When is the denial inequality an apartness relation?

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Abstract

In the framework of Bishop’s constructive mathematics, we show that the De Morgan law for \( \Pi^0_1 \)-formulas holds if and only if the denial inequality on the set of Cauchy-reals is an apartness relation.

A binary relation \( \triangleright \triangleleft \) on a set \( X \) is called apartness relation if for all \( x, y, z \in X \) the following holds:

a) \( \neg (x \triangleright \triangleleft x) \)

b) \( x \triangleright \triangleleft y \Rightarrow y \triangleright \triangleleft x \)

c) \( x \triangleright \triangleleft y \Rightarrow x \triangleright \triangleleft z \lor y \triangleright \triangleleft z \)

We work with the Cauchy reals as introduced in [1], that is the set \( \mathbb{R} \) of Cauchy sequences \( (x_n) \) of rationals such that

\[ \forall m, n \left( |x_m - x_n| \leq m^{-1} + n^{-1} \right). \]

The real number 0 is represented by the sequence \( (z_n) \) with \( z_n = 0 \) for all \( n \).

For two reals \( x, y \) we define equality by

\[ x = y \overset{def}{\iff} \forall n \left( |x_n - y_n| \leq 2^{-n} \right), \]

which is equivalent to

\[ \forall k \exists n_0 \forall n \geq n_0 \left( |x_n - y_n| \leq k^{-1} \right). \]  \hspace{1cm} (1)

The negation of equality, the so-called denial inequality is given by

\[ x \neq y \overset{def}{\iff} \neg (x = y) \]
and clearly fulfills a) and b). We present a logical axiom which is equivalent to the statement: ‘The denial inequality is an apartness relation.’

A formula $\Phi$ is called a $\Pi^0_1$-formula if there exists a binary sequence $\alpha$ such that

$$\Phi \iff \forall n \left( \alpha_n = 0 \right).$$

Consider the following axioms:

DA  The denial inequality is an apartness relation.

$\Pi^0_1$-DML For all $\Pi^0_1$-formulas $\Phi$ and $\Psi$, $\neg (\Phi \land \Psi) \Rightarrow \neg \Phi \lor \neg \Psi$.

The latter is an instance of the the De Morgan law.

**Proposition 1.** The axioms $\Pi^0_1$-DML and DA are equivalent.

**Proof.** Assume $\Pi^0_1$-DML. It suffices\(^1\) to show that

$$x \neq 0 \lor y \neq 0$$

holds for arbitrary but fixed $x, y \in \mathbb{R}$ with $x \neq y$. Define binary sequences $\alpha, \beta$ by

$$\alpha_n = 1 \overset{\text{def}}{\iff} |x_n| > 2^{n-1}$$

and

$$\beta_n = 1 \overset{\text{def}}{\iff} |y_n| > 2^{n-1}.$$ 

Define further $\Phi \overset{\text{def}}{=} \forall n \left( \alpha_n = 0 \right)$ and $\Psi \overset{\text{def}}{=} \forall n \left( \beta_n = 0 \right)$. We can see that that

- $\Phi \iff x = 0$
- $\Psi \iff y = 0$

So the characterisation of equality given by (1) yields $\neg (\Phi \land \Psi)$, which implies $\neg \Phi \lor \neg \Psi$, which in turn implies (2).

Now assume DA and fix binary sequences $\alpha, \beta$ such that

$$\neg (\forall n \left( \alpha_n = 0 \right) \land \forall n \left( \beta_n = 0 \right)).$$

\(^1\)Since $=$ is compatible with $+$ and $-$, we can get from the special case to the general case as follows:

$$x \neq y \Rightarrow x - z \neq y - z \Rightarrow x - z \neq 0 \lor y - z \neq 0 \Rightarrow x \neq z \lor y \neq z$$
We have to show that
\[ \neg \forall n (\alpha_n = 0) \vee \neg \forall n (\beta_n = 0). \] (4)

Define binary sequences \( \alpha' \) and \( \beta' \) by
\[
\alpha' n = 1 \overset{\text{def}}{\iff} \alpha n = 1 \wedge \forall k < n (\alpha k = 0 \wedge \beta k = 0)
\]
and
\[
\beta' n = 1 \overset{\text{def}}{\iff} \beta n = 1 \wedge \forall k < n (\alpha k = 0 \wedge \beta k = 0) \wedge \alpha_n = 0.
\]
Without loss of generality, we may assume that
\[ \alpha' 0 = \beta' 0 = 0. \]
Define sequences \( x \) and \( y \) by
\[
x_0 = y_0 = 0,
\]
and for positive \( n \),
\[
x_n = \begin{cases} 
k^{-1} & \text{if there exists } k \leq n \text{ with } \alpha' k = 1 \\
0 & \text{else}
\end{cases}
\]
and
\[
y_n = \begin{cases} 
k^{-1} & \text{if there exists } k \leq n \text{ with } \beta' k = 1 \\
0 & \text{else}
\end{cases}
\]
Note that
\[ x \] and \( y \) are real numbers
\[ x = 0 \iff \forall n (\alpha' n = 0) \]
\[ y = 0 \iff \forall n (\beta' n = 0) \]
\[ x = y \Rightarrow x = 0 \wedge y = 0 \]
\[ \neg (\forall n (\alpha' n = 0) \wedge \forall n (\beta' n = 0)) \]
The last statement follows from (3). So \( x \) and \( y \) are real numbers with \( x \neq y \). By DA, we obtain
\[ x \neq 0 \vee y \neq 0. \]
The case \( x \neq 0 \) implies \( \neg \forall n (\alpha' n = 0) \), which in turn implies \( \neg \forall n (\alpha n = 0) \).
The case \( y \neq 0 \) implies \( \neg \forall n (\beta n = 0) \). This concludes the proof of (4).
A formula $\Phi$ is called a $\Sigma^0_1$-formula if there exists a binary sequence $\alpha$ such that
$$\Phi \iff \exists n (\alpha n = 1).$$
A formula $\Phi$ is a $\Delta^0_1$-formula if there exist binary sequences $\alpha, \beta$ such that
$$\Phi \iff \exists n (\alpha n = 1) \iff \forall n (\beta n = 0).$$

The law of excluded middle, which says that for $\theta$-formulas we have $\Phi \lor \neg \Phi$, is denoted by $\theta$-LEM, for $\theta \in \{\Delta^0_1, \Sigma^0_1, \Pi^0_1\}$. The Markov Principle is the statement
$$\neg \forall n (\alpha n = 0) \Rightarrow \exists n (\alpha n = 1)$$
for binary sequences $\alpha$. It is easy to see that that
$$\Sigma^0_1\text{-LEM} \Rightarrow \Pi^0_1\text{-LEM} \Rightarrow \Sigma^0_1\text{-DML} \Rightarrow \Pi^0_1\text{-DML} \Rightarrow \Delta^0_1\text{-LEM}$$
and
$$\Sigma^0_1\text{-LEM} \Rightarrow \text{MP} \Rightarrow \Pi^0_1\text{-DML} \Rightarrow \Delta^0_1\text{-LEM}.$$ 

Note further that $\Pi^0_1\text{-DML}$ is equivalent to the axiom $\text{MP}^\lor$ in [2] and to the axiom LLPE in [4].

Overall, the observation that DA and $\Pi^0_1\text{-DML}$ are equivalent yields a rather clear position of DA in the landscape of constructive reverse mathematics. In particular, a proof of $\text{MP}^\lor$ in BISH would imply the inconsistency of Brouwer’s intuitionistic mathematics [3]. Moreover we obtain the following.

**Corollary 1.** The following statement on the Cauchy-reals are equivalent:

i) $\forall x, y (x \neq y \rightarrow x \leq y \lor y \leq x)$

ii) $\forall x, y, z (x \neq y \rightarrow x \neq z \lor y \neq z)$

**Proof.** The first of these statements is equivalent to $\text{MP}^\lor$, see [3, 4].

**References**


