

MALOA conference, lecture 5

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- ▶ cardinal arithmetic.

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Definition of iterated forcing on the board.

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- ▶ MA implies not CH
- ▶ and verifying what else MA implies is a purely mathematical endeavour.

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$\text{MA}(\omega_1)$ Suppose that \mathbb{P} is a ccc forcing notion and $\{D_\alpha : \alpha < \omega_1\}$ is a family of dense sets in \mathbb{P} . Then there is a filter G in \mathbb{P} which intersects all D_α .

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Easton's product of Cohen forcing (not iteration).

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Foreman and Woodin, 1980s *Modulo (large) large cardinals it is possible that the GCH fails everywhere.*

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Magidor (1978) proved that it can be a singular of countable cofinality (modulo large cardinals).

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A property of an infinite cardinal is called a *large cardinal property* if any cardinal κ satisfying it must be regular but limit (so the existence of such a cardinal is not provable in ZFC).

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For example, inaccessible cardinals are like that, but measurable ones are not.

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Is there some ZFC truth?

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Theorem

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SUSIFA $_{\theta, \partial, \lambda}(\kappa, u)$ stands for:

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there are disjoint subsets S_1 and S_2 of ∂ such that S_1 is unbounded, S_2 is a club and ...

- (a) for $i \in S_1$, \mathbf{V}_{i+1} satisfies SFA(κ) and $\mathbf{V}_{i+1} \models "2^\kappa > |\mathcal{P}(\kappa^+)|^{\mathbf{V}_i}"$,
- (b) for each $i \in S_2$, $u_i = u \upharpoonright \mathbf{V}_i \in (U_\infty)^{\mathbf{V}_i}$,
- (c) there is a sequence $\bar{A} = \langle A_i : i \in S_2 \rangle$ such that each $A_i \in \mathcal{F}(u)$ and for all $B \in \mathcal{F}(u_i)$, $|B \setminus A_i| < \kappa$, and $A_i \in \mathbf{V}_{i+1}$.