to the definition of "finite totality", by discussing elementary facts concerning not necessarily consistent finite sets. In section 4 we define finite totality, prove finite density with Theorem 4.7, and characterize finite totality in a noninductive way in Theorem 4.9. In section 5 we establish density with Theorem 5.10, and we list some of its direct consequences, among them the "separation property" in Proposition 5.14. We end in section 6 with comments on the literature and future work.

2 Nonflat domains via coherent information systems

We concentrate on a type system supporting arrow types over inductive base types.[2] We use ξ as a dummy type variable. Write $\overrightarrow{\rho} \to \sigma$ to mean $\rho_1 \to \cdots \to \rho_r \to \sigma$ for some $r \geqslant 0$ associated to the right; in case $r = 0$ the vector is empty.

- For every vector $\overrightarrow{\xi}$ of length r, the expression $\overrightarrow{\xi} \to \xi$ is a *constructor type (of arity r)*.

[2] In this section we omit proofs and details, for which the reader may consult [44, Part 3] and [46, Part I]. In relation to the former, in particular, note that we will be working within the *nonparametric* and *finitary* fragment of the system.

- If $\kappa_1, \dots, \kappa_k$ are constructor types for $k > 0$ and one of them nullary, then $\mu_\xi(\kappa_1, \dots, \kappa_k)$ is a *type*. We think of such types as inductively defined *base types* or *algebras*.
- If ρ, σ are types then $\rho \to \sigma$ is a *type*; these are the usual *higher types*.

Note that constructor types only serve to build base types, and are not themselves admitted as types. Examples of base types are

- the *unit type* $\mathbb{U} := \mu_\xi(\xi)$ with a single nullary constructor,
- the *type of boolean values* $\mathbb{B} := \mu_\xi(\xi, \xi)$, with constructors for the truth $\mathtt{tt} : \mathbb{B}$ and the falsity $\mathtt{ff} : \mathbb{B}$,
- the *type of natural numbers* $\mathbb{N} := \mu_\xi(\xi, \xi \to \xi)$, with constructors for the zero $\mathtt{0} : \mathbb{N}$ and the successor $\mathtt{S} : \mathbb{N} \to \mathbb{N}$,
- the *type of (extended) derivations* $\mathbb{D} := \mu_\xi(\xi, \xi, \xi \to \xi, \xi \to \xi \to \xi)$, with constructors for an axiom $\mathtt{0} : \mathbb{D}$, another axiom $\mathtt{1} : \mathbb{D}$, a one-premise rule $\mathtt{S} : \mathbb{D} \to \mathbb{D}$, and a two-premise rule $\mathtt{B} : \mathbb{D} \to \mathbb{D} \to \mathbb{D}$ (this algebra is simple yet nontrivial enough to provide us with examples as we go along).

We will write ι to denote an arbitrary *base* type and ρ, σ to denote arbitrary types in general.

A *(Scott) information system* [45, 47] is a triple $(\mathrm{Tok}, \mathrm{Con}, \vdash)$, where Tok is a countable set of *tokens*, Con is a collection of finite sets of tokens which we call *consistent sets* or *(formal) neighborhoods*, and $\vdash$ is a subset of $\mathrm{Con} \times \mathrm{Tok}$, the *entailment*. These are subject to the axioms

$$
\begin{aligned}
&\{a\} \in \mathrm{Con}, \\
&U \subseteq V \wedge V \in \mathrm{Con} \to U \in \mathrm{Con}, \\
&U \in \mathrm{Con} \wedge a \in U \to U \vdash a, \\
&U \vdash V \wedge V \vdash c \to U \vdash c, \\
&U \vdash b \to U \cup \{b\} \in \mathrm{Con},
\end{aligned}
$$

where $U \vdash V$ stands for $U \vdash b$ for all $b \in V$. From the latter follows vacuously that $U \vdash \varnothing$ for all U, while $\varnothing \in \mathrm{Con}$ follows from the first two axioms. We may refer to the fifth axiom as *propagation (of consistency through entailment)*.

For finite sets of tokens Γ which are not necessarily consistent we write Fin, so $\mathrm{Con} \subseteq \mathrm{Fin}$. An information system is called *coherent* when in addition to the above it satisfies

$$
\bigvee_{a,a' \in U} \{a, a'\} \in \mathrm{Con} \to U \in \mathrm{Con} \tag{1}
$$

for all $U \in \mathrm{Fin}$. By the coherence and the second axiom above, it follows that the consistency of a token set is equivalent to the consistency of its pairs. Drawing on this property, we often write $a \asymp b$ for $\{a, b\} \in \mathrm{Con}$, and even $U \asymp V$ for $U \cup V \in \mathrm{Con}$ (which is also often written $U \uparrow V$). In the following we restrict our attention to coherent systems.

Given two coherent information systems ρ and σ, we form their *function space* $\rho \to \sigma$: define its tokens by $\langle U, b \rangle \in \mathrm{Tok}$ if $U \in \mathrm{Con}_\rho$ and $b \in \mathrm{Tok}_\sigma$, its consistency by $\langle U,b\rangle \asymp \langle U',b'\rangle$ if $U \asymp_\rho U'$ implies $b \asymp_\sigma b'$, and its entailment by $W \vdash \langle U,b\rangle$ if $WU \vdash_\sigma b$, where

$$b \in WU := \exists_{U' \in \mathrm{Con}_\rho} \big(\langle U',b\rangle \in W \wedge U \vdash_\rho U'\big).$$

The last operation is called *neighborhood application*. We will revisit it in some depth in section 3.3 where we will also show that it is monotone in both arguments, that is, that $U \vdash U'$ implies $WU \vdash WU'$ and that $W \vdash W'$ implies $WU \vdash WU'$, for all appropriate U,U',W,W'.

Fact 2.1. *The function space of two coherent systems is itself a coherent information system.*

An *ideal* (or *element*) of an information system ρ is a possibly infinite token set $x \subseteq \mathrm{Tok}$, such that $U \in \mathrm{Con}$ for every $U \subseteq_f x$ (consistency), and $U \vdash b$ for some $U \subseteq_f x$ implies $b \in x$ (deductive closure). If x is an ideal of ρ, we write $x : \rho$ or $x \in \mathrm{Ide}_\rho$. Note that there is an empty ideal $\perp_\rho = \varnothing$ for every ρ.

By a *(Scott–Ershov) domain* we mean here a countably based directed complete partial order with a least element, which is additionally algebraic and bounded complete. A domain is *coherent* [37], if every set of compacts has a least upper bound exactly when each of its *pairs* has a least upper bound. Write $b \in \overline{U}$ if and only if $U \vdash b$ (it is $\overline{\varnothing} := \varnothing$).

Fact 2.2 (Representation theorem)**.** *Let* $\rho = (\mathrm{Tok}_\rho, \mathrm{Con}_\rho, \vdash_\rho)$ *be a coherent information system. Then* $(\mathrm{Ide}_\rho, \subseteq, \varnothing)$ *is a coherent domain with compacts given by* $\{\overline{U} \mid U \in \mathrm{Con}_\rho\}$. *Conversely, every coherent domain can be represented by a coherent information system.*

An *approximable mapping* between two information systems ρ and σ is a relation $r \subseteq \mathrm{Con}_\rho \times \mathrm{Con}_\sigma$ that generalizes entailment in the following sense: $\langle \varnothing, \varnothing\rangle \in r$; if $\langle U,V_1\rangle, \langle U,V_2\rangle \in r$ then $\langle U, V_1 \cup V_2\rangle \in r$; and if $U \vdash_\rho U'$, $\langle U',V'\rangle \in r$, and $V' \vdash_\sigma V$, then $\langle U,V\rangle \in r$. One can show [45] that there is a bijective correspondence between the approximable mappings from ρ to σ and the ideals of the function space $\rho \to \sigma$, and moreover establish the categorical equivalence between domains with Scott continuous functions and information systems with approximable mappings. The equivalence is preserved if we restrict ourselves to the coherent case on both sides [22].

The *Scott topology* on Ide_ρ is given by the collection $\{\nabla U \mid U \in \mathrm{Con}_\rho\}$, where ∇U is the set $\{x : \rho \mid U \subseteq x\}$ of all ideals above U. A set $\mathscr{U} \subseteq \mathrm{Ide}_\rho$ of ideals is *Scott open* when it is closed under supersets (Alexandrov condition) and for every $x \in \mathscr{U}$ there is a $U \subseteq x$ such that $\overline{U} \in \mathscr{U}$ (Scott condition). One can furthermore show that an ideal-mapping f sending ideals from Ide_ρ to ideals in Ide_σ is *Scott continuous* when it is *monotone* and satisfies the *principle of finite support* (also called *approximation principle*) for all $x : \rho$, that is,

$$\forall_{b \in \mathrm{Tok}_\sigma} (b \in f(x) \to \exists_{U \in \mathrm{Con}_\rho} (U \subseteq x \wedge b \in f(\overline{U}))). \tag{FS}$$

Finally, it can be shown that the ideals $\mathrm{Ide}_{\rho\to\sigma}$ and the Scott continuous ideal-mappings $\mathrm{Ide}_\rho \to \mathrm{Ide}_\sigma$ are in a bijective correspondence, a fact that justifies the nondiscriminating notation $f : \rho \to \sigma$.

Now we proceed to assign an information system to each type. Every higher type is naturally assigned a function space, so it suffices to discuss the information systems for base types, that is, for algebras. Let ι be an algebra, with at least one nullary constructor if it is to be nontrivial. We add to it an extra nullary pseudoconstructor $*_\iota$ (or just $*$) to denote partiality.

- If C is an r-ary constructor and $a_i \in \mathrm{Tok}_\iota$ for $i = 1, \dots, r$ then $Ca_1 \cdots a_r \in \mathrm{Tok}_\iota$.[3] For its *head constructor* write $\mathrm{hd}(Ca_1 \cdots a_r) = C$; for its *i-th component token* write $a(i)$, that is, $(Ca_1 \cdots a_r)(i) = a_i$ for $i = 1, \dots, r$.
- It is $a \asymp_\iota *$ and $* \asymp_\iota a$ for all $a \in \mathrm{Tok}_\iota$. Furthermore, if C is an r-ary constructor and $a_i \asymp_\iota b_i$ for $i = 1, \dots, r$ then $Ca_1 \cdots a_r \asymp_\iota Cb_1 \cdots b_r$. Finally, it is $U \in \mathrm{Con}_\iota$ if $a \asymp_\iota a'$ for all $a, a' \in U$.
- It is $U \vdash_\iota *$ for all $U \in \mathrm{Con}_\iota$. Furthermore, if C is an r-ary constructor, every $U_i \in \mathrm{Con}_\iota$ is inhabited and $U_i \vdash_\iota b_i$ for $i = 1, \dots, r$, then $U \vdash_\iota Cb_1 \cdots b_r$ for all $U \in \mathrm{Con}_\iota$ which are *sufficient for C on $U_1, \dots, U_r$*, in the sense that for each $i = 1, \dots, r$ and each $a_i \in U_i$ there exists an $a \in U$ such that $\mathrm{hd}(a) = C$ and $a(i) = a_i$. Finally, if $U \vdash_\iota b$, then also $U \cup \{*\} \vdash_\iota b$.

Note that the definition of Con_ι incorporates (1), so it follows that $\varnothing \vdash_\iota \{*\}$. Write $U \sim V$ for $U \vdash V \wedge V \vdash U$. Concerning the notion of sufficiency, note that (a) it is $U \sim_\iota CU_1 \cdots U_r$, whenever U is sufficient for C on $U_1, \dots, U_r$, where the *constructor application* is defined by

$$CU_1 \cdots U_r := \{Ca_1 \cdots a_r \mid a_1 \in U_1, \dots, a_r \in U_r\},$$

and (b) in case C is a proper constructor, U is sufficient for C on $U_1, \dots, U_r$ if and only if $U \cup \{*\}$ is, if and only if $U \backslash \{*\}$ is. More generally, every neighborhood U which is *nontrivial* (meaning $U \not\sim_\iota \varnothing$) is equivalent to one of the form $CU_1 \cdots U_r$: if

$$U \backslash \{*\} = \{Ca_{11} \cdots a_{r1}, \dots, Ca_{1m} \cdots a_{rm}\},$$

we gather all i-th component tokens into a neighborhood, the *i-th component neighborhood* $U(i) := \{a_{i1}, \dots, a_{im}\}$ of U, and let $U_i := U(i)$ for every $i = 1, \dots, r$. Finally, the finite set $CU_1 \cdots U_r$ is consistent if every U_i is consistent.

Fact 2.3. *Let ι be an algebra given by constructors. The triple $(\mathrm{Tok}_\iota, \mathrm{Con}_\iota, \vdash_\iota)$ is a coherent information system.*

3 Finite sets

Recall that the first step in our strategy is to decide on a reasonable definition of "finite totality", one that will already embody the totalization mechanism for density on the one hand, and that will be susceptible to a canonical extension to a total ideal on the other. To this end it turns out that we need for finite token sets an operation akin to application, but defined using consistency rather than entailment. The examination of the behavior of this operation leads us to consider "transitive elements", that is, elements that witness local transitivity within a not necessarily transitive relation. This is how we come to spend some space discussing not necessarily consistent finite sets in some generality, while we postpone the actual definition of total neighborhoods until section 4.

[3] Throughout the text we adopt the polish notation for tokens for typographical convenience.

3.1 Entailment and consistency for finite sets

There is the trivial syntactical reason to look at finite sets in general and not just at the consistent ones: the latter presuppose the former by definition—in particular, the thematization of finite sets is unavoidable in implementation endeavors like [20]. But finite sets may play a natural and important role within purely semantical arguments as well—to mention a naive example, think of the subtokens $a_1,\dots,a_r$ of a base-type token $a = Ca_1\cdots a_r$. In this section we will hardly cover anything more than what we will need later, with the possible exception of Lemma 3.2, which we included for the sake of some points in §3.3.

As we already mentioned, we write Fin_ρ instead of $\mathscr{P}_f(\mathrm{Tok}_\rho)$, so $\Gamma \in \mathrm{Fin}_\rho$ means that Γ is a *finite* set of tokens, not necessarily consistent. If $\Theta = \{\langle U_j, b_j\rangle \mid j = 1,\dots,l\} \in \mathrm{Fin}_{\rho\to\sigma}$, write $L(\Theta)$ for $\bigcup_j U_j \in \mathrm{Fin}_\rho$ (notice that this is a *flattening*), and $R(\Theta)$ for $\bigcup_j \{b_j\} \in \mathrm{Fin}_\sigma$. Furthermore, if $U \in \mathrm{Con}_\rho$ and $\Delta \in \mathrm{Fin}_\sigma$, write $\langle U,\Delta\rangle$ for $\{\langle U,b\rangle \mid b \in \Delta\} \in \mathrm{Fin}_{\rho\to\sigma}$ (note that $\langle U,\varnothing_\sigma\rangle = \varnothing_{\rho\to\sigma}$).

Lemma 3.1. *Let $\Theta,\Theta' \in \mathrm{Fin}_{\rho\to\sigma}$. It is $L(\Theta\cup\Theta') = L(\Theta)\cup L(\Theta')$ and $R(\Theta\cup\Theta') = R(\Theta)\cup R(\Theta')$. Furthermore, if $\Theta\subseteq\Theta'$ then it is $L(\Theta)\subseteq L(\Theta')$ as well as $R(\Theta)\subseteq R(\Theta')$.*

A *neighborhood in $\Gamma \in \mathrm{Fin}_\rho$* is a subset $U \subseteq \Gamma$, which happens to be consistent; write $U \in \mathrm{Con}_\Gamma$. The empty set and the singletons of Γ are always in Con_Γ. Say that Γ *entails Γ'* (*as a finite set*), and write $\Gamma \vdash^F_\rho \Gamma'$, when

$$\forall_{U'\in\mathrm{Con}_{\Gamma'}}\ \exists_{U\in\mathrm{Con}_\Gamma}\ U \vdash_\rho U'.$$

This is obviously a direct generalization of the notion $U \vdash_\rho U'$ for neighborhoods, and a bit more sophisticated than the notion "$\Gamma \vdash_\rho \Gamma'$ if and only if for every $a' \in \Gamma'$ there is some $U \in \mathrm{Con}_\Gamma$ such that $U \vdash_\rho a'$" (equivalently, "if and only if $\Gamma \vdash^F_\rho a'$ for all $a' \in \Gamma'$"); contrary to the case of consistent sets, although $\Gamma \vdash^F_\rho \Gamma'$ implies $\Gamma \vdash^F_\rho a$ for all $a \in \Gamma'$, the converse is not true in general, since for example $\{\mathtt{B00},\mathtt{B11}\} \vdash^F_{\mathbb{D}} \mathtt{B0}{*}$ and $\{\mathtt{B00},\mathtt{B11}\} \vdash^F_{\mathbb{D}} \mathtt{B}{*}\mathtt{1}$, but $\{\mathtt{B00},\mathtt{B11}\} \nvdash^F_{\mathbb{D}} \{\mathtt{B0}{*},\mathtt{B}{*}\mathtt{1}\}$.

Similarly, say that Γ and Γ' are *consistent* (*as finite sets*), and write $\Gamma \asymp^F_\rho \Gamma'$, when

$$\forall_{U\in\mathrm{Con}_\Gamma}\ \forall_{U'\in\mathrm{Con}'_\Gamma}\ U \asymp_\rho U'.$$

Again, this is a generalization of consistency between neighborhoods which proves more important for generally non-consistent finite sets than the simple notion "$\Gamma \asymp_\rho \Gamma'$ if and only if $\{a,a'\} \in \mathrm{Con}_\rho$ for all $a \in \Gamma$ and $a' \in \Gamma'$" (which we may nevertheless occasionally use); and again, it is obvious that $\Gamma \asymp^F_\rho \Gamma'$ implies $\Gamma \asymp_\rho \Gamma'$, but the converse does not hold in general. Note that in the case of $\asymp^F_\rho$ we generally don't have reflexivity; in fact, it is trivial that $\Gamma \asymp^F_\rho \Gamma$ if and only if $\Gamma \in \mathrm{Con}_\rho$. An example of consistency between inconsistent finite sets is $\{\mathtt{B0}{*},\mathtt{B1}{*}\} \asymp^F_{\mathbb{D}} \{\mathtt{B}{*}\mathtt{0},\mathtt{B}{*}\mathtt{1}\}$.

Reflexivity of consistency is the only property that the triple $(\mathrm{Fin}_\rho, \asymp^F_\rho, \vdash^F_\rho)$ lacks in order to constitute a Scott information system.

Lemma 3.2. *The entailment between finite sets is reflexive and transitive and the consistency between finite sets is symmetric and propagates through entailment, that is,*

1. $\forall_{\Gamma\in\mathrm{Fin}}\Gamma \vdash^F \Gamma$,

2. $\forall_{\Gamma,\Delta,\Theta\in\mathrm{Fin}}(\Gamma \vdash^F \Delta \wedge \Delta \vdash^F \Theta \to \Gamma \vdash^F \Theta)$,

3. $\forall_{\Gamma,\Delta\in\mathrm{Fin}}(\Gamma \asymp^F \Delta \to \Delta \asymp^F \Gamma)$,

4. $\forall_{\Gamma,\Delta,\Theta\in\mathrm{Fin}}(\Gamma \asymp^F \Delta \wedge \Delta \vdash^F \Theta \to \Gamma \asymp^F \Theta)$.

Proof. We just show the propagation property. Let $\Gamma,\Delta,\Theta \in \mathrm{Fin}$ be such that $\Gamma \asymp^F \Delta$ and $\Delta \vdash^F \Theta$. Consider $U \in \mathrm{Con}_\Gamma$ and $W \in \mathrm{Con}_\Theta$; by the assumptions, there exists a $V \in \mathrm{Con}_\Delta$ with $V \vdash W$ and $U \asymp V$; by propagation on Con we get $U \asymp W$. □

3.2 Maximal and transitive neighborhoods

Think of some finite set Γ of type ρ and suppose that we wish to assign σ-values b_i to neighborhoods U_i of Con_Γ (for some $i \in I$) in a way that the finite set $\{\langle U_i, b_i\rangle \mid i \in I\}$ at type $\rho \to \sigma$ will be consistent. Some reflection shows that it suffices to pair the “maximal” neighborhoods of Γ with the given values of σ, but we can actually do better than that: we can relax the requirement of maximality by requiring instead that we assign the given arbitrary values already to those neighborhoods which are, so to speak, maximal enough or “almost maximal”, in the sense that they are below exactly one maximal in Γ; these are exactly the “transitive neighborhoods” in Γ.

Call $U \in \mathrm{Con}_\Gamma$ a *maximal neighborhood in* Γ, and write $U \in \mathrm{Con}_\Gamma^{\max}$, when it is maximal with respect to the entailment relation, that is, when

$$\forall_{U'\in\mathrm{Con}_\Gamma}(U' \vdash_\rho U \to U \vdash_\rho U').$$

Call U *(consistency) transitive in* Γ, and write $U \in \mathrm{Con}_\Gamma^{\mathrm{ctr}}$, when it satisfies the property

$$\forall_{U_1,U_2\in\mathrm{Con}_\Gamma}(U_1 \asymp_\rho U \asymp_\rho U_2 \to U_1 \asymp_\rho U_2).$$

We can reformulate this by introducing the notation $\widetilde{U}$ for the *consistency closure* of U, that is, for the set $\{a \in \mathrm{Tok}_\rho \mid U \asymp_\rho a\}$ (it is clear that, while it encompasses the deductive closure, the consistency closure of a neighborhood is *not* in general an ideal, because consistency may fail); then U is transitive in Γ when $\widetilde{U} \cap \Gamma \in \mathrm{Con}_\rho$.

More generally, call $U \in \mathrm{Con}_\rho$ *transitive for* Γ or just Γ*-transitive (in* ρ*)*, and write $U \in \mathrm{Con}_{\rho|\Gamma}^{\mathrm{ctr}}$, if, again, $U_1 \asymp_\rho U \asymp_\rho U_2$ implies $U_1 \asymp_\rho U_2$ for all $U_1, U_2 \in \mathrm{Con}_\Gamma$; obviously, $\mathrm{Con}_\Gamma \subseteq \mathrm{Con}_{\rho|\Gamma}$.

It is clear that every maximal in a finite set is also transitive in it. It is also immediate that consistency between neighborhoods, restricted to $\mathrm{Con}_\Gamma^{\mathrm{ctr}}$ (but not to $\mathrm{Con}_{\rho|\Gamma}^{\mathrm{ctr}}$!), becomes an equivalence relation. Still trivially, but importantly, we have the following.

Lemma 3.3 (Upward closedness of transitivity)**.** *Let* ρ *be a type and* $\Gamma \in \mathrm{Fin}_\rho$*. For any* $U, U' \in \mathrm{Con}_\rho$*, if* $U \in \mathrm{Con}_{\rho|\Gamma}^{\mathrm{ctr}}$ *and* $U' \vdash_\rho U$*, then* $U' \in \mathrm{Con}_{\rho|\Gamma}^{\mathrm{ctr}}$.

Proof. Let $U_1, U_2 \in \mathrm{Con}_\Gamma$ be such that $U_1 \asymp_\rho U' \asymp_\rho U_2$. By propagation it is $U_1 \asymp_\rho U \asymp_\rho U_2$, so $U_1 \asymp_\rho U_2$. □

It is often handy to check for extremality (that is, maximality or transitivity) on the level of tokens.

Lemma 3.4 (Extremality through tokens)**.** *Let* ρ *be an arbitrary type,* $\Gamma \in \mathrm{Fin}_\rho$*, and* $U \in \mathrm{Con}_\Gamma$.

1. *It is* $U \in \mathrm{Con}_\Gamma^{\max}$ *if and only if* $U \asymp_\rho a$ *implies* $U \vdash_\rho a$ *for* $a \in \Gamma$.

2. *It is* $U \in \mathrm{Con}_{\rho|\Gamma}^{\mathrm{ctr}}$ *if and only if* $a_1 \asymp_\rho U \asymp_\rho a_2$ *implies* $a_1 \asymp_\rho a_2$ *for* $a_1, a_2 \in \Gamma$.

Proof. For 1. From left to right, assume that U is maximal, and let $a \in \Gamma$ be such that $U \asymp_\rho a$. Then $U \cup \{a\} \vdash_\rho U$, and by the maximality of U we get $U \sim_\rho U \cup \{a\}$, which gives us $U \vdash_\rho a$. For the other way around, let $U' \in \mathrm{Con}_\Gamma$ be such that $U' \vdash_\rho U$; then $U \asymp_\rho U'$, and by the assumption we get that $U \vdash_\rho U'$, so U is indeed maximal.

For 2. From left to right, assume that U is transitive for Γ, and let $a_1, a_2 \in \Gamma$ be such that $U \asymp_\rho a_i$ for both i. Then $U \asymp_\rho \{a_i\}$, and by the transitivity of U we get $a_1 \asymp_\rho a_2$. For the other way around, let $U_1, U_2 \in \mathrm{Con}_\Gamma$ be such that $U \asymp_\rho U_i$, for both i, and $a_i \in U_i$; then $a_1 \asymp_\rho a_2$ by the assumption, so $U_1 \asymp_\rho U_2$, and U is indeed Γ-transitive. □

The lemma makes the significance of extremality in a finite set quite apparent. In particular, it is good to know that two maximals in a finite set are either equivalent or inconsistent (a fact that we can put even more bluntly like this: if U is maximal and deductively closed in Γ, then for each $a \in \Gamma$ it is either $a \in U$ or $a \not\asymp_\rho U$).

Lemma 3.5. *Let* ρ *be any type. For all* $\Gamma \in \mathrm{Fin}_\rho$ *and* $U \in \mathrm{Con}_\Gamma^{\max}$*, if* $U' \in \mathrm{Con}_\rho$ *is such that* $U' \vdash_\rho U$ *then* $U' \in \mathrm{Con}_{\Gamma \cup U'}^{\max}$.

Proof. Let $a \in \Gamma \cup U'$ be such that $a \asymp_\rho U'$; since $U' \vdash_\rho U$, it is $a \asymp_\rho U$. In case $a \notin \Gamma$ it is $a \in U'$; in case $a \in \Gamma$, it is $U \vdash_\rho a$ by Lemma 3.4.1; in both cases it follows that $U' \vdash_\rho a$, so U' is maximal in $\Gamma \cup U'$ by Lemma 3.4.1. □

Lemma 3.6 (Maximal extensions)**.** *Let* ρ *be a type and* $\Gamma \in \mathrm{Fin}_\rho$.

1. *For any* $U \in \mathrm{Con}_\Gamma$*, it is* $U \in \mathrm{Con}_\Gamma^{\mathrm{ctr}}$ *if and only if there is exactly one* $\hat{U} \in \mathrm{Con}_\Gamma^{\max}$*, up to equientailment, such that* $\hat{U} \vdash_\rho U$.

2. *For any* $U \in \mathrm{Con}_\rho$*, it is* $U \in \mathrm{Con}_{\rho|\Gamma}^{\mathrm{ctr}}$ *if and only if, whenever there exist* $U_0 \in \mathrm{Con}_\Gamma$ *with* $U \asymp_\rho U_0$*, there exists a* $\hat{U} \in \mathrm{Con}_\Gamma^{\max}$ *such that* $U \asymp_\rho U_0$ *implies* $\hat{U} \vdash_\rho U_0$ *for all* $U_0 \in \mathrm{Con}_\Gamma$.

Proof. For 1, from left to right, assume that $U \in \mathrm{Con}_\Gamma^{\mathrm{ctr}}$ and let $U_1, U_2 \in \mathrm{Con}_\Gamma^{\max}$ be such that $U_i \vdash_\rho U$ for both $i = 1, 2$. By the propagation of consistency, it is $U_1 \asymp_\rho U \asymp_\rho U_2$; by the assumption it is $U_1 \asymp_\rho U_2$; by the maximality of U_1 and U_2, it follows from Lemma 3.4.1 that $U_1 \sim_\rho U_2$.

For the other direction, assume that U is such that any two maximal neighborhoods in Γ that entail it are equivalent, and let $U_1, U_2 \in \mathrm{Con}_\Gamma$ be such that $U_1 \asymp_\rho U \asymp_\rho U_2$. Then for any two $U_1^m, U_2^m \in \mathrm{Con}_\Gamma^{\max}$, with $U_i^m \vdash_\rho U \cup U_i$, by the assumption, we must have $U_1^m \sim_\rho U_2^m$; it follows that $U_1 \asymp_\rho U_2$, by the propagation of consistency.

For 2, let $U \in \mathrm{Con}_\rho$. Assume that $U \in \mathrm{Con}_{\rho|\Gamma}^{\mathrm{ctr}}$ and $U \asymp_\rho U_i$ for some $U_i \in \mathrm{Con}_\Gamma$, where $i > 0$. Gather all these U_i in the neighborhood $U_0 := \bigcup_i U_i$; it is of course $U_0 \in \mathrm{Con}_\Gamma$. Then there is at least one maximal $\hat{U} \in \mathrm{Con}_\Gamma^{\max}$ such that $\hat{U} \vdash_\rho U_0 \vdash_\rho U_i$ for all i.

Conversely, assume that U satisfies

$$\exists_{U_0 \in \mathrm{Con}_\Gamma} U \asymp_\rho U_0 \to \exists_{\hat{U} \in \mathrm{Con}_\Gamma^{\max}} \forall_{U_0 \in \mathrm{Con}_\Gamma} (U \asymp_\rho U_0 \to \hat{U} \vdash_\rho U_0),$$

and let $U_1, U_2 \in \mathrm{Con}_\Gamma$ be such that $U_1 \asymp_\rho U \asymp_\rho U_2$. From the assumption we get a maximal $\hat{U} \in \mathrm{Con}_\Gamma^{\max}$ with $\hat{U} \vdash_\rho U_i$ for each i, so $U_1 \asymp_\rho U_2$. □

We will say *the maximal extension of U in Γ*, if $U \in \mathrm{Con}^{\mathrm{ctr}}_{\Gamma}$, for the unique (up to equivalence) maximal neighborhood entailing U; this we denote by $\hat{U}$, as in the statement of the above Lemma. But note that in the case of transitive neighborhoods *outside* Γ uniqueness is not guaranteed: an example with two maximals at type $\mathbb{D}$ provide the finite set $\Gamma = \{\mathtt{S}*, \mathtt{S0}, \mathtt{S1}\}$ and the neighborhood $U = \{\mathtt{SS}*\}$.

3.3 Upper and middle application

The notion of application fx of some higher-type term f to some input term x, both appropriately typed, is interpreted as “the information that we hold on x suffices to draw the information fx on the output, given the information that we have on f”. In section 2, in the definition of neighborhood application, we saw that when we bring this notion down to the finite level it is *entailment* that we read into “suffices”, but for our purposes it will come in handy to consider a different version of application between neighborhoods, where we replace entailment by consistency.

Let $\Theta \in \mathrm{Fin}_{\rho\to\sigma}$ and $U \in \mathrm{Con}_\rho$. The *(upper) application* $\Theta \,\dot{}\, U$ gathers all values $b \in \mathrm{Tok}_\sigma$ whose arguments U_b fall under U:

$$b \in \Theta \,\dot{}\, U := \underset{U_b \in \mathrm{Con}_\rho}{\exists} \left(\langle U_b, b \rangle \in \Theta \wedge U \vdash_\rho U_b \right).$$

Note that this trivially generalizes the neighborhood application of section 2; from now on we will always write $W \,\dot{}\, U$ instead of WU. The *middle application* $\Theta \cdot U$ is defined by

$$b \in \Theta \cdot U := \underset{U_b \in \mathrm{Con}_\rho}{\exists} \left(\langle U_b, b \rangle \in \Theta \wedge U \asymp_\rho U_b \right).$$

It follows immediately from the definition that the middle application yields at least as much information as the upper one does, namely $\Theta \,\dot{}\, U \subseteq \Theta \cdot U$.

In the case of a consistent left argument, we can make the following easy observations.

Lemma 3.7. *Let ρ, σ be arbitrary types, $W \in \mathrm{Con}_{\rho\to\sigma}$, $U \in \mathrm{Con}_\rho$ and $b \in \mathrm{Tok}_\sigma$.*

1. *It is $W \vdash_{\rho\to\sigma} \langle U, b \rangle$ if and only if $W \,\dot{}\, U \vdash_\sigma b$.*
2. *It is $W \asymp_{\rho\to\sigma} \langle U, b \rangle$ if and only if $W \cdot U \asymp_\sigma b$.*

Note that in 3.7.2 the finite set $W \cdot U$ may not be consistent, but we still did not write $W \cdot U \asymp^F_\sigma b$; here we’re just saying that every pair $\{b^W, b\}$ will be consistent, for $b^W \in W \cdot U$.

Lemma 3.8 (Application)**.** *Let ρ, σ be arbitrary types.*

1. *Application is* consistently defined*, that is, if $W \in \mathrm{Con}_{\rho\to\sigma}$ and $U \in \mathrm{Con}_\rho$, then $W \,\dot{}\, U \in \mathrm{Con}_\sigma$.*
2. *Application is monotone in the right argument, in particular, if $\Theta \in \mathrm{Fin}_{\rho\to\sigma}$ and $U, U' \in \mathrm{Con}_\rho$, with $U \vdash_\rho U'$, then $\Theta \,\dot{}\, U' \subseteq \Theta \,\dot{}\, U$.*
3. *Application is monotone in the left argument, that is, if $\Theta, \Theta' \in \mathrm{Fin}_{\rho\to\sigma}$ with $\Theta \vdash^F_{\rho\to\sigma} \Theta'$ and $U \in \mathrm{Con}_\rho$, then $\Theta \,\dot{}\, U \vdash^F_\sigma \Theta' \,\dot{}\, U$.*

Proof. For 1, let $W \in \mathrm{Con}_{\rho\to\sigma}$ and $U \in \mathrm{Con}_\rho$, and consider $b_1, b_2 \in W \cdot U$. By the definition, there must be $\langle U_1, b_1\rangle, \langle U_2, b_2\rangle \in W$, such that $U \vdash_\rho U_1 \cup U_2$; it follows that $U_1 \asymp_\rho U_2$, so the consistency of W ensures that $b_1 \asymp_\sigma b_2$.

For 2, let $\Theta \in \mathrm{Fin}_{\rho\to\sigma}$ and $U, U' \in \mathrm{Con}_\rho$, and assume that $U \vdash_\rho U'$. Consider a $b \in V$; by the definition there exists a $U_b \in L(\Theta)$ with $\langle U_b, b\rangle \in \Theta$ and $U' \vdash_\rho U_b$; the assumption immediately gives $U \vdash_\rho U_b$, so $b \in \Theta \cdot U$ as well.

For 3, let $\Theta, \Theta' \in \mathrm{Con}_{\rho\to\sigma}$ and $U \in \mathrm{Con}_\rho$, and assume that $\Theta \vdash^F_{\rho\to\sigma} \Theta'$. Consider a $V' \in \mathrm{Con}_{\Theta' \cdot U}$; for each $b' \in V'$ there is a $U_{b'} \in \mathrm{Con}_\rho$ such that $\langle U_{b'}, b'\rangle \in \Theta'$ and $U \vdash_\rho U_{b'}$; the set $W' := \{\langle U_{b'}, b'\rangle \in \Theta \mid b' \in V' \wedge U \vdash_\rho U_{b'}\}$ is consistent in Θ'. By the assumption there exists some $W \in \mathrm{Con}_\Theta$ such that $W \vdash_{\rho\to\sigma} W'$. Since for each $\langle U_{b'}, b'\rangle \in W'$ it is $W \cdot U_{b'} \vdash_\sigma b'$, by 2 we get $W \cdot U \vdash_\sigma W \cdot U_{b'}$, hence $W \cdot U \vdash_\sigma b'$, that is, $W \cdot U \vdash_\sigma V'$ and since $W \cdot U \in \mathrm{Con}_{\Theta \cdot U}$, we're done. □

The following gives us conservative extensions of a neighborhood by way of extending its set of arguments.

Lemma 3.9. *Let $W \in \mathrm{Con}_{\rho\to\sigma}$ and $\Gamma \in \mathrm{Fin}_\rho$ such that $L(W) \subseteq \Gamma$. Then*

$$W \sim_{\rho\to\sigma} \bigcup_{U \in \mathrm{Con}_\Gamma} \langle U, W \cdot U\rangle.$$

Proof. From left to right, let $U \in \mathrm{Con}_\Gamma$ and $b \in W \cdot U$. There exists a $U_b \in \mathrm{Con}_{L(\Theta)}$ with $\langle U_b, b\rangle \in W$ and $U \vdash_\rho U_b$. Then $\langle U_b, b\rangle \vdash_{\rho\to\sigma} \langle U, b\rangle$. The other way around is obvious, since $W \subseteq \bigcup_{U \in \mathrm{Con}_\Gamma} \langle U, W \cdot U\rangle$.[4] □

Turning our attention to middle application, the first thing we want to know is how it fares compared to Lemma 3.8.

Lemma 3.10 (Middle application)**.** *Let ρ, σ be arbitrary types.*

1. *Middle application is consistently defined* for transitive right arguments*, that is, if $W \in \mathrm{Con}_{\rho\to\sigma}$ and $U \in \mathrm{Con}^{\mathrm{ctr}}_{\rho|L(W)}$, then $W \cdot U \in \mathrm{Con}_\sigma$.*
2. *Middle application is* antimonotone *in the right argument, in particular, if $\Theta \in \mathrm{Fin}_{\rho\to\sigma}$ and $U, U' \in \mathrm{Con}_\rho$ with $U \vdash_\rho U'$, then $\Theta \cdot U \subseteq \Theta \cdot U'$.*
3. *Middle application between neighborhoods is monotone in the left argument* for transitive right arguments*, that is, if $W, W' \in \mathrm{Con}_{\rho\to\sigma}$ are such that $W \vdash_{\rho\to\sigma} W'$ and $U \in \mathrm{Con}^{\mathrm{ctr}}_{\rho|L(W)\cup L(W')}$, then $W \cdot U \vdash_\sigma W' \cdot U$.*

Proof. To show 1, let $U \in \mathrm{Con}^{\mathrm{ctr}}_{\rho|L(W)}$ and $b_1, b_2 \in W \cdot U$. Then there exist U_1, U_2 with $\langle U_i, b_i\rangle \in W$ and $U_i \asymp_\rho U$. The transitivity of U implies $U_1 \asymp_\rho U_2$, and the consistency of W ensures that $b_1 \asymp_\sigma b_2$.

For 2, assume U and U' such that $U \vdash_\rho U'$, and let $b \in \Theta \cdot U$. There is some U_b such that $\langle U_b, b\rangle \in \Theta$ and $U_b \asymp_\rho U$. By propagation we get $U_b \asymp_\rho U'$, so $b \in \Theta \cdot U'$.

For 3. By 1, the assumption that $U \in \mathrm{Con}^{\mathrm{ctr}}_{\rho|L(W)\cup L(W')}$ ensures that the result of both middle applications is a neighborhood (in general, if $\Gamma \subseteq \Gamma'$ then $\mathrm{Con}^{\mathrm{ctr}}_{\rho|\Gamma'} \subseteq \mathrm{Con}^{\mathrm{ctr}}_{\rho|\Gamma}$). Let $b' \in W' \cdot U$. By the definition of middle application, there exists a $\langle U', b'\rangle \in W'$, such that $U' \asymp_\rho U$. Since $W \vdash_{\rho\to\sigma} W'$, there is a subneighborhood $\{\langle U_i, b_i\rangle \mid i = 1, \ldots, m\} \subseteq W$, such that for all $i = 1, \ldots, m$ it is $U' \vdash_\rho U_i$ and $\{b_i \mid i = 1, \ldots, m\} \vdash_\sigma b'$; by propagation it follows that $U \asymp_\rho U_i$ for all i. This means that $b_i \in W \cdot U$ for all i, so $W \cdot U \vdash_\sigma b'$, and we're done. □

[4] The proof in fact shows that the equientailment is linear (U entails b *linearly* when $\{a\} \vdash b$ for some $a \in U$).

Lemma 3.11. *Let $\Theta \in \mathrm{Fin}_{\rho\to\sigma}$.*

1. *For all $U, U' \in \mathrm{Con}^{\mathrm{ctr}}_{L(\Theta)}$, if $U \asymp_\rho U'$ then $\Theta \cdot U = \Theta \cdot U'$.*
2. *For all $U \in \mathrm{Con}^{\mathrm{ctr}}_{L(\Theta)}$, it is $\Theta \cdot U = \Theta \,\dot{}\, \hat{U}$.*
3. *For all $U \in \mathrm{Con}^{\max}_{L(\Theta)}$ and $U' \in \mathrm{Con}_\rho$, if $U \asymp_\rho U'$ then $\Theta \cdot U \subseteq \Theta \cdot U'$.*

Proof. For the first statement, assume that $U \asymp_\rho U'$ and let $b \in R(\Theta)$. It is $b \in \Theta \cdot U$ if and only if there is some U_b with $\langle U_b, b\rangle \in \Theta$ and $U \asymp_\rho U_b$. Since U is transitive in $L(\Theta)$, we get $U' \asymp_\rho U_b$ by the assumption, so $b \in \Theta \cdot U'$. The converse is similar.

For the second statement, let $U \in \mathrm{Con}^{\mathrm{ctr}}_{L(\Theta)}$. By the definition of middle application, if $b \in \Theta \cdot U$, then there is some U_b with $\langle U_b, b\rangle \in \Theta$, such that $U_b \asymp_\rho U$; since U is transitive, by the maximality of its maximal extension it follows that $\hat{U} \vdash_\rho U_b$, so the definition of application gives us $b \in \Theta \,\dot{}\, \hat{U}$. For the other way around, if $b \in \Theta \,\dot{}\, \hat{U}$, then there is a U_b with $\langle U_b, b\rangle \in \Theta$, such that $\hat{U} \vdash_\rho U_b$; then $U \asymp_\rho U_b$ by propagation, so $b \in \Theta \cdot U$, by the definition of middle application.

For the third statement, let U be maximal in $L(\Theta)$ and U' some neighborhood with $U \asymp_\rho U'$. For every $b \in \Theta \cdot U$, by the definition of middle application, there is some U_b with $\langle U_b, b\rangle \in \Theta$, such that $U \asymp_\rho U_b$, which by maximality means that $U \vdash_\rho U_b$; by propagation we get $U' \asymp_\rho U_b$, so $b \in \Theta \cdot U'$, and we're done. ☐

We close the section with a hint on how extremality evolves over types.

Lemma 3.12. *Let $\Theta \in \mathrm{Fin}_{\rho\to\sigma}$ and $W \in \mathrm{Con}_\Theta$. It is $W \in \mathrm{Con}^{\mathrm{ctr}}_\Theta$ if one of the following holds.*

1. *For all $U \in \mathrm{Con}^{\mathrm{ctr}}_{L(\Theta)}$ it is $W \cdot U \in \mathrm{Con}^{\mathrm{ctr}}_{\Theta \cdot U}$.*
2. *For all $U \in \mathrm{Con}^{\max}_{L(\Theta)}$ it is $W \cdot U \in \mathrm{Con}^{\mathrm{ctr}}_{\Theta \cdot U}$.*

Proof. For the first criterion, let $\langle U_i, b_i\rangle \in \Theta$ be such that $\langle U_i, b_i\rangle \asymp_{\rho\to\sigma} W$ for $i = 1, 2$, and assume that $U_1 \asymp_\rho U_2$. Consider a $U \in \mathrm{Con}^{\mathrm{ctr}}_{L(\Theta)}$ with $U \asymp_\rho U_1 \cup U_2$; then $b_i \in \Theta \cdot U$ for each i. From $\langle U_i, b_i\rangle \asymp_{\rho\to\sigma} W$, by Lemma 3.10.1, we get $b_i \asymp_\sigma W \cdot U$ for both i, so by the assumption we get $b_1 \asymp_\sigma b_2$.

To get the second criterion, it suffices to show that

$$\underset{U \in \mathrm{Con}^{\max}_{L(\Theta)}}{\forall} W \cdot U \in \mathrm{Con}^{\mathrm{ctr}}_{\Theta \cdot U} \to \underset{U \in \mathrm{Con}^{\mathrm{ctr}}_{L(\Theta)}}{\forall} W \cdot U \in \mathrm{Con}^{\mathrm{ctr}}_{\Theta \cdot U}.$$

Assume that W is such that $W \cdot U \in \mathrm{Con}^{\mathrm{ctr}}_{\Theta \cdot U}$, for all $U \in \mathrm{Con}^{\max}_{L(\Theta)}$, and let $U \in \mathrm{Con}^{\mathrm{ctr}}_{L(\Theta)}$. Consider the maximal extension $\hat{U}$ of U. On the one hand it is $\hat{U} \in \mathrm{Con}^{\max}_{L(\Theta)}$, so $W \cdot \hat{U} \in \mathrm{Con}^{\mathrm{ctr}}_{\Theta \cdot \hat{U}}$ by the assumption. On the other hand it is $\hat{U} \in \mathrm{Con}^{\mathrm{ctr}}_{L(\Theta)}$ with $\hat{U} \asymp_\rho U$, so $\Theta \cdot \hat{U} = \Theta \cdot U$ and $W \cdot \hat{U} = W \cdot U$, by Lemma 3.11.1. It follows that $W \cdot U \in \mathrm{Con}^{\mathrm{ctr}}_{\Theta \cdot U}$, so we can apply the previous criterion and we're done. ☐

4 Totality of neighborhoods

In this section we take the first two steps of the strategy that we outlined in the introduction. A total object of type $\rho \to \sigma$ is represented by a possibly infinite token set which (a) is an ideal, that is, consistent and deductively closed, (b) admits all totals of type

ρ as arguments—a property we think of as "omniception", for lack of a less pompous but as grammatically smooth synonym for *admission* or *acceptance of all*—, and (c) responds to every total argument with a total value at type σ. To bring the notion down to the finite level we dispose of half of the demand (a), namely, that the set of tokens be deductively closed, and we reinterpret "admittance" and "response" in (b) and (c) in terms of consistency rather than entailment. The first move, which is clearly dictated by the demand of finiteness, in some sense causes the reaction of the second move: what we lose by denying deductive closure we have to regain with the wider and more tolerant scope of consistency.

4.1 Finite density

At type ρ, call *side extension* of a neighborhood U any neighborhood U' which is consistent to U. We give a name to this rather mundane notion just to point to its intended use: trivially, if U' is a side extension of U, then $U' \cup U$ is an extension of U, and this is exactly how we will work towards finding total extensions of neighborhoods.

Lemma 4.1. *Let ρ and σ be types. For every $W \in \mathrm{Con}_{\rho\to\sigma}$, the finite set $\bigcup_{U\in\mathrm{Con}^{\mathrm{ctr}}_{L(W)}}\langle U,(W\cdot U)'\rangle$, where V' denotes a fixed side extension of $V \in \mathrm{Con}_\sigma$, is a side extension of W.*

Proof. To show the consistency of the finite set, let $\langle U_i, b_i\rangle$ be such that $U_i \in \mathrm{Con}^{\mathrm{ctr}}_{L(W)}$ and $b_i \in (W\cdot U_i)'$, for $i = 1,2$. If $U_1 \asymp_\rho U_2$, then $W\cdot U_1 = W\cdot U_2$ by Lemma 3.11.1, hence $(W\cdot U_1)' = (W\cdot U_2)'$, and $b_1 \asymp_\sigma b_2$.

To show the side extension, let $\langle U,b\rangle \in W$ and $\langle U',b'\rangle$ be such that $U' \in \mathrm{Con}^{\mathrm{ctr}}_{L(W)}$ and $b' \in (W\cdot U)'$. If $U \asymp_\rho U'$ then by the definition of middle application it is $b \in W\cdot U$. But $W\cdot U \asymp_\sigma (W\cdot U)'$ by assumption, so $b \asymp_\sigma b'$. $\square$

As we mentioned in section 1, a *total token* at a base type ι is a token $a \in \mathrm{Tok}_\iota$ which consists exclusively of proper constructors; write $a \in \mathrm{Tok}^g_\iota$. It is $Ca_1\cdots a_r \in \mathrm{Tok}^g_\iota$ if and only if C is a proper constructor of arity r and $a_i \in \mathrm{Tok}^g_\iota$ for all $i = 1,\dots,r$. So $\mathtt{SB{*}0} \notin \mathrm{Tok}^g_{\mathbb{D}}$ but $\mathtt{SB10} \in \mathrm{Tok}^g_{\mathbb{D}}$. Define *total neighborhoods* inductively over types:

$$U \in \mathrm{Con}^g_\iota := \exists_{a\in\mathrm{Tok}^g_\iota} U \vdash_\iota a,$$

$$W \in \mathrm{Con}^g_{\rho\to\sigma} := \forall_{P\in\mathrm{Con}^g_\rho} W\cdot P \in \mathrm{Con}^g_\sigma.$$

On the other hand, define *weakly omniceptive finite sets* explicitly by

$$\Gamma \in \mathrm{Fin}^{wo}_\rho := \forall_{P\in\mathrm{Con}^g_\rho} \exists_{U^P\in\mathrm{Con}^{\mathrm{ctr}}_\Gamma} P \asymp_\rho U^P.$$

At a type $\rho \to \sigma$ call Θ a *(strongly) omniceptive finite set*, and write $\Theta \in \mathrm{Fin}^o_{\rho\to\sigma}$, if

$$\Theta \in \mathrm{Fin}^{wo}_{\rho\to\sigma} \wedge \forall_{U\in\mathrm{Con}^{\max}_{L(\Theta)}} (U \in \mathrm{Con}^g_\rho \wedge \Theta\cdot U \in \mathrm{Fin}^o_\sigma \wedge \langle U,\Theta\cdot U\rangle \subseteq \Theta),$$

which intuitively says that, beyond weak omniception, Θ must meet certain requirements of finite totality, preservation of omniception, and closure under middle application for each of its left maximals. By convention, we set $\mathrm{Fin}^o_\iota := \mathrm{Fin}^{wo}_\iota$ for arbitrary base types.

Call a type ρ *finitely dense* if every neighborhood U at ρ has a total side extension, and *finitely omniceptive* if every finite set has an omniceptive extension. Moreover, call it *finitely total-transitive* if every total neighborhood U is *transitive (in ρ)*. The latter just means that $U_1 \asymp_\rho U \asymp_\rho U_2$ implies $U_1 \asymp_\rho U_2$ for all $U_1, U_2 \in \mathrm{Con}_\rho$, and we write $U \in \mathrm{Con}_\rho^{\mathrm{ctr}}$. Here's a lemma to set the intuition straight.

Lemma 4.2 (Compactness of transitivity)**.** *Let ρ be a type. A neighborhood is transitive in ρ if and only if it is transitive for every finite set of ρ.*

Proof. For the less trivial direction, let $U \in \mathrm{Con}_\rho$ be such that $U \in \mathrm{Con}_{\rho|\Gamma}^{\mathrm{ctr}}$ for every $\Gamma \in \mathrm{Fin}_\rho$, and let $U_1, U_2 \in \mathrm{Con}_\rho$ be such that $U_1 \asymp_\rho U \asymp_\rho U_2$. Set $\Gamma := U_1 \cup U \cup U_2$; then it is $U_1 \asymp_\rho U_2$ by Γ-transitivity. □

To start off the main argument we need two elementary definitions. The *size* $\|a\|$ of a base-type token $a \in \mathrm{Tok}_\iota$ counts the proper constructors of the token: $\|*\| = 0$ and $\|Ca_1 \cdots a_r\| = 1 + \|a_1\| + \cdots + \|a_r\|$. The *supremum* or *eigentoken* $\mathsf{sup}(U)$ of a base-type neighborhood $U \in \mathrm{Con}_\iota$ is defined by $\mathsf{sup}(\varnothing_\iota) = *_\iota$ and $\mathsf{sup}(\{a_1, \ldots, a_m\}) = \mathsf{sup}^t(\cdots \mathsf{sup}^t(a_1, a_2) \cdots, a_m)$, where $\mathsf{sup}^t(a, *) = a$ and $\mathsf{sup}^t(Ca_1 \cdots a_r, Cb_1 \cdots b_r) = C\mathsf{sup}^t(a_1, b_1) \cdots \mathsf{sup}^t(a_r, b_r)$.

Proposition 4.3. *Every base type is finitely total-transitive, finitely dense, and finitely omniceptive.*

Proof. Let ι be any base type with a distinguished nullary constructor 0. For the transitivity of total neighborhoods, let $P \in \mathrm{Con}_\iota^g$ and $U_1, U_2 \in \mathrm{Con}_\iota$ be such that $U_i \asymp_\iota P$ for each i. Then $P \vdash_\iota U_i$, for both $i = 1, 2$, since, as is easy to see, total tokens are maximal at base types, so $U_1 \asymp_\iota U_2$.

We turn to finite density by firstly considering tokens: the trivial token $*$ is consistent to 0, and if $a_1^g, \ldots, a_r^g$ are total tokens consistent to $a_1, \ldots, a_r$ respectively, then $Ca_1^g \cdots a_r^g$ is a total token consistent to $Ca_1 \cdots a_r$, for an r-ary constructor C. Then if $U \in \mathrm{Con}_\iota$, the neighborhood $U^g := \overline{\mathsf{sup}(U)^g}$ is obviously a total neighborhood consistent to (above, even) U.

Now for the finite omniception. If Γ is trivial (that is, if it carries no proper information), then set $\Gamma^o := \{*\}$. If not, let $m := \max\{\|\mathsf{sup}(U)\| \mid U \in \mathrm{Con}_\Gamma\}$, and set $\Gamma^o := \{a \in \mathrm{Tok}_\iota \mid \|a\| \leqslant m\}$. Again, it is easy to convince ourselves that this is a sufficient choice by construction. □

Proposition 4.4 (Finite total-transitivity)**.** *Let ρ and σ be finitely total-transitive types. If ρ is finitely dense then $\rho \to \sigma$ is finitely total-transitive.*

Proof. Let $T \in \mathrm{Con}_{\rho\to\sigma}^g$ and $W_1, W_2 \in \mathrm{Con}_{\rho\to\sigma}$, with $W_1 \asymp_{\rho\to\sigma} T \asymp_{\rho\to\sigma} W_2$. Consider pairs $\langle U_i, b_i \rangle \in W_i$, $i = 1, 2$, and assume that $U_1 \asymp_\rho U_2$. By the finite density at ρ, there exists a $P \in \mathrm{Con}_\rho^g$, such that $P \asymp_\rho U_1 \cup U_2$. By the assumptions at ρ and Lemma 3.10.1 we get $b_1 \asymp_\sigma T \cdot P \asymp_\sigma b_2$. But $T \cdot P$ is total, so the assumption at σ gives $b_1 \asymp_\sigma b_2$. □

Proposition 4.5 (Finite density)**.** *Let ρ and σ be types. If ρ is finitely omniceptive and σ finitely dense and finitely total-transitive, then $\rho \to \sigma$ is finitely dense.*

Proof. Let $W \in \mathrm{Con}_{\rho\to\sigma}$ be any neighborhood. By finite omniception at ρ we get a $\Gamma \in \mathrm{Fin}_\rho^o$ with $L(W) \subseteq \Gamma$. Consider the neighborhood $W^o := \bigcup_{U \in \mathrm{Con}_\Gamma} \langle U, W \cdot U \rangle$; by Lemma 3.9 it is $W \sim_{\rho\to\sigma} W^o$. Now set

$$W^g := \bigcup_{U \in \mathrm{Con}_{L(W^o)}^{\mathrm{ctr}}} \langle U, (W^o \cdot U)^g \rangle,$$

with the help of density at σ; note that $L(W^g) = L(W^o) = \Gamma$. This is a side extension of W^o (therefore of W as well) by Lemma 4.1.

To show that it is total, let $P \in \mathrm{Con}^g_\rho$. Since $L(W^g)$ is omniceptive (in fact, that it is weakly omniceptive is enough), there is some $U^P \in \mathrm{Con}^{\mathrm{ctr}}_\Gamma$ such that $P \asymp_\rho U^P$. It is $\langle U^P, (W^o \cdot U^P)^g \rangle \subseteq W^g$ by construction, and $W^g \cdot P = (W^o \cdot U^P)^g$, since, by transitivity of total neighborhoods at σ, the value $W^o \cdot P$ is independent from the choice of U^P. $\square$

Proposition 4.6 (Finite omniception)**.** *Let ρ and σ be finitely total-transitive types. If ρ is finitely dense and σ finitely omniceptive, then $\rho \to \sigma$ is finitely omniceptive.*

Proof. Let $\Theta \in \mathrm{Fin}_{\rho\to\sigma}$ be any finite set. Extend it as follows:

$$\Theta^o := \Theta \cup \bigcup_{U \in \mathrm{Con}^{\max}_{L(\Theta)}} \langle U^g, (\Theta \cdot U)^o \rangle,$$

with the use of finite density at ρ and finite omniception at σ.

If we show that this is weakly omniceptive, then it will be omniceptive immediately by construction (based on Lemma 3.5). Let $T \in \mathrm{Con}^g_{\rho\to\sigma}$. For every $U \in \mathrm{Con}^{\max}_{L(\Theta)}$ it is $T \cdot U^g \in \mathrm{Con}^g_\sigma$, and since $(\Theta \cdot U)^o$ is omniceptive, there will be some $V^{T\cdot U^g} \in \mathrm{Con}^{\mathrm{ctr}}_{(\Theta\cdot U)^o}$, such that $T \cdot U^g \asymp_\sigma V^{T\cdot U^g}$ $(\star)$. Fix these side extensions and set

$$W^T := \bigcup_{U \in \mathrm{Con}^{\max}_{L(\Theta^o)}} \langle U^g, V^{T\cdot U^g} \rangle.$$

It is $W^T \subseteq \Theta^o$ by construction. Moreover, it is $T \asymp_{\rho\to\sigma} W^T$: let $\langle U, b \rangle \in T$ and $\langle U', b' \rangle \in W^T$ be such that $U \asymp_\rho U'$; it is $b \in T \cdot U'$ and $b' \in V^{T\cdot U'}$, so $b \asymp_\sigma b'$ by $(\star)$. Since T is total, W^T is a neighborhood by transitivity of total neighborhoods, which we get for $\rho \to \sigma$ by Proposition 4.4. Finally, it is transitive in Θ^o by Lemma 3.12.2, since for every $U \in \mathrm{Con}^{\max}_{L(\Theta)}$ it is by construction $W^T \cdot U = V^{T\cdot U}$, which is transitive in $\Theta \cdot U$ by omniception. $\square$

Theorem 4.7. *Every type is finitely omniceptive, finitely total-transitive, and, in particular, finitely dense.*

Proof. We get this by mutual induction over types from Propositions 4.3, 4.4, 4.5, and 4.6. $\square$

4.2 Totality of transitive neighborhoods

There is plenty of evidence to suggest that total neighborhoods at ρ are to Con_ρ what transitive neighborhoods in Γ are to Con_Γ. For one, Theorem 4.7 shows that total neighborhoods are transitive. Furthermore, an immediate corollary of total transitivity is that consistency, restricted to the total neighborhoods, becomes an equivalence relation, that is,

$$\forall_{P_1, P_2, P_3 \in \mathrm{Con}^g_\rho} \left(P_1 \asymp_\rho P_2 \asymp_\rho P_3 \to P_1 \asymp_\rho P_3 \right) .$$

Here are further examples of using total transitivity, which include some more evidence to this effect.

Lemma 4.8. *Let ρ and σ be types. Let $\Theta \in \mathrm{Fin}_{\rho\to\sigma}$, $P, P' \in \mathrm{Con}^g_\rho$, and $U, U' \in \mathrm{Con}_\rho$.*

1. *For every $U^P \in \mathrm{Con}_{L(\Theta)}$ with $P \asymp_\rho U^P$, it is $\Theta \cdot P \subseteq \Theta \cdot U^P$. Moreover, it is $\Theta \cdot P = \Theta \cdot U^P$ whenever $U^P \in \mathrm{Con}^{\max}_{L(\Theta)}$.*

2. *If $P \asymp_\rho P'$ then $\Theta \cdot P = \Theta \cdot P'$.*

3. *If $U \in \mathrm{Con}^g_\rho$ and $U' \vdash_\rho U$, then $U' \in \mathrm{Con}^g_\rho$.*

4. *If $T \in \mathrm{Con}^g_{\rho\to\sigma}$ and $U \in \mathrm{Con}^{\mathrm{ctr}}_{L(T)}$ then $T \cdot U \in \mathrm{Con}^g_\sigma$.*

Proof. For 1, let $b \in \Theta \cdot P$. Then, by the definition of middle application, there is some U with $\langle U, b\rangle \in \Theta$, such that $U \asymp_\rho P$. From $U \asymp_\rho P \asymp_\rho U^P$ we get $U \asymp_\rho U^P$ by Theorem 4.7, so the definition of middle application yields that $b \in \Theta \cdot U^P$. Moreover, if U^P is actually maximal in $L(\Theta)$, then by Lemma 3.11.3 we immediately get $\Theta \cdot U^P \subseteq \Theta \cdot P$.

For 2, assume that $P \asymp_\rho P'$ and let $b \in \Theta \cdot P$. By the definition of middle application, there is some U with $\langle U, b\rangle \in \Theta$, such that $U \asymp_\rho P$. By Theorem 4.7 and the assumption it is $U \asymp_\rho P'$, so $b \in \Theta \cdot P'$.

For 3. At a base type ι if $U \in \mathrm{Con}^g_\iota$, then there exists a total token a such that $U \vdash_\iota a$. The transitivity of entailment yields what we need. At a higher type $\rho \to \sigma$, let $W \in \mathrm{Con}^g_{\rho\to\sigma}$ and $W' \vdash_{\rho\to\sigma} W$. Let further $P \in \mathrm{Con}^g_\rho$. By Theorem 4.7, P is transitive for $L(W)$, so by the left monotonicity of middle application on transitive arguments (Lemma 3.10.3), we have $W' \cdot P \vdash_\sigma W \cdot P$, and by the totality of W we get $W \cdot P \in \mathrm{Con}^g_\sigma$, so the induction hypothesis at σ finishes the job.

For 4. By Theorem 4.7 there exists some $P^U \in \mathrm{Con}^g_\rho$ with $P^U \asymp_\rho U$. By 1 it is $T \cdot P^U \subseteq T \cdot U$, where $T \cdot U$ is consistent by Lemma 3.10.1. It follows by 3 that $T \cdot U \in \mathrm{Con}^g_\sigma$. ☐

Note in particular that Lemma 4.8.3 is analogous to Lemma 3.3 (both of them actually anticipate Lemma 5.2).

We now show that the correspondence between transitivity and finite totality is complete.

Theorem 4.9 (Explicit finite totality)**.** *At every type, a neighborhood is total if and only if it is transitive.*

Proof by induction over types. The rightward direction we have of course from Theorem 4.7. For the other direction, we have to show that, at each type, every transitive neighborhood must be total.

At a base type ι, assume that $U \in \mathrm{Con}^{\mathrm{ctr}}_\iota$. Obviously, it is $U \not\asymp_\iota \{*\}$, so there will be a constructor C and tokens $a_1, \ldots, a_r \in \mathrm{Tok}_\iota$ such that $U \sim_\iota Ca_1 \cdots a_r$. By Lemma 3.4.2, since U is transitive, for any two tokens $b_1, b_2 \in \mathrm{Tok}_\iota$ we will have $b_1 \asymp_\iota U \asymp_\iota b_2$ imply $b_1 \asymp_\iota b_2$. Then for $i = 1, \ldots, r$ it is

$$\begin{aligned} b_{1i} \asymp_\iota a_i \asymp_\iota b_{2i} &\Rightarrow Ca_1 \cdots b_{1i} \cdots a_r \asymp_\iota U \asymp_\iota Ca_1 \cdots b_{2i} \cdots a_r \\ &\overset{\mathrm{ctr}}{\Rightarrow} Ca_1 \cdots b_{1i} \cdots a_r \asymp_\iota Ca_1 \cdots b_{2i} \cdots a_r \\ &\Rightarrow b_{1i} \asymp_\iota b_{2i}, \end{aligned}$$

which by induction hypothesis yields $a_i \in \mathrm{Tok}^g_\iota$. It follows that $Ca_1 \cdots a_r$ itself is a total token, so U is a total neighborhood.

At type $\rho \to \sigma$, assume that $W \in \mathrm{Con}^{\mathrm{ctr}}_{\rho\to\sigma}$, and let $P \in \mathrm{Con}^{g}_{\rho}$. For any $b_1, b_2 \in \mathrm{Tok}_\sigma$ it is

$$\begin{aligned} b_1 \asymp_\sigma W \cdot P \asymp_\sigma b_2 &\Leftrightarrow \langle P, b_1\rangle \asymp_{\rho\to\sigma} W \asymp_{\rho\to\sigma} \langle P, b_2\rangle \\ &\overset{\mathrm{ctr}}{\Rightarrow} \langle P, b_1\rangle \asymp_{\rho\to\sigma} \langle P, b_2\rangle \\ &\Rightarrow b_1 \asymp_\sigma b_2, \end{aligned}$$

which means that $W \cdot P$ is transitive in σ, so by the induction hypothesis at σ we get $W \cdot P \in \mathrm{Con}^{g}_{\sigma}$, and by the definition of finite totality it is $W \in \mathrm{Con}^{g}_{\rho\to\sigma}$, as we wanted. $\square$

The theorem indicates that our notion of finite totality is a robust one, and indeed, we will see in the next section that it is very well suited to our purposes. Interestingly, we will also see that its equivalence to transitivity is peculiar to the finitary level: in Proposition 5.12 the respective correspondence for ideals is shown to be tilted.

5 Elevating totality to ideals

The last step in our strategy is to find a canonical extension of a total neighborhood to a total ideal. The natural candidate would be the deductive closure of a neighborhood, but again, closure under entailment proves to be too strict for our purposes. Instead, based on the transitivity of total neighborhoods, we will use closure under consistency.

5.1 Density

The notion of continuity that we employ in our setting implies that if we're given an estimate V on a value $f(x)$ then we can find an adequate estimate U_V on the argument x of f; let us highlight this elementary fact as we will need it later on.

Lemma 5.1 (Finite support)**.** *Let $f : \rho \to \sigma$ and $x : \rho$. For every $V \in \mathrm{Con}_\sigma$ with $V \subseteq f(x)$ there exists a $U_V \in \mathrm{Con}_\rho$ such that $\langle U_V, V\rangle \subseteq f$.*

Proof. From (FS) it follows directly that if $b \in f(x)$ then there exists a $U_b \subseteq x$ such that $\langle U_b, b\rangle \in f$ due to the deductive closure of f. Assuming then that V is such that $V \subseteq f(x)$, it is $\langle U_V, V\rangle \subseteq f$ for $U_V := \bigcup_{b\in V} U_b$, again by the deductive closure of f. $\square$

An ideal $x : \rho$ is a *total ideal*, for which we write $G_\rho(x)$ or $x \in G_\rho$, if it conforms to the following inductive clauses.

$$G_\iota(x) := \underset{U\in\mathrm{Con}^{g}_{\iota}}{\exists} U \subseteq x \,,$$

$$G_{\rho\to\sigma}(f) := \underset{x:\rho}{\forall} (G_\rho(x) \to G_\sigma(fx)) \,.$$

Totality of ideals is upwards closed.

Lemma 5.2 (Extension lemma)**.** *At type ρ, if $x, y : \rho$ are such that $G_\rho(x)$ and $x \subseteq y$ then $G_\rho(y)$.*

Proof. At a base type ι, let $G_\iota(x)$ and $y : \iota$ be two ideals with $x \subseteq y$. Then there is a total token $a \in \mathrm{Tok}^{g}_{\iota}$, such that $a \in x$, so also $a \in y$. At a higher type $\rho \to \sigma$, let $G_{\rho\to\sigma}(f)$, $g : \rho \to \sigma$, and assume that $f \subseteq g$. We want to show that g is also total, so consider an arbitrary x with $G_\rho(x)$. By the totality of f we have that $G_\sigma(fx)$, and since it is straightforward to see that $fx \subseteq gx$, we get $G_\sigma(gx)$ by the induction hypothesis at σ. $\square$

The main argument starts with the following obvious observation.

Lemma 5.3. *At every type, if a neighborhood is transitive then its consistency closure is an ideal (and the converse holds as well).*

Proof. Let ρ be a type and $U \in \mathrm{Con}_\rho^{\mathrm{ctr}}$. By transitivity, every two tokens in the consistency closure of U will be consistent, and the consistency closure is already deductively closed: $U' \subseteq \widetilde{U}$ means $U \asymp_\rho U'$ by definition, so if $U' \vdash_\rho a$, then propagation yields $U \asymp_\rho a$, hence $a \in \widetilde{U}$ as well. The converse is also direct to show. $\square$

By Theorem 4.9, it is an immediate consequence of the previous that the consistency closure of every total neighborhood is an ideal, so it suffices to show that, for a given $P \in \mathrm{Con}_\rho^g$, it must be $G_\rho(\widetilde{P})$. Consider the following statements for an arbitrary type ρ.

$$\forall_{\Gamma \in \mathrm{Fin}_\rho^o} \forall_{x \in G_\rho} \exists_{U^x \in \mathrm{Con}_\Gamma^{\mathrm{ctr}}} U^x \asymp_\rho x, \tag{O}$$

$$\forall_{U, U' \in \mathrm{Con}_\rho} \forall_{x \in G_\rho} (U' \nasymp_\rho U \asymp_\rho x \to \exists_{U_0 \in \mathrm{Con}_\rho} (U \vdash_\rho U_0 \subseteq x \wedge U' \nasymp_\rho U_0)), \tag{W}$$

$$\forall_{P \in \mathrm{Con}_\rho^g} \widetilde{P} \in G_\rho, \tag{C}$$

$$\forall_{U \in \mathrm{Con}_\rho} \exists_{x \in G_\rho} U \subseteq x. \tag{D}$$

The first one is an expression of infinitary *omniception*, as it states that an omniceptive finite set accepts each total *ideal* by being consistent to it with one of its transitive neighborhoods. The second expresses *inconsistency preserving witnessing* of the consistency between a total ideal and a neighborhood; the claimed witness is stronger than the neighborhood itself, since it lies below both the total ideal and the neighborhood, and in a sense to be made clearer after Lemma 5.8 below, it provides the missing feature from omniception that we need to achieve totality on the level of ideals. The third one is the crux of our strategy, as it says that the *consistency closure* of a total neighborhood is a total ideal, and the fourth one, of course, is *density*.

Proposition 5.4 (Conditional density)**.** *Let ρ be a type. If* (C) *holds in ρ then also* (D) *holds in ρ.*

Proof. Let U be any neighborhood at type ρ. By Theorem 4.7 there exists a total neighborhood P^U such that $U \asymp_\rho P^U$. Then $U \subseteq \widetilde{P^U}$ by definition, whereas $\widetilde{P^U} \in G_\rho$ by (C). We set $x := \widetilde{P^U}$ and we are done. $\square$

Lemma 5.5. *Every base type satisfies* (O), (W), (C), *and* (D).

Proof. Let ι be some base type. To show (O), consider an omniceptive finite set Γ and a total ideal x. By the totality of x there's some $P \in \mathrm{Con}_\iota^g$ such that $P \subseteq x$, and by the omniception of Γ there is some $U^P \in \mathrm{Con}_\Gamma^{\mathrm{ctr}}$ such that $U^P \asymp_\iota P$. Set $U^x := U^P$. Then for every $U \subseteq x$ it is $U^x \asymp_\iota P \asymp_\iota U$, which implies $U^x \asymp_\iota U$ by the total transitivity of ι (Proposition 4.3), so $U^x \asymp_\iota x$.[5]

To show (W), let U and U' be neighborhoods and x be a total ideal, such that $U' \nasymp_\iota U \asymp_\iota x$. By the totality of x there exists a total neighborhood P such that $P \subseteq x$.

[5] Notice again that we only needed *weak* omniception from Γ. Furthermore, observe that in the flat setting this argument would fail due to the requirement of finiteness of Γ.

It is of course $P \asymp_\iota U$, which, since total tokens are maximal at base types, implies that $P \vdash_\iota U$. This in turn implies that $U \subseteq x$ by the deductive closure of x, so we may set $U_0 := U$, which trivially meets the stated requirements.

To show (C), let P be some total neighborhood. Then there's some total token $a \in \mathrm{Tok}_\iota$ with $P \vdash a$; a fortiori it is $P \asymp_\iota a$, so $a \in \widetilde{P}$ by the definition of consistency closure. Since by Lemma 5.3 the set $\widetilde{P}$ is an ideal, we conclude that it is in fact total.

Finally, that every base type is dense we get from Proposition 5.4, since (C) already holds. □

Proposition 5.6 (Omniception)**.** *Let ρ and σ be types. If* (C) *holds in ρ and* (O) *holds in σ then* (O) *holds in $\rho \to \sigma$.*

Proof. Let $\Theta \in \mathrm{Fin}^o_{\rho\to\sigma}$ and $f \in G_{\rho\to\sigma}$. By the finite omniception of Θ we know that each $U \in \mathrm{Con}^{\max}_{L(\Theta)}$ is a total neighborhood, so by (C) at ρ we have $\widetilde{U} \in G_\rho$. By the totality of f we have that $f(\widetilde{U}) \in G_\sigma$, so there will be some $V^{f(\widetilde{U})} \in \mathrm{Con}^{\mathrm{ctr}}_{\Theta\cdot U}$ such that $V^{f(\widetilde{U})} \asymp_\sigma f(\widetilde{U})$, because $\Theta \cdot U$ is omniceptive by the finite omniception of Θ and (O) at σ. Based on these, we may set

$$W^f := \bigcup_{U \in \mathrm{Con}^{\max}_{L(\Theta)}} \langle U, V^{f(\widetilde{U})} \rangle.$$

It is $W^f \in \mathrm{Con}^{\mathrm{ctr}}_\Theta$ by Lemma 3.12.2. Furthermore, let $\langle U_0, b_0 \rangle \in W$ and $\langle U, b \rangle \in f$ be such that $U \asymp_\rho U_0$; then $U \subseteq \widetilde{U_0}$ (remember that U_0 is a total neighborhood) and consequently $\overline{U} \subseteq \widetilde{U_0}$ by the propagation of consistency; by the monotonicity of f we get $f(\overline{U}) \subseteq f(\widetilde{U_0})$, so since $b \in f(\overline{U})$ it must also be $b \in f(\widetilde{U_0})$; but $f(\widetilde{U_0}) \asymp_\sigma V^{f(\widetilde{U_0})}$ and $b_0 \in V^{f(\widetilde{U_0})}$, so $b \asymp_\sigma b_0$, as we wanted. □

Proposition 5.7 (Witnessing)**.** *Let ρ and σ be types. If* (D) *holds in ρ and* (W) *holds in σ then* (W) *holds in $\rho \to \sigma$.*

Proof. Let $f \in G_{\rho\to\sigma}$ and $W, W' \in \mathrm{Con}_{\rho\to\sigma}$ be such that $W' \not\asymp_{\rho\to\sigma} W \asymp_{\rho\to\sigma} f$. For $i = 1, \dots, m$, let $U'_i \in \mathrm{Con}_{L(W')}$ and $U_i \in \mathrm{Con}_{L(W)}$ run through *all* witnessing pairs of inconsistency between W' and W, that is, cover all the cases where

$$U'_i \asymp_\rho U_i \wedge W' \cdot U'_i \not\asymp_\sigma W \cdot U_i.$$

By (D) at ρ, for each i there exists an $x_i \in G_\rho$ such that $U'_i \cup U_i \subseteq x_i$. By the consistency of (upper) application, for every such x_i it is $W \cdot U_i \asymp_\sigma f(x_i)$, and by (W) at σ there exists some $V_{i0} \in \mathrm{Con}_\sigma$ such that $W \cdot U_i \vdash_\sigma V_{i0} \subseteq f(x_i)$ and $W' \cdot U'_i \not\asymp_\sigma V_{i0}$. By Lemma 5.1, there exists some $U_{V_{i0}} \subseteq x_i$ for every i such that $\langle U_{V_{i0}}, V_{i0} \rangle \subseteq f$. Letting $U_{i0} := U_{V_{i0}} \cup U'_i \cup U_i$, by the deductive closure of f it follows that $\langle U_{i0}, V_{i0} \rangle \subseteq f$. Since by the hypotheses at σ for every i it is

$$\langle U_i, W \cdot U_i \rangle \vdash_{\rho\to\sigma} \langle U_{i0}, V_{i0} \rangle \subseteq f \wedge \langle U'_i, W' \cdot U'_i \rangle \not\asymp_{\rho\to\sigma} \langle U_{i0}, V_{i0} \rangle,$$

it follows that

$$W \vdash_{\rho\to\sigma} \bigcup_{i=1}^m \langle U_{i0}, V_{i0} \rangle \subseteq f \wedge W' \not\asymp_{\rho\to\sigma} \bigcup_{i=1}^m \langle U_{i0}, V_{i0} \rangle,$$

so we may set $W_0 := \bigcup_{i=1}^m \langle U_{i0}, V_{i0} \rangle$ and we're done. □

We may generalize the property (W) to account for inconsistency preserving witnesses of the consistencies between a total ideal and neighborhoods in a finite set.

Lemma 5.8. *At a type ρ, the statement* (W) *is equivalent to the following: Let $\Gamma \in \mathrm{Fin}_\rho$ and $x \in G_\rho$; for all $U \in \mathrm{Con}_\Gamma$ with $U \asymp_\rho x$ there exists a neighborhood $N_{U,\Gamma,x} \in \mathrm{Con}_\rho$ such that*

$$U \vdash_\rho N_{U,\Gamma,x} \subseteq x \wedge \underset{U' \in \mathrm{Con}_\Gamma}{\forall} (U' \not\asymp_\rho U \to U' \not\asymp_\rho N_{U,\Gamma,x}). \tag{W'}$$

Proof. Let Γ be a finite set, U some neighborhood of Γ and x a total ideal. Assume that (W) holds, and furthermore that $U_1, \dots, U_m \in \mathrm{Con}_\Gamma$ are all neighborhoods in Γ such that $U_i \not\asymp_\rho U$ for $i = 1, \dots, m$. Then for each such i there is a neighborhood $U_{0i} \in \mathrm{Con}_\rho$ such that $U_i \not\asymp_\rho U_{0i}$ and $U \vdash_\rho U_{0i} \subseteq x$. Setting $N_{U,\Gamma,x} := \bigcup_{i=1}^m U_{0i}$ we're done. In the other way around, let U and U' be two neighborhoods and x a total ideal, such that $U' \not\asymp_\rho U \asymp_\rho x$, and assume that (W') holds for all finite sets Γ, neighborhoods $U \subseteq \Gamma$ and total ideals x. Setting $U_0 := N_{U,U \cup U',x}$ we're done. □

So *if* Γ accepts a total ideal x at all, even if with a nontransitive neighborhood U, *then* it could be safely side extended to include a common part $N_{U,\Gamma,x}$ of $\overline{U}$ and x; enriched in this way Γ would now accept x in the strong sense of inclusion. This is exactly what we need to exploit by taking the consistency closure of a higher-type total neighborhood, provided its list of arguments is omniceptive. But let us get to the details without further ado.

Proposition 5.9 (Closure)**.** *Let ρ and σ be types. If* (O) *and* (W) *hold in ρ and* (C) *holds in σ then* (C) *holds in $\rho \to \sigma$.*

Proof. Let $T \in \mathrm{Con}^g_{\rho\to\sigma}$ and $x \in G_\rho$. We show that $\widetilde{T}(x) \in G_\sigma$. Based on Lemma 3.9, we may assume that $L(T) \in \mathrm{Fin}^o_\rho$ without harming generality. By (O) at ρ there exists a $U^x \in \mathrm{Con}^{\mathrm{ctr}}_{L(T)}$ such that $U^x \asymp_\rho x$. By Lemma 4.8.4 it is $T \cdot U^x \in \mathrm{Con}^g_\sigma$, and by (C) at σ it is $\widetilde{T \cdot U^x} \in G_\sigma$. So in order to show that $\widetilde{T}(x) \in G_\sigma$, it suffices to show that $\widetilde{T \cdot U^x} \subseteq \widetilde{T}(x)$ and invoke Lemma 5.2.

Let then $b \in \mathrm{Tok}_\sigma$ be such that $b \in \widetilde{T \cdot U^x}$. This means that $b \asymp_\sigma T \cdot U^x$. By Lemma 3.7.2 we have $\langle U^x, b\rangle \asymp_{\rho\to\sigma} T$. By (W) at ρ and Lemma 5.8, there exists a neighborhood $U^x_0 := N_{U^x,L(T),x} \in \mathrm{Con}_\rho$ such that

$$U^x \vdash_\rho U^x_0 \subseteq x \wedge \underset{U' \in \mathrm{Con}_{L(T)}}{\forall} (U' \not\asymp_\rho U^x \to U' \not\asymp_\rho U^x_0);$$

it is $\langle U^x_0, b\rangle \asymp_{\rho\to\sigma} T$, because for every $\langle U', b'\rangle \in T$ with $U' \asymp_\rho U^x_0$ it has to be $U' \asymp_\rho U^x$ from the above, therefore $b \asymp_\sigma b'$ follows by $\langle U^x, b\rangle \asymp_{\rho\to\sigma} T$. We have found a $U_b := U^x_0 \in \mathrm{Con}_\rho$ such that $\langle U_b, b\rangle \asymp_{\rho\to\sigma} T$ and $U_b \subseteq x$; but this means by definition that $b \in \widetilde{T}(x)$, and we're done. □

Theorem 5.10 (Density)**.** *Every type satisfies* (O)*,* (W)*, and* (C)*, and in particular, every type is dense.*

Proof. It follows by a mutual induction over types by Lemma 5.5 and Propositions 5.4, 5.6, 5.7, and 5.9. □

As a closing remark, we should note that the witness which we provide is actually the *maximal* total extension of a given neighborhood, in the sense that if, for a type ρ, $U \in \mathrm{Con}_\rho$ is some neighborhood, $U^g \in \mathrm{Con}^g_\rho$ is the witness provided by Theorem 4.7, and $x \in G_\rho$ is such that $U \subseteq x$, then $x \subseteq \widetilde{U^g}$.

5.2 Nontotality of transitive ideals

In the same way as we did with finite totality and transitivity in Theorem 4.9, we would like to know if we can connect totality and transitivity on the level of ideals, and possibly obtain an explicit characterization of totality in terms of consistency. We see now that this is not as straightforward as one might expect, and the problem seems to lie in nonflatness.

This becomes clear if we draw from the expositions [3, 5, 6], where various explicit characterizations of *abstract totality* are mentioned (note the qualifier). The most relevant characterization for us there is that an element x is *abstractly total* if and only if it is *almost maximal* (which is classically equivalent to its having a unique maximal extension [6]). In our setting, this means that $y_1 \supseteq x \subseteq y_2$ implies $y_1 \asymp y_2$ for all y_1 and y_2. At the same time, we call x *transitive* if $y_1 \asymp x \asymp y_2$ implies $y_1 \asymp y_2$ for all y_1, y_2. We immediately see the following.

Lemma 5.11. *At every type, an ideal is almost maximal if and only if it is transitive.*

Proof. That transitivity implies almost maximality is clear. To see the converse let x be almost maximal and $y_1 \asymp x \asymp y_2$. Then $x \subseteq y_i \cup x$ for each i and we get $y_1 \cup x \asymp y_2 \cup x$ by almost maximality, which yields $y_1 \asymp y_2$. □

For the following we express the transitivity of x through tokens, similarly to Lemma 3.4.

Proposition 5.12 (Total-transitivity)**.** *At any type, total ideals are transitive, but not the other way around.*

Proof. At a base type ι, let a_1, a_2 be tokens and x a total ideal, such that $a_1 \asymp_\iota x \asymp_\iota a_2$. There exists a total neighborhood P with $P \subseteq x$, so the assumption yields $a_1 \asymp_\iota P \asymp_\iota a_2$, which implies $a_1 \asymp_\iota a_2$ by the *finite* total transitivity of ι (Proposition 4.3).

At a higher type $\rho \to \sigma$, let $\langle U_1, b_1\rangle$, $\langle U_2, b_2\rangle$ be tokens and f be a total ideal, such that $\langle U_1, b_1\rangle \asymp_{\rho\to\sigma} f \asymp_{\rho\to\sigma} \langle U_2, b_2\rangle$. Assume furthermore that $U_1 \asymp_\rho U_2$. By Theorem 5.10 there exists a total ideal $x : \rho$ such that $U_1 \cup U_2 \subseteq x$. Since f is itself total, the ideal $f(x) : \sigma$ must also be total, and by the induction hypothesis at σ it must also be transitive. Now, applying all terms of the assumption to x we obtain $b_1 \asymp_\sigma f(x) \asymp_\sigma b_2$, which then yields $b_1 \asymp_\sigma b_2$.

For the converse, a counterexample is the transitive nontotal ideal $\infty = \{\mathrm{S}^m * \mid m \geqslant 0\}$ of type $\mathbb{N}$. □

So, somewhat paradoxically, total ideals are abstractly total but there exist abstractly total ideals which are not total. This is of course explained by the fact that abstract totality abstracts totality as this manifests in hierarchies over *flat* base types, where no infinities like ∞ can arise. On another note, the fact that totality and transitivity coincide on the finite level seems to suggest that density in nonflat-based hierarchies is indeed cleanest explained by explicitly finitary witnesses.

5.3 Noncontinuity of totalization

The witness for density that we have provided in the previous is a mapping of the sort $\mathsf{tot} : \mathrm{Con}_\rho \to \rho$.[6] It is easy to see that this is not a "continuous" mapping—that is, it

[6] Such mixed typings of terms appear often and naturally in considerations within information systems, and should be accounted for in a theory of partial computable functionals together with their approximations as in [20].

does not extend to an ideal of type $\rho \to \rho$—since it can not be expected to preserve consistency: consider the neighborhoods $\{\mathtt{S}*\}$ and $\{\mathtt{SS}*\}$ at type $\mathbb{N}$; these are consistent to each other, but

$$\mathtt{tot}(\{\mathtt{S}*\}) \ni \mathtt{S0} \not\asymp_{\mathbb{N}} \mathtt{SS0} \in \mathtt{tot}(\{\mathtt{SS}*\}) \; .$$

This counterexample is general enough to convince us that this shortcoming is not particular to our witness.

Lemma 5.13. *There is no consistency-preserving mapping* $t : \mathrm{Con}_{\mathbb{N}} \to \mathbb{N}$ *such that* $U \subseteq t(U)$ *and* $t(U) \in G_{\mathbb{N}}$ *for all* $U \in \mathrm{Con}_{\mathbb{N}}$.

Proof. If such a mapping existed it should be $t(U_1) \asymp_{\mathbb{N}} t(U_2)$ for any two neighborhoods $U_1, U_2 \subseteq \infty$. Fixing such a U_1 with $t(U_1) = \overline{\{\mathtt{S}^n 0\}}$ for some n and setting $U_2 := \{\mathtt{S}^{n+1}*\}$ we get $t(U_1) \not\asymp_{\mathbb{N}} t(U_1)$, a contradiction. □

5.4 Separation

One of Berger's key insights in [2], which permeates all subsequent approaches that our work is based upon (including our own), was that the notion of totality can be clarified if density is viewed together with an accompanying notion of "separation" (also called "codensity"): intuitively, a type ρ is considered to feature the *separation property*, if any two open sets of conflicting information can be told apart by a total "predicate" of type $\rho \to \mathbb{B}$. His argument proceeded by mutual induction for both properties of density and separation over all finite types. What we did instead in our mutual inductive arguments above was in effect to replace the notion of "separation of neighborhoods by infinite total ideals" by notions of "acceptance of total ideals by finite sets". In our exposition separation follows as a simple corollary of density.

Following [44], call a type ρ *separating* if

$$\bigvee_{U,U' \in \mathrm{Con}_\rho} (U \not\asymp_\rho U' \to \bigexists_{f \in G_{\rho \to \mathbb{B}}} \langle U, \mathtt{tt} \rangle \in f \ni \langle U', \mathtt{ff} \rangle),$$

and *finitely separating* if

$$\bigvee_{U,U' \in \mathrm{Con}_\rho} (U \not\asymp_\rho U' \to \bigexists_{T \in \mathrm{Con}^g_{\rho \to \mathbb{B}}} \langle U, \mathtt{tt} \rangle \asymp_{\rho \to \mathbb{B}} T \asymp_{\rho \to \mathbb{B}} \langle U', \mathtt{ff} \rangle).$$

Proposition 5.14 (Separation)**.** *Every type is finitely separating, and consequently separating.*

Proof. If U and U' are inconsistent a ρ, then the finite set $\{\langle U, \mathtt{tt} \rangle, \langle U', \mathtt{ff} \rangle\}$ is a neighborhood at $\rho \to \mathbb{B}$, and by Theorem 4.7 there will exist some $T \in \mathrm{Con}^g_{\rho \to \mathbb{B}}$ which side extends it. Consequently, by Theorem 5.10 the total ideal $\widetilde{T}$ will extend it. □

6 Notes

We gave a new proof of the Kreisel density theorem for finite types interpreted over nonflat inductive base types given as algebras by constructors. We introduced a notion of *totality for neighborhoods* and connected it to the usual notion of totality for ideals: given a neighborhood one may first totalize it in an explicitly finitary way to obtain a *total neighborhood*, which then extends trivially to a total ideal by means of consistency; the resulting ideal, though generated by a compact element, is the maximal totalization of the given neighborhood. Additionally, we saw that traditional characterizations of totality inspired from the flat setting, although in a sense reflected on the level of neighborhoods, do not carry over to the nonflat setting for ideals. Here we gather notes on the above, on related literature, and on future work.

The density theorem in the literature

As already pointed out in the introduction, the density problem was addressed for the first time by Kreisel [25] and also Kleene [24]. In his phd thesis [2], Ulrich Berger recast and solved the density problem within domain theory, generalizing results of Yuri Ershov [13, 14] and paralleling work of Dag Normann [30]—see [3, 46] for an account in english. A proof which does not thematize separation is given by Dag Normann in [33], while a modern approach from a viewpoint of an all-encompassing theory of higher-type computability can be found in the recent volume [26] by John Longley and Dag Normann. The density theorem is a fundamental result with several deep and far-reaching applications, like the choice theorem [25, 3, 42], Kreisel's representation theorem [25, 29, 32], a generalized Kreisel–Lacombe–Shoenfield theorem [3], Normann's theorem [34, 35, 38], and Escardó's theory of exhaustive search [15, 16], as well as extensions and generalizations, for example to dependent and universe domains [4], to Scott's equilogical spaces [1], or even to an account of totality independently of density [31]—see also [5, 33, 6, 36]. It would be natural to seek among these studies for ones that would benefit from the possibility of explicitly finitary totalization. Existence of such cases would further justify the extension of the results presented here to richer type systems, starting with the one adopted in [44], and possibly moving on to the type systems covered in [4].

Related work

The problem of finding a proof of density theorem "by compacts" occurred to the author back in the early 2011, and since then tackling it has primarily provided an incentive to develop the theory of nonflat information systems for semantics (see [23] for examples of collateral results). A partial result in the direction of finite witnesses for density was presented in [21], where, in contrast to the present approach, it was shown that one may first prove a version of finite separation at every type and then use this as a lemma to prove density (a version of our Proposition 5.7 also appears there); that approach provided a satisfactory finitary explanation of separation but not of totalization. Meanwhile, an alternative bottom-up approach to the density theorem, which grew independently but turned out to be similar in spirit to ours, was carried out by Davide Rinaldi in [39]. Rinaldi offers a nonflat semantics which is topological rather than domain-theoretic: he uses certain formal topologies [41], for which he proves that they are equivalent to *unary information systems*; these are information systems where in addition neighborhoods always have eigentokens, that is, for every $U \in \mathrm{Con}$ there exists some $a \in \mathrm{Tok}$ such that $U \sim \{a\}$. In our setting this is true of base types, but not of higher types. To adapt Rinaldi's semantics in a way that clearly matches broader categories of information systems than just the unary ones, and look at a formal-topological proof of density by compacts anew, would not only be instructive, but it could also provide a more elegant proof.

Towards a common study of totality and cototality

Recently, "cototal ideals", that is, total ideals together with infinities like ∞ at type $\mathbb{N}$, have been used to model stream-like objects at base types arising from initial algebras, offering an alternative to versions of semantics simultaneously based on initial algebras and final coalgebras [40, 18, 17]; for this line of work, rooted in [7, 8, 11], see [9, 44, 27, 28, 10]. In view of the mismatch between transitivity and totality in a nonflat setting which we described in section 5.2, it looks like a refinement is possible, where totality should feature an increased degree of finiteness and should be studied hand in hand with an appropriate notion of cototality: beside more or less obvious differences of the two at base types (based on the proof of Lemma 5.13, for example, one could expect continuous "cototalizations" to exist), their interplay at higher types remains terra incognita at the time of this writing.

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