Elaborating on Ishihara's 'WKL implies FAN' proof

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Abstract

In 1990 Hajime Ishihara proved indirectly that the weak König's lemma implies the fan theorem. Here we reproduce a direct proof Ishihara provided for the same implication, which he presented in 2004 in Munich.

This is a work done within a coordinated attempt of the Mathematical Logic Group in Munich to formalize Ishihara's arguments in theorem proving environments, an aim which was met in late spring by Stefan Schimanski on Coq and Nikolaus Thiel on Minlog.

1 Preliminaries

The well-known binary coding of natural numbers by

$$m \mapsto \langle a_1, \ldots, a_n \rangle \qquad \text{iff} \qquad m = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$$

where for every $i = 1, \ldots, n$ it is $a_i \in \{0,1\}$, allows us to actually identify $\mathbb{N}$ with the set $\{0,1\}^\star$ of all *finite binary sequences*:

$$\{0,1\}^\star = \{\langle a_1, \ldots, a_n \rangle \in \{0,1\}^n \mid n \in \mathbb{N}\}$$

where for the *empty sequence* we write $\langle\rangle$. We use m, n, i, k for elements of $\mathbb{N}$ and α, β, γ, δ when we want to stress their $\{0,1\}^\star$-sequential nature. We denote by $\{0,1\}^{\mathbb{N}}$ the set of all *infinite binary sequences*; we use ψ, ω for elements of $\{0,1\}^{\mathbb{N}}$. The *catenation* of two sequences $\langle a_1, \ldots, a_m \rangle$ and $\langle b_1, \ldots, b_n \rangle$ is defined in the standard way by

$$\langle a_1, \ldots, a_n \rangle * \langle b_1, \ldots, b_m \rangle = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle$$

The *length* of a sequence α is defined by

$$|\langle\rangle| = 0 \qquad \& \qquad |\alpha * \langle a \rangle| = |\alpha| + 1$$

The k-th *initial segment* (also *prefix* or *restriction*) of $\alpha = \langle a_1, \ldots, a_n \rangle$ is defined by

$$\bar{\alpha}(k) = \begin{cases} \langle a_1, \ldots, a_k \rangle & \text{if } k \leq n \\ \alpha & \text{otherwise} \end{cases}$$

We will write just $\bar{\alpha}$ for some initial segment of α and we will say that α is an *extension* of any $\bar{\alpha}$.[1] The set $\{0,1\}^\star$ now becomes a poset by the usual *prefix ordering*:

$$\alpha \preceq \beta \qquad \text{iff} \qquad \exists k.\ \alpha = \bar{\beta}(k)$$

[1] It is trivially $|\bar{\alpha}(k)| = k$, for $k \leq |\alpha|$ and $\bar{\alpha}(|\alpha|) = \alpha$.

Let A be a subset of $\{0,1\}^\star$. We say that A is *decidable* when

$$\forall\alpha.\ \alpha \in A\ \vee\ \alpha \notin A$$

in the sense that there exists a *decidability algorithm* which solves the membership problem of A. We say that A is *closed under restriction* when

$$\beta \in A \wedge \alpha \preceq \beta\ \rightarrow\ \alpha \in A$$

Finally A is a *(binary) tree* when it is decidable and closed under restriction. We will refer to the poset $\{0,1\}^\star$ as the *full tree.*

A tree T is *infinite* if

$$\forall k \exists \alpha \in \{0,1\}^k.\ \alpha \in T$$

An *infinite branch in T* is an $\omega \in \{0,1\}^{\mathbb{N}}$ such that

$$\forall k.\ \bar{\omega}(k) \in T$$

We will just write "infinite branch" for an "infinite branch in the full tree".

The first statement which is involved in Ishihara's implication is the *weak König's lemma*:

Statement 1 (WKL). *Every infinite tree has an infinite branch.*

A subset B of $\{0,1\}^\star$ is called a *bar* when all infinite sequences eventually meet B, that is

$$\forall \omega \exists k.\ \bar{\omega}(k) \in B$$

It follows that B cannot be empty.[2] A bar B is said to be *uniform* if there is a height up to which all infinite branches are sure to have met B, that is

$$\exists k \forall \omega \exists i \leq k.\ \bar{\omega}(i) \in B$$

We will say that B is uniform by k. The second statement we need is the *fan theorem*:

Statement 2 (FAN). *Every decidable bar is uniform.*

2 Ishihara's implication

Ishihara's reasoning is based on two lemmata: (i) the construction of an infinite extension of a (possibly non-infinite) tree and (ii) the equivalence of WKL to the existence of a 'projection axis' for any tree.

For the mental model of the first lemma: if we are given a 'finite' tree T, an economical way of extending it to an infinite one $\hat{T}$ is to pick up its highest leaves and consider their full-tree expansions. In case T is already an infinite one, such an extension can and need not be made.

For the second lemma: the observation is that WKL yields a handy means of handling the lengths of nodes in a tree T, namely a 'projection axis', that is an infinite branch ω, whereon whenever we project nodes of T we still remain in T. If the tree is infinite the axis is one of its infinite branches, else it is an infinite extension of one of the highest nodes in T, that is an infinite branch of the corresponding $\hat{T}$.

[2] A set B which is a bar defines in a natural way a tree T_B which is prevented from being infinite because of B; we may think of B as *barring* the tree T_B from being infinite. This will be made clear in the proof of the main theorem (see page 4).

Let then T be an arbitrary tree; induce the set $\hat{T}$ which consists of T together with all finite extensions of its highest leaves:

$$\hat{T} = \{\beta \in \{0,1\}^\star \mid \beta \in T \vee (\exists \bar{\beta} \in T.\ \forall \alpha \in \{0,1\}^{\leq|\beta|}.\ \alpha \in T \to |\alpha| \leq |\bar{\beta}|)\}$$

Lemma 1. *If T is a tree then the set $\hat{T}$ is an infinite tree.*

Proof. $\hat{T}$ is decidable: This follows directly from the definition, where only the decidable T and finite lengths are involved.

$\hat{T}$ is closed under restriction: Let $\beta \in \hat{T}$ and γ be an arbitrary prefix. Since T is decidable, we have

$$\gamma \in T \vee \gamma \notin T$$

In the first case we have $\gamma \in \hat{T}$. In the second case it must also be that $\beta \notin T$. Let $\bar{\beta}$ be the highest leaf prefix which is provided by the definition of $\hat{T}$, for which

$$\alpha \in T \to |\alpha| \leq |\bar{\beta}|$$

Since $\gamma \notin T$ and $\forall \alpha \in \{0,1\}^{\leq|\gamma|}.\ \alpha \in T \to |\alpha| \leq |\bar{\beta}|$, that is since $\bar{\beta}$ serves as a highest leaf prefix for γ too, we have $\gamma \in \hat{T}$.

$\hat{T}$ is an infinite tree: Let k be an arbitrary length. By decidability of T we have

$$(\exists \beta \in \{0,1\}^k.\ \beta \in T) \vee (\forall \beta \in \{0,1\}^k.\ \beta \notin T)$$

In the first case we have $\exists \beta \in \{0,1\}^k.\ \beta \in \hat{T}$. In the second case choose a highest leaf $\delta \in T$[3] and let

$$\beta = \delta * \langle \underbrace{0, \ldots, 0}_{k-|\delta|} \rangle$$

Obviously $\beta \in \{0,1\}^k \wedge \beta \in \hat{T}$. □

Lemma 2. *The following statements are equivalent:*

1. *Every tree has a projection axis, that is for every tree T*

$$\exists \omega \in \{0,1\}^{\mathbb{N}}.\ \alpha \in T \to \bar{\omega}(|\alpha|) \in T$$

2. *Statement 1.*

Proof. (1) ⇒ (2): Let T be an infinite tree and ω be a projection axis of T. Since it is infinite

$$\forall k \exists \alpha \in \{0,1\}^k.\ \alpha \in T$$

We can project this α on ω and still be in T:

$$\forall k \exists \alpha \in \{0,1\}^k.\ \bar{\omega}(|\alpha|) \in T$$

which yields

$$\forall k.\ \bar{\omega}(k) \in T$$

So ω is an infinite branch in T.

(2) ⇒ (1): Let T be an arbitrary tree. The set $\hat{T}$ it induces is an infinite tree by lemma 1, so by (2) it will have an infinite branch ω. Let $\alpha \in T$. By decidability of T it is

$$\bar{\omega}(|\alpha|) \in T \vee \bar{\omega}(|\alpha|) \notin T$$

[3]This choice is decidable since it is bounded by k.

Suppose it was $\bar{\omega}(|\alpha|) \notin T$. By definition of $\hat{T}$ there would exist a highest leaf prefix of $\bar{\omega}(|\alpha|)$, say $\delta \in T$. But then we would have

$$|\delta| \leq |\bar{\omega}(|\alpha|)| = |\alpha| \qquad \text{and} \qquad |\alpha| \leq |\delta|$$

that is $|\delta| = |\bar{\omega}(|\alpha|)|$, and since the first is a prefix of the other

$$T \ni \delta = \bar{\omega}(|\alpha|) \notin T$$

which is a contradiction. $\square$

We are now able to prove the main

Theorem. *Statement 1 implies statement 2.*

Proof. Let B be a decidable bar. If the empty sequence $\langle\rangle$ belongs to B then B is trivially uniform by 0, since $\langle\rangle$ is a prefix of every sequence.

Assume that $\langle\rangle \notin B$. Define

$$T_B = \{\alpha \mid \gamma \preceq \alpha \to \gamma \notin B\}$$

This set is evidently a tree, so by lemma 2 it will have a projection axis ω. Since B is a bar

$$\exists k.\ \bar{\omega}(k) \in B$$

Let now ψ be any infinite branch and suppose that we had $\forall i \leq k.\ \bar{\psi}(i) \notin B$. By definition of T_B it would be $\bar{\psi}(k) \in T_B$, so its projection on ω would still lie in T_B, that is $\bar{\omega}(|\bar{\psi}(k)|) \in T_B$. But then

$$B \ni \bar{\omega} \in T_B$$

which is a contradiction. So B is uniform by k. $\square$

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