CONSTRUMATH

Constructive Mathematics: Proof and Computation

Marie Curie International Research Staff Exchange Scheme

Project Summary

Although constructive proofs—that is, proofs which actually show how to construct/compute/find the objects whose existence is under scrutiny—have long been highly regarded in mathematics, it is only in the last few decades that significant numbers of mathematicians and theoretical computer scientists have made concerted efforts to develop deep mathematics constructively. There are now several approaches to constructivity in mathematics. At one end of the spectrum we have the Brouwer–Bishop approach in which the underlying logic is changed from classical to so-called "intuitionisti" logic, the exclusive use of which (together with a constructively appropriate foundation such as the formal theory of natural numbers, constructive ZF set theory, or Martin-Löf's type theory) guarantees that all proofs are fully constructive. At the other end there are systems like recursive function theory, Weihrauch's TTE, and that of Pour-El and Richards that use the full power of classical logic within a carefully prescribed formal algorithmic framework. In between lie, for example, the recursive constructive analysis of the Markov School, which uses intuitionistic logic within the recursive-function-theoretic framework, and domain theory, an approach that neatly combines theory and practice (for example, with software for solving differential equations). By constructive mathematics (CM) within CONSTRUMATH, we mean mathematics with intuitionistic logic (IL) and some appropriate set-theoretic foundation. If we add to CM Brouwer's continuity principle and fan theorem (FT), we obtain the intuitionistic model (INT) of CM. If we add to CM Church's thesis and Markov's principle, we obtain Markov's recursive model (RUSS). Constructive mathematics, in its various forms, has its practical side. Every constructive proof of the existence of an ob ject x with a property P embodies algorithms for the construction of an ob ject x and for the demon- stration that it actually has the property P. These algorithms can be ex- tracted from the proofs and then implemented on a computer. Moreover, the original constructive proof is a proof that the algorithm meets its specfica- tions (that is, does the job it is supposed to do). Several groups of computer scientists world-wide have been carrying out the extraction and implementa- tion of algorithms from constructive proofs over the past twenty-five years. Recently another aspect of constructivity has begun to attract researchers' attention: constructive reverse mathematics, in

which intuitionistic logic is used to determine exactly which principles are both necessary and sufficient for the constructive proof of a particular result. To some extent this idea has been around since Brouwer, whose "Brouwerian counterexamples" typ- ically show that in order to prove a certain classical theorem P one needs, and it suffices to use, some weak (but still non-constructive) form of the law of excluded middle. However, more recent research in constructive re- verse mathematics has concentrated on what are the minimal constructive principles that are needed to prove P. The ultimate goal of CONSTRUMATH stretches towards shaping a more coherent and solidly based body of constructive mathematics, in the sense that classical, non-constructive mathematics can be said to be coherent or well-based. To work towards this goal, the participants will focus on their respective areas of expertise and their interrelations.