Lectures on Constructive Mathematics

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Lecture 2:

Functional Analysis

Separation theorems

Suppose we start at a point ξ in the interior of a located subset C of a normed space X and move linearly towards a point z in the metric complement of C. Are we able to tell when we are crossing the boundary

$$\partial C = \overline{C} \cap \overline{\sim C}$$

of C?

In general, the constructive answer is *no*. However, our geometric intuition suggests that when C is convex, we might succeed in pinpointing boundary crossing points.

Proposition: *Let C be an open convex subset of a Banach space X such that $C \cup -C$ is dense in X, and let $\xi \in C$. For each $z \in -C$ and each $t \in [0,1]$ write*

$$z_t \equiv t\xi + (1-t)\,z.$$

Then

(i) $\gamma(\xi, z) \equiv \inf \{t \in [0,1] : z_t \in C\}$ *exists, and* $0 < \gamma(\xi, z) < 1$;

(ii) $z_{\gamma(\xi,z)}$ *is the unique intersection of the segment* $[\xi, z]$ *with* ∂C;

(iii) *if* $\gamma(\xi, z) < t \leqslant 1$, *then* $z_t \in C$; *and*

(iv) *if* $0 \leqslant t < \gamma(\xi, z)$, *then* $z_t \in -C$.

Moreover, the mapping $(\xi, z) \rightsquigarrow z_{\gamma(\xi,z)}$ *of* $C \times -C$ *into* ∂C *is continuous at each point of* $C \times -C$.

For fixed $\xi \in C$, we call the mapping $z \rightsquigarrow z_{\gamma(\xi,z)}$ in the foregoing proposition the **boundary crossing map of** C *relative to* ξ.

A subset C of a vector space X over $\mathbf{K}$ is called a **cone** if for all $x, y \in C$ and all $t > 0$, both $x + y$ and tx belong to C.

In that case, C is convex.

The closure of a cone is a cone, as is the intersection of two cones.

If K is a convex subset of X, then the set

$$c(K) = \{tx : x \in K, t > 0\}$$

is a cone, the **cone generated by the convex set** K.

If X is a normed space and K is open, then so is $c(K)$.

If K is a bounded, located, convex subset of X such that $\rho(0, K) > 0$, then $c(K)$ is located.

A linear subset H of a normed space X is called a **hyperplane** if there exist an **associated vector** $x_0 \in X$ and a positive number c such that

- ▷ $\|x - x_0\| \geqslant c$ for each $x \in H$, and
- ▷ each $x \in X$ is represented (uniquely) in the form $x = tx_0 + y$ with $t \in \mathbf{K}$ and $y \in H$.

The **kernel**, $\ker(u) = u^{-1}(0)$, of a nonzero bounded linear functional on X is a hyperplane.

Proposition: *Let X be a normed space, and H a hyperplane in X with associated vector x_0. Then there exists a unique bounded linear functional u on X such that* $\ker u = H$ *and* $u(x_0) = 1$.

A **half space** of a normed space X is a convex subset K such that ∂K is a hyperplane and the set

$$\{x \in X : x \in K \vee -x \in K\}$$

is dense in X.

We are now ready for the **basic separation theorem**:

Let X be a separable normed space, K_0 a bounded, located, open, convex subset of X such that $\rho(0, K_0) > 0$, and x_0 a point of X such that $-x_0 \in K_0$. Then there exists an open half-space K of X such that $K_0 \subset K$, $\rho(x_0, K) > 0$, and ∂K is a located subspace of X that is a hyperplane with associated vector x_0.

The proof illustrates an important observation about classical proofs using Zorn's lemma: for separable spaces it is often possible to replace such a proof by a constructive one that uses an induction argument.

The basic idea of the constructive proof is this. Given a dense sequence $(x_n)_{n \geqslant 1}$ in X, carry out a succession of located convex enlargements of K_0 such that for $n \geqslant 1$,

- ▷ the cone generated by the nth enlargement K_n is close to at least one of the points x_n and $-x_n$, and
- ▷ the union of the cones $c(K_n)$ is the desired open half-space.

The idea may seem simple, but the details are very complicated.

The **full separation theorem:**

Let A and B be bounded convex subsets of a separable normed space X such that the algebraic difference

$$\{y - x : x \in A, y \in B\}$$

is located and the mutual distance

$$d \equiv \inf \{\|y - x\| : x \in A, y \in B\}$$

is positive. Then for each $\varepsilon > 0$ there exists a normed linear functional u on X, with norm 1, such that

$$\mathrm{Re}\, u(y) > \mathrm{Re}\, u(x) + d - \varepsilon$$

for all $x \in A$ and $y \in B$.

Corollary: *Let x be an element of a nontrivial separable normed space X, and let $\varepsilon > 0$. Then there exists a normed linear functional u on X such that $\|u\| = 1$ and $u(x) > \|x\| - \varepsilon$.*

Proof: If $x \neq 0$, apply the separation theorem with $A = \{0\}$ and $B = \{x\}$.

In the general case, choose a nonzero vector y such that $\|x - y\| < \varepsilon/2$, and construct a normed linear functional u on X such that $\|u\| = 1$ and $u(y) > \|y\| - \varepsilon/2$. Then

$$u(x) \geqslant u(y) - |u(x) - u(y)| > \|y\| - \frac{\varepsilon}{2} - \|x - y\| > \|x\| - \varepsilon.$$

The previous proposition is used in the proof of the **Hahn–Banach theorem**:

Let v be a nonzero bounded linear functional on a linear subset Y of a separable normed linear space X such that $\ker v$ *is located in X. Then for each $\varepsilon > 0$ there exists a normed linear functional u on X such that $\|u\| < \|v\| + \varepsilon$ and $u(y) = v(y)$ for each $y \in Y$.*

In the constructive context we deal only with the extension of linear functionals on subspaces of a separable normed space. The standard classical proofs extending the theorem to nonseparable normed spaces depend on Zorn's lemma and are therefore nonconstructive.

In **RUSS** there is an example where it is impossible to obtain an extended linear function u such that $\|u\| = \|v\|$.

Ishihara has shown that such an extension can be found when the norm function on X is Gâteaux differentiable.

The Hahn–Banach theorem has some surprising applications, like the following (whose classical proof is almost trivial).

Proposition: *Let $x_1, \ldots, x_n$ be elements of an infinite-dimensional normed space X, and let $\varepsilon > 0$. Then there exist linearly independent elements $e_1, \ldots, e_n$ of X such that $\|x_k - e_k\| < \varepsilon$ for each k.*

Proof: First construct a finite-dimensional subspace V of $\operatorname{span}\{x_1, \ldots, x_n\}$ such that for each i there exists $y_i \in V$ with $\|x_i - y_i\| < \varepsilon/2$. Embed V in an n-dimensional subspace W of X.

WLOG $y_1 \neq 0$. Set $e_1 \equiv y_1$.

Suppose we have found $e_1, \ldots, e_k$ in W such that $\|y_i - e_i\| < \varepsilon/2$ for $1 \leqslant i \leqslant k < n$. Let $V_k \equiv \operatorname{span}\{e_1, \ldots, e_k\}$.

Construct a normed linear functional u on W such that $u(V_k) = \{0\}$ and $\|u\| = 1$.

Pick $z \in W$ such that $\|z\| = \varepsilon/2$ and $u(z) > \varepsilon/3$.

If $u(y_{k+1}) \neq 0$, then $\rho(y_{k+1}, V_k) > 0$ and we set $e_{k+1} \equiv y_{k+1}$.

If $u(y_{k+1}) < \varepsilon/3$, then $u(y_{k+1} - z) \neq 0$, $\rho(y_{k+1} - z, V_k) > 0$, and we set $e_{k+1} \equiv y_{k+1} - z$.

Locally Convex Spaces

A **locally convex space** consists of a linear space X over $\mathbf{K}$, a family $(p_i)_{i \in I}$ of seminorms on X, and the equality and compatible inequality defined by

$$\begin{aligned} x = y &\iff \forall_{i \in I} \left(p_i \left(x - y\right) = 0\right), \\ x \neq y &\iff \exists_{i \in I} \left(p_i \left(x - y\right) > 0\right). \end{aligned}$$

The corresponding **locally convex topology** on X is the family τ_X of all subsets of X that are unions of sets of the form

$$U(a, F, \varepsilon) = \left\{ x \in X : \sum_{i \in F} p_i \left(x - a\right) < \varepsilon \right\}$$

where $a \in X$, F is an inhabited finitely enumerable subset of I, and $\varepsilon > 0$.

With natural modifications, we can extend notions from normed to locally convex spaces.

For example, a subset S of the locally convex space $\left(X, (p_i)_{i \in I}\right)$ is said to be **located** (in X) if

$$\inf \left\{ \sum_{i \in F} p_i (x - y) : y \in S \right\}$$

exists for each $x \in X$ and each finitely enumerable subset F of I.

Consider the linear space $\mathcal{B}(X, Y)$ of all bounded linear mappings between the locally convex spaces X and Y.

This set becomes a locally convex space when endowed with the seminorms p_x defined by

$$p_x(T) = \|Tx\| \quad (x \in X, \ T \in \mathcal{B}(X, Y)) .$$

We denote the unit ball of $\mathcal{B}(X, Y)$ by $\mathcal{B}_1(X, Y)$ or just $\mathcal{B}_1$. When $X = Y$, we usually write $\mathcal{B}(X)$ and $\mathcal{B}_1(X)$ rather than $\mathcal{B}(X, Y)$ and $\mathcal{B}_1(X, Y)$.

In the special case where Y is the ground field $\mathbf{K}$, we obtain the space of all bounded linear functionals on X; this space is called the **dual** of X, and is denoted by X^*; its unit ball is denoted by X_1^*. The topology associated with the family of seminorms $(p_x)_{x \in X}$ on X^* is called the **weak*** **topology** on X^*.

When we are dealing with, for example, total boundedness relative to the locally convex structure on X^*, we speak of *weak***-total boundedness*.

Banach–Alaoglu theorem: *If X is a separable normed space, then X_1^* is weak*-complete and weak*-totally bounded.*

It is straightforward to prove the weak*-completeness of X_1^*.

Weak*-total boundedness of X_1^* is a lot trickier to establish; we sketch the ideas.

Let $F = \{x_1, \ldots, x_m\}$ be a finitely enumerable subset of X, let

$$M > 4 + \max\{\|x_i\| : 1 \leqslant i \leqslant m\},$$

and let $0 < \varepsilon < 1$.

Construct a finite-dimensional subspace X_0 of X such that for $1 \leqslant i \leqslant m$, $\rho(x_i, X_0) < \varepsilon/m$ and therefore there exists $y_i \in X_0$ with $\|x_i - y_i\| < \varepsilon/m$.

If $X_0 = \{0\}$, then life is easy. So we assume that X_0 has positive dimension. Then every element of X_0^* is normed, and X_0^*, taken with the operator norm, is a finite-dimensional Banach space. Hence its unit ball is compact relative to the operator norm.

Each nonzero element of X_0^* has its kernel located in X_0; since X_0 is locally compact, this kernel is locally compact and hence is located *in the space* X. It follows that the Hahn–Banach theorem can be applied to extend nonzero bounded linear functionals from X_0 to X.

Let $\left\{u_1^0, \ldots, u_n^0\right\}$ be an ε/m-approximation to the unit ball of X_0^* in the operator norm, such that $0 < \left\|u_k^0\right\| < 1$ for each k.

Use the Hahn–Banach theorem to construct normed linear functionals $u_1, \ldots, u_n$ in X_1^* such that $u_k(x) = u_k^0(x)$ for each $x \in X_0$ and each k.

Given $u \in X_1^*$, we can find k such that $\left|u(x) - u_k^0(x)\right| < \varepsilon/m$ for all $x \in X_0$ with $\|x\| \leqslant 1$. Then

$$\sum_{i=1}^{m} |u(x_i) - u_k(x_i)| < M\varepsilon.$$

Thus $\{u_1, \ldots, u_n\}$ is an $M\varepsilon$-approximation to X_1^* relative to F.

This technique of cutting down to a finite-dimensional subspace and then applying the Hahn-Banach theorem is fundamental in the constructive theory of duality.

Let X be a normed space. For a fixed vector $x \in X$, the linear functional $u \rightsquigarrow u(x)$ on X^* is weak*-uniformly continuous on X_1^*.

Any element of X^{**} (the dual of X^*) that is uniformly continuous on X_1^* can be approximated arbitrarily closely by functionals of this special form.

Proposition: *Let X be a separable normed space, and ϕ a linear function on X^* that is weak*-uniformly continuous on the unit ball X_1^*. Then for each $\varepsilon > 0$ there exists $x \in X$ such that $\|x\| < 3\,\|\phi\|$ and*

$$|\phi(u) - u(x)| < \varepsilon \quad (u \in X_1^*)\,.$$

If X is complete, then this approximation can be made exact.

Theorem: *Let X be a separable Banach space, and ϕ a linear functional on X^* that is weak*-uniformly continuous on X_1^*. Then there exists $x \in X$ such that $\phi(u) = u(x)$ for each $u \in X^*$.*

Proof: We may assume that $\|\phi\| < 1$. Recursively applying the preceding proposition, construct a sequence $(x_n)_{n\geqslant 1}$ of vectors in X such that for each n,

$$\left|\phi(u) - \sum_{k=1}^{n} u(x_k)\right| < \frac{1}{2^n} \qquad (u \in X_1^*) \tag{1}$$

and $\|x_n\| < 3/2^{n-1}$.

The series $\sum\limits_{n=1}^{\infty} x_n$ then converges to an element x of the complete space X.

Using the linearity and continuity of u, and letting $n \longrightarrow \infty$ in (1), we obtain the desired conclusion.

Let H be a nontrivial Hilbert space. One of the topologies on $\mathcal{B}(H)$ that is important in operator-algebra theory is the **weak-operator topology** τ_w: the locally convex topology defined by the seminorms of the form $T \rightsquigarrow |\langle Tx, y\rangle|$ with x, y in H.

Classically, the sets of the type

$$\left\{ T \in \mathcal{B}(H) : \sum_{i,j=1}^{n} \left| \left\langle Te_i, e_j \right\rangle \right| < \delta \right\},$$

with $\delta > 0$ and $\{e_1, \ldots, e_n\}$ a set of pairwise orthogonal unit vectors in H, form a base of weak-operator neighbourhoods of 0 in $\mathcal{B}(H)$.

This is not the case constructively. However, it is constructively provable that the sets of the stated type form a base of weak-operator neighbourhoods of 0 *in the unit ball* $\mathcal{B}_1(H)$.

Proposition: *The unit ball $\mathcal{B}_1(H)$ of $\mathcal{B}(H)$ is τ_w-totally bounded.*

Proof: Let $\{e_1, \ldots, e_n\}$ be a finite set of pairwise orthogonal unit vectors generating a finite-dimensional subspace H_0 of H. It will suffice to prove that $\mathcal{B}_1(H)$ is totally bounded with respect to the seminorm

$$p_{jk} : T \rightsquigarrow \sum_{j,k=1}^{n} \left| \left\langle Te_j, e_k \right\rangle \right| .$$

Let P be the projection of H on H_0. Then $\mathcal{B}(H_0)$ is a finite-dimensional Banach space, and hence has a totally bounded unit ball, relative to the operator norm. Let $\left\{T_1^0, \ldots, T_m^0\right\}$ be an ε/n^2-approximation to $\mathcal{B}_1(H_0)$, and consider any $T \in \mathcal{B}_1(H)$.

The restriction $(PT)_0$ of PT to H_0 belongs to $\mathcal{B}_1(H_0)$, so there exists i such that $\left\|(PT)_0 - T_i^0\right\| < \varepsilon/n^2$. Also, $T_i^0 P \in \mathcal{B}_1(H)$. Thus if $1 \leqslant j,k \leqslant n$, then

$$\begin{aligned}\left|\left\langle \left(T - T_i^0 P\right) e_j, e_k \right\rangle\right| &= \left|\left\langle \left(T - T_i^0\right) e_j, P e_k \right\rangle\right| = \left|\left\langle P\left(T - T_i^0\right) e_j, e_k \right\rangle\right| \\ &= \left|\left\langle \left((PT)_0 - T_i^0\right) e_j, e_k \right\rangle\right| \leqslant \left\|(PT)_0 - T_i^0\right\| < \frac{\varepsilon}{n^2}.\end{aligned}$$

Hence $\sum_{j,k=1}^{n} \left|\left\langle \left(T - T_i^0 P\right) e_j, e_k \right\rangle\right| < \varepsilon$. We now see that $\left\{T_1^0 P, \ldots, T_m^0 P\right\}$ is an ε-approximation to $\mathcal{B}_1(H)$ relative to the seminorm p_{jk}.

Classically, a linear functional ϕ on $\mathcal{B}(H)$ is τ_w-continuous if and only if it has the following special continuity property.

SC: *There exist* $\delta > 0$ *and a set* $\{e_1, \ldots, e_n\}$ *of pairwise orthogonal unit vectors in* H *such that for each* $T \in B(H)$, *if* $\sum_{i,j=1}^{n} \left| \left\langle Te_i, e_j \right\rangle \right| < \delta$, *then* $|\phi(T)| < 1$.

Constructively, techniques like those used to characterise the linear functionals on a dual space that are weak*-uniformly continuous on the unit ball of the dual enable us to prove the

Proposition: *Let H be a nontrivial Hilbert space, and let ϕ be a linear functional on $\mathcal{B}(H)$ with the property* **SC**. *Then for each $\varepsilon > 0$ there exist a finite set $\{e_1, \ldots, e_n\}$ of pairwise orthogonal unit vectors in H and elements c_{jk} $(1 \leqslant j, k \leqslant n)$ of $\mathbf{K}$, such that*

$$\left| \phi(T) - \sum_{j,k=1}^{n} c_{jk} \left\langle Te_j, e_k \right\rangle \right| < \varepsilon$$

for all $T \in \mathcal{B}_1(H)$.