Homogeneous structures, Lecture Notes, Fischbachau MALOA Workshop, September 5–11, 2010

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Introduction

I aim to give the flavour of a selection of topics based around Fraïssé's notion of *homogeneous structure.* This is an area connecting ideas from model theory, permutation group theory, combinatorics, descriptive set theory, complexity theory, and other subjects. I will touch superficially a number of these subjects, with little in depth. The background assumed will be basic first order logic (languages, structures, interpretations, compactness), but of course, some familiarity with other subjects will help. The notes below contain no original material.

Outline of lectures.

1. Homogeneous and ω-categorical structures. Fraïssé amalgamation. Examples and classification results.

2. Model theory of homogeneous and ω-categorical structures.

3. Automorphisms of homogeneous and ω-categorical structures. Polish groups. Recovering a structure from its automorphism group.

4. Structural Ramsey theory and homogeneous structures.

5. Ramsey theory, homogeneous structures, and topological dynamics.

1 Basic of homogeneous structures

Throughout, L will denote a first order language, usually assumed to have countably many relation, function and constant symbols. We say it is a *relational language* if it has no function or constant symbols, and that it is a *finite relational language* if in addition it has just finitely many relation symbols. I shall use symbols $M, N, \ldots$ for L-structures, using in general the same symbol for a structure and its domain. Given an L-structure M, the *theory* of M, denoted $\mathrm{Th}(M)$ is the set of L-sentences (i.e. L-formulas without free variables) which are true in M. We say that M, N are *elementarily equivalent*, and write $M \equiv N$, if $\mathrm{Th}(M) = \mathrm{Th}(N)$.

Definition 1.1 A first order structure M is *ω-categorical* if $|M| = \aleph_0$ and any countably infinite L-structure N with $\mathrm{Th}(M) = \mathrm{Th}(N)$ is isomorphic to M. A first order theory is ω-categorical if it is the theory of some ω-categorical structure.

Example 1.2 1. The structure $(\mathbb{Q}, <)$ is ω-categorical, by Cantor's back-and-forth argument: the theory includes the axioms of dense linear orders without endpoints, and by the back-and-forth argument, any two countable dense linear orderings without endpoints are isomorphic. For an example of a back-and-forth argument, see the proof of Proposition 1.5 below.

2. The structure (M, E), where $|M| = \aleph_0$ and E is an equivalence relation on M with infinitely many infinite equivalence classes and no finite classes, is ω-categorical. Given two countably infinite models (M, E) and (N, E), we may match up the E-classes of M with those of N, and choose a bijection between each E-class of M and the corresponding one of N. The union of these bijections is an isomorphism $M \to N$.

3. Let $\mathbb{F}_q$ be the field of q elements (q a prime power) and V be an $\aleph_0$-dimensional vector space over $\mathbb{F}_q$, viewed as a structure $(V, +, 0, (f_q)_{q \in \mathbb{F}_q})$. Then V is $\aleph_0$-categorical.

Now suppose that L is a relational language.

Definition 1.3 A structure M over a relational language L is *homogeneous* if $|M| = \aleph_0$ and every isomorphism between finite substructures of M extends to an automorphism of M.

Example 1.4 The examples $(\mathbb{Q}, <)$ and (M, E) (E an equivalence relation with infinitely many classes, all infinite) are homogeneous (Exercise). On the other hand, if M had infinitely many infinite E-classes together with a single class of size one, then it would not be homogeneous, since the class of size one is isomorphic to any other singleton, but cannot be moved by an automorphism.

Proposition 1.5 *Let M be a homogeneous structure over a finite relational language. Then M is ω-categorical.*

Proof. Let $N \equiv M$ be countably infinite. We build an isomorphism $M \to N$ by a back-and-forth argument, as the union of a sequence $(f_k : k > 0)$ of finite partial isomorphisms. So let $M = \{m_i : i \in \mathbb{N}\}$ and $N = \{n_i : i \in \mathbb{N}\}$. The key point is that for any finite substructure U of M, there is a sentence expressing that U embeds in M, and this sentence is true of N too, so U embeds also in N.

To start the construction, by the last paragraph there is some $n_i \in N$ such that $\{m_0\} \cong \{n_i\}$. So define $f_0(m_0) = n_i$ (we can make it canonical by choosing i minimal subject to this).

Now at step $2k$, we have a partial isomorphism $f_{2k-1} : U \to V$, where $U \subset M$ and $V \subset N$ have size at most $2k-1$. If $m_k \in \mathrm{dom}(f_{2k-1})$, put $f_{2k} = f_{2k-1}$. If not, find some $g_{2k} : U \cup \{m_k\} \to N$, and put $U' := g_{2k}(U)$. The map $g_{2k} \circ f_{2k-1}^{-1} : V \to U'$ is an isomorphism between finite substructures of N, so extends to an automorphism h of N. Let $n_i := h^{-1}(g_{2k}(m_k))$. Let $f_{2k} := f_{2k-1} \cup \{(m_k, n_i)\}$. Then f_{2k} is a partial isomorphism extending f_{2k-1}.

At odd steps, to construct f_{2k+1}, if $n_k \in \mathrm{ran}(f_{2k})$ put $f_{2k+1} = f_{2k}$, and otherwise, do the above argument with M and N reversed to put n_k into $\mathrm{ran}(f_{2k+1})$. At the end, put $f := \bigcup(f_k : k \in \mathbb{N})$. Then $f : M \to N$ is an isomorphism. $\square$

Next, we give a method for constructing richer examples of homogeneous structures, compared to those described so far.

Definition 1.6 Let L be a relational language.

(i) A class $\mathcal{C}$ of finite L-structures has the *joint embedding property* (JEP) if for any $A_1, A_2 \in \mathcal{C}$, there is $D \in \mathcal{C}$ and embeddings $f_1 : A_1 \to D$ and $f_2 : A_2 \to D$.

(ii) An *Age* is a class of finite L-structures, containing arbitrarily large finite members, and closed under isomorphism and substructure and having (JEP).

(iii) If M is an infinite L-structure, then the *age* of M, written $\mathrm{Age}(M)$, is the collection of all finite L-structures which embed into M.

(iv) An age $\mathcal{C}$ has the *amalgamation property* (AP), and is an *amalgamation class*, if, for any $A, B_1, B_2 \in \mathcal{C}$ and embeddings $f_1 : A \to B_1$ and $f_2 : A \to B_2$, there is $D \in \mathcal{C}$ and embeddings $g_1 : B_1 \to D$ and $g_2 : B_2 \to D$ such that $g_1 \circ f_1 = g_2 \circ f_2$ (on A). We say $\mathcal{C}$ has the *disjoint amalgamation property* if, here, D, g_1, g_2 can be chosen so that $g_1(B_1) \cap g_2(B_2) = g_1 \circ f_1(A)$, that is, no extra identifications are made.

The following result is due originally to Fraïssé [7].

Theorem 1.7 *Let L be a relational language.*

(i) If M is a countably infinite L-structure, then $\mathrm{Age}(M)$ *is an age. Conversely, given an age $\mathcal{A}$ over a countable relational language L, there is a countably infinite L-structure M with* $\mathrm{Age}(M) = \mathcal{A}$.

(ii) If M is a homogeneous L-structure, then $\mathrm{Age}(M)$ *has the amalgamation property.*

(iii) If $\mathcal{C}$ is an amalgamation class, then there is a unique (up to isomorphism) countably infinite homogeneous L-structure M with $\mathrm{Age}(M) = \mathcal{C}$.

The structure M arising in (iii) is known as the *Fraïssé limit* of $\mathcal{C}$.

Proof. (i) The first assertion is immediate. Let $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ be an age over L. We build M as a union of a chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ of members of

$\mathcal{A}$. To build the chain, put $M_0 = A_0$, and for each i let M_{i+1} be any member of $\mathcal{A}$ which contains copies of both A_{i+1} and M_i. This exists by (JEP), and we may suppose that $M_i \leq M_{i+1}$.

(ii) Let $A, B_1, B_2 \in \mathrm{Age}(M)$ with $f_i : A \to B_i$. There are embeddings $h_i : B_i \to M$. There is an isomorphism $k := h_1 \circ f_1 \circ f_2^{-1} \circ h_2^{-1} : h_2(f_2(A)) \to h_1(f_1(A))$, and by homogeneity k is extended by an automorphism k' of $\mathrm{Aut}(M)$. Put $D := h_1(f_1(B_1) \cup k'(h_2(f_2(B_2)))$. Put $g_1 = h_1 \circ f_1$ and $g_2 = k' \circ h_2 \circ f_2$.

(iii) The uniqueness assertion follows by a back-and-forth argument. For existence, we aim to build M so that for any $A, B \in \mathcal{C}$ with $A \leq B$, and any embedding $f : A \to M$, there is an embedding $g : B \to M$ extending f. (Given that M has this property, a back-and-forth argument yields that M is homogeneous.) Essentially, there are countably many such triples (f, A, B) to consider. We build M as a union of a chain $M_0 \leq M_1 \leq M_2 ...$ of members of M. At some stages, we use (JEP) just to ensure that $\mathrm{Age}(M) = \mathcal{C}$. At other stages, we consider a triple (f, A, B) with $f : A \to M_i$. We then amalgamate M_i and B over A (via the embeddings $f : A \to M_i$ and $\mathrm{id} : A \to B$) to obtain M_{i+1}, and we may assume the corresponding embedding $M_i \to M_{i+1}$ is the identity. □

Remark 1.8 1. Fraïssé's Theorem is the standard way to build homogeneous structures. Typically, one starts with an age $\mathcal{C}$, and aims to prove it has the amalgamation property. For this, given structures $B_1, B_2 \in \mathcal{C}$ with a common substructure A, we aim to build a structure D in $\mathcal{C}$ with domain $B_1 \cup B_2$, possibly, if necessary, identifying certain elements of $B_1 \setminus A$ with elements of $B_2 \setminus A$.

2. There are many variations on the Fraïssé construction.

First, we restricted to relational languages just for simplicity. One may work with more general first order languages with function and constant symbols, and replace the notion of 'finite structure' by 'finitely generated' structure.

Second, it is possible to formulate a version of Fraïssé's theorem, where one works with a restricted class of structures, and embeddings, satisfying reasonable properties; see e.g. [6]. This, with the notion of 'self-sufficient embedding', was exploited beautifully by Hrushovski in constructing several counterexamples to long-standing conjectures. In particular, he constructed an ω-categorical stable structure which is not ω-stable, and a non locally modular strongly minimal set which does not interpret an infinite field.

Example 1.9 1. Let $\mathcal{C}$ be the class of all finite (loopless, undirected) graphs, viewed as structures in a language with a single binary relation symbol. Then $\mathcal{C}$ is an amalgamation class. Given finite graphs B_1 and B_2 with a common subgraph A, we may form a graph on $B_1 \cup B_2$ (keeping $B_1 \setminus A$ and $B_2 \setminus A$ disjoint) however we like. For example, we may insist that there are no edges between $B_1 \setminus A$ and $B_2 \setminus A$. Let Γ denote the Fraïssé limit of $\mathcal{C}$. It can be seen that Γ is the unique countably infinite graph which has the following property $(*)_n$ for each $n \in \mathbb{N}^{>0}$

$(*)_n$ For any disjoint sets of vertices U, V of size n, there is a unique vertex adjacent to all members of U and to no members of V.

Let σ_n be a first order sentence expressing that $(*)_n$ holds. Then $\{\sigma_n : n > 0\}$ (together with axioms saying that R is symmetric and irreflexive) axiomatise $\mathrm{Th}(\Gamma)$. Also, it can be shown that for any n and k , if a_k is the number of graphs on vertex set $\{1, \ldots, k\}$ which satisfy σ_n, and b_k is the number of graphs on $\{1, \ldots, k\}$, then $\frac{a_k}{b_k} \to 1$ as $k \to \infty$. In particular, since the above sentences axiomatise $\mathrm{Th}(\Gamma)$, one obtain's Fagin's famous result, that for any sentence about graphs, the proportion of finite graphs which satisfy it tends to 0 or 1 (a 'zero-one law'), and those for which the limit is 1 are exactly those true of Γ. Furthermore, there is a natural probability measure on the collection of all graphs with domain $\mathbb{N}$, such that with probability 1, such a graph is isomorphic to Γ. We therefore call Γ *the random graph.*

There are many explicit constructions of the random graph. For example, consider the graph whose vertex set is $\mathbb{N}$, with vertex x joined to vertex y if and only if 2^x occurs in the binary expansion of y (or vice versa). This graph satisfies each axiom $(*)_n$, so is isomorphic to the random graph.

2. Fix $n \geq 3$, and let K_n denote the complete graph on n vertices. Let $\mathcal{C}_n$ denote the collection of all finite graphs which do not embed K_n (i.e. K_n is not isomorphic to an induced subgraph). Then $\mathcal{C}_n$ is an amalgamation class: amalgamate as in (1), requiring that $B_1 \cap B_2 = A$ and there are no edges between $B_1 \setminus A$ and $B_2 \setminus A$; in the union, any complete induced subgraph must lie in B_1 or in B_2, so cannot embed K_n. The resulting Fraïssé limit is Γ_n, the *universal homogeneous K_n-free graph.*

3. Let $\mathcal{C}$ be the collection of all finite partially ordered sets $(X, <)$. Then $\mathcal{C}$ has the amalgamation property: when amalgamating B_1 and B_2 over A, arrange that $B_1 \cap B_2 = A$, and let $<$ on $B_1 \cup B_2$ be the transitive closure of the union of the relations on B_1 and B_2; one must check that this is irreflexive. The Fraïssé limit is the *countable universal homogeneous poset.* Likewise, the collection of all finite total orders has the amalgamation property: argue at the above, at the end taking any totally ordered extension of the induced partial order of $B_1 \cup B_2$. The Fraïssé limit in this case is isomorphic to $(\mathbb{Q}, <)$.

4. Now consider *digraphs* as structures with a single binary relation R (sometimes just denoted $\to$) such that $\forall x \neg Rxx \wedge \forall x \forall y (Rxy \to \neg Ryx)$ holds. In particular, a partial order may be viewed as a digraph. A *tournament* is a digraph satisfying also the condition $\forall x \forall y (x \neq y \to (Rxy \vee Ryx))$.

First observe that the collection of all finite tournaments has the amalgamation property. Its Fraïssé limit, the *random tournament*, has many properties, such as a zero-one law, in common with the random graph.

Let S be any set of finite tournaments. Consider the class $\mathcal{C}_S$ of all finite digraphs which do not embed any member of S. Then $\mathcal{C}_S$ has the amalgamation property: when amalgamating B_1 and B_2 over A, we ensure $B_1 \cap B_2 = A$ and that if $x \in B_1 \setminus A$ and $y \in B_2 \setminus A$ then $\neg Rxy \wedge \neg Ryx$ (so any tournament embedding in the union must embed in B_1 or B_2).

Henson found an infinite set S of finite tournaments such that no member of S embeds in any other member. It follows that if S_1 and S_2 are distinct subsets of S then $\mathcal{C}_{S_1}$ and $\mathcal{C}_{S_2}$ are distinct amalgamation classes, so their Fraïssé limits are non-isomorphic. In particular, there are $2^{\aleph_0}$ non-isomorphic homogeneous digraphs.

5. We may view rational metric spaces (i.e. metric spaces with rational distances) as relational structures: for each $q \in \mathbb{Q}^{\geq 0}$ let R_q be a binary relation symbol read informally as $d(x, y) = q$. The collection of all finite rational metric spaces, with isometric embeddings, is then an amalgamation class. To see the amalgamation property, given a subspace A of $B_1 \cap B_2$, arrange $B_1 \cap B_2 = A$ and for $x \in B_1$ and $y \in B_2$ put $d(x, y) = \inf_{a \in A}(d(x, a) + d(y, a))$. The Fraïssé limit $\mathbb{U}_0$ is known as the *universal homogeneous rational metric space.* Its completion is $\mathbb{U}$, the Urysohn space. It is the unique complete separable metric space which is both homogeneous (any isometry between finite subspaces of $\mathbb{U}$ extends to an isometry of $\mathbb{U}$) and universal (embeds every complete separable metric space)

Finally, we mention some classification theorems for homogeneous structures.

Theorem 1.10 (Lachlan, Woodrow [13]) *Let Γ be a homogeneous graph. Then Γ or its complement is isomorphic to the random graph, the universal homogeneous K_n-free graph (for some $n \geq 3$) of to a disjoint union of complete graphs, all of the same size.*

Theorem 1.11 (Schmerl [20]) *Let $(P, <)$ be a homogeneous poset. Then P is isomorphic to $(\mathbb{Q}, <)$, to the universal homogeneous poset, to an infinite antichain, or, for some n with $1 \leq n \leq \aleph_0$, to a partial order obtained from $(\mathbb{Q}, <)$ be replacing each $q \in \mathbb{Q}$ by an antichain of size n (there are in fact two possible constructions here).*

Theorem 1.12 (Lachlan [14]) *(i) There is a homogeneous tournament T whose domain is a countably infinite dense subset of the unit circle, with no antipodal pairs, such that $x \to y$ iff $\arg(y/x) < \pi$.*

(ii) The only homogeneous tournaments are $(\mathbb{Q}, <)$, the universal homogeneous (or random*) tournament, and the tournament T in (i) above.*

The above list of results culminated in Cherlin's classification of the homogeneous digraphs in [5].

2 Model theory of homogeneous and ω-categorical structures.

Recall that if M is an L-structure, then an L-structure N is a substructure of M, written $N \leq M$, if any elements of M interpreting constant symbols lie in

N, the functions of N are the restrictions of the functions of M (so in particular N is closed under these functions), and for any n and relation R of arity n,

$$\{\bar{a} \in N^n : N \models R\bar{a}\} := \{\bar{a} \in N^n : M \models R\bar{a}\}.$$

If M, N are L-structures, with $M \leq N$, then M is an elementary substructure of N, written $M \preceq N$, if for every formula $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$, we have

$$M \models \phi(\bar{a}) \Leftrightarrow N \models \phi(\bar{a}).$$

If M is an L-structure, and $\bar{a} \in M^n$, then the (complete) type of $\bar{a}$, denoted $\mathrm{tp}(\bar{a})$ or $\mathrm{tp}^M(\bar{a})$, is $\{\psi(\bar{x}) : M \models \psi(\bar{a})\}$.

Fairly standard arguments with compactness give:

Lemma 2.1 *Let M be an L-structure, and let p be a set of L-formulas in free variables $x_1, \ldots, x_n$. Then the following are equivalent.*

(i) There is N with $M \preceq N$ and $a_1, \ldots, a_n \in N$, such that $p = \mathrm{tp}^N(\bar{a})$.

(ii) p is a maximal consistent set of L-formulas in $x_1, \ldots, x_n$ which contains $\mathrm{Th}(M)$.

We often refer to a type p satisfying the conditions of Lemma 2.1 as a type (or n-type) of $\mathrm{Th}(M)$. We say p is *realised* in M if there is $\bar{a} \in M^n$ with $p = \mathrm{tp}^M(\bar{a})$. Frequently, if $A \subset M$, we expand L to the language $L(A)$ containing constant symbols for elements of A. A type of $\mathrm{Th}((M, (a)_{a\in A}))$ in the language $L(A)$ will be referred to as a type of $\mathrm{Th}(M)$ *over* A. If T denotes the theory of M in L, then we write T_A for the theory of $(M, (a)_{a\in A})$ in $L(A)$. We write $S_n(T)$ for the set of all n-types of T. This is a compact totally disconnected topological space, where the basic open sets are determined by formulas $\phi(x_1, \ldots, x_n)$ and have the form $[\phi] := \{p \in S_n(T) : \phi \in p\}$. In particular, the topological compactness of $S_n(T)$ follows from the compactness theorem of first order logic. The space $S_n(T)$ may be viewed as the Stone space of the Boolean algebra of all formulas in $x_1, \ldots, x_n$ up to equivalence modulo T, that is, it is the space of ultrafilters of this Boolean algebra. (These observations may be viewed as exercises.)

Above, what we are calling a *type* is sometimes in the literature called a *complete type*, with the term *type* being reserved for arbitrary subsets of complete types (also called *partial types*).

The type p of T is *isolated* if $\{p\}$ is an isolated point in the above topological space; that is, $p = [\phi]$ for some $\phi \in p$. This means exactly that for every $\psi \in p$, $T \models \forall x_1 \ldots \forall x_n(\phi(\bar{x}) \to \psi(\bar{x}))$.

For any structure M, we denote by $\mathrm{Aut}(M)$ the automorphism group of M.

Theorem 2.2 (Ryll-Nardzewski Theorem) *Let M be a countably infinite structure, and put $T := \mathrm{Th}(M)$. Then the following are equivalent.*

(i) M is ω-categorical.

(ii) All types of T are isolated.

(iii) For each n, T has finitely many n-types.

(iv) For each n, there are finitely many formulas in variables $x_1, \ldots, x_n$ up to T-equivalence.

(v) For each n, M realises finitely many n-types of T.

(vi) For each n, $\mathrm{Aut}(M)$ has finitely many orbits in its induced action on M^n.

We shall mainly be using the equivalence $(i) \Leftrightarrow (vi)$. This of course yields another proof of Proposition 1.5. For over a finite relational language there are finitely many non-isomorphic structures on n elements, and if two n-tuples in the homogeneous L-structure M carry isomorphic L-structures then by homogeneity they lie in the same orbit of $\mathrm{Aut}(M)$.

The above theorem is essentially a corollary of Vaught's Omitting Types Theorem. The latter can be proved in the manner of the proof of the compactness theorem by adding Henkin constants, though the proof requires some care (it is omitted here).

Theorem 2.3 (Omitting Types Theorem) *Let T be a complete theory in countable language, and $\{p_i : i \in \mathbb{N}\}$ a countable set of non-isolated types of T. Then there is a countable model M of T which does not realise any of the p_i.*

Sketch (not very efficient!) proof of Theorem 2.2.

$(i) \Rightarrow (ii)$. If (ii) is false, then T has a non-isolated type p, so by Theorem 2.3 it has a countable model M not realising p. But T also has a countable model N realising p (e.g. by the downward Löwenheim-Skolem Theorem). The models M, N cannot be isomorphic, contradicting ω-categoricity.

$(ii) \Rightarrow (i)$. Suppose (ii), and let $M, N \models T$ be countable. We build an isomorphism $\alpha : M \to N$ by a back-and-forth argument. The key is to be able to extend finite elementary maps, so suppose $\bar{a} \in M^n$, $\bar{b} \in N^n$ with $\alpha(\bar{a}) = \bar{b}$, and that $a \in M \setminus \{a_1, \ldots, a_n\}$. Let $\phi(\bar{x})$ isolate $\mathrm{tp}^M(\bar{a})$ and $\psi(\bar{x}, y)$ isolate $\mathrm{tp}^M(\bar{a}a)$. So $T \models \forall \bar{x}(\phi(\bar{x}) \to \exists y \psi(\bar{x}, y))$. Now as α is elementary, $N \models \phi(\bar{b})$, so $N \models \exists y \psi(\bar{b}, y)$. Choose $b \in N$ with $N \models \psi(\bar{b}, b)$, and extend α by putting $\alpha(a) = b$.

$(ii) \Rightarrow (iii)$. Observe that a compact topological space in which all points are isolated must be finite.

$(iii) \Leftrightarrow (iv)$. Immediate as $S_n(T)$ is the Stone space of the Boolean algebra of formulas in $x_1, \ldots, x_n$ up to T-equivalence.

$(iii) \Rightarrow (v)$. Immediate.

$(v) \Rightarrow (ii)$. Suppose $p = \{\phi_i : i \in \mathbb{N}\}$ is a non-isolated type. For each $j \in \mathbb{N}$ the formula $\psi_j := \bigwedge_{i<j} \phi_i$ does not isolate p and it follows that there is $k_j > j$ such that $\psi_j \cup \{\neg \phi_{k_j}\}$ is consistent, so is realised by some $\bar{a}_j$ in M. Now choose natural numbers $i_0 < i_1 < \ldots$ such that $i_{k+1} > k_{i_j}$. Then $\bar{a}_{i_0}, \bar{a}_{i_1}, \ldots$ all realise distinct n-types in M.

$(vi) \Rightarrow (v)$. Immediate, as tuples in the same orbit satisfy the same formulas.

$(ii) \Rightarrow (vi)$. Show by a back-and-forth argument that if $\bar{a}, \bar{b} \in M^n$ then there is an automorphism α of M with $\alpha(\bar{a}) = \bar{b}$. We extend partial elementary maps $M \to M$ as in the proof of $(ii) \Rightarrow (i)$. $\Box$

Remark 2.4 Observe that the above proof shows that if M is ω-categorical and $\bar{a}, \bar{b} \in M^n$ realise the same type, then there is $\alpha \in \mathrm{Aut}(M)$ with $\alpha(\bar{a}) = \bar{b}$.

If M is a first order structure, and $n \in \mathbb{N}$, then a subset $X \subseteq M^n$ is a *definable set* is there is a formula $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ and $a_1, \dots, a_m \in M$ such that $X = \{\bar{x} \in M^n : M \models \phi(\bar{x}, \bar{a})\}$. The set X is A-definable, where $A \subset M$, if the parameters $a_1, \dots, a_m$ can be chosen from A. In particular, a set is $\emptyset$-definable if it is definable by a formula without parameters. Much of the emphasis in modern model theory is on the definable sets in structures: understanding their combinatorics and geometry in rather general situations, or understanding them in detail in particular structures. A key tool is quantifier elimination.

Definition 2.5 A complete first order theory T has *quantifier elimination* (QE) if for every formula $\phi(x_1, \dots, x_n)$ there is a quantifier-free formula $\psi(x_1, \dots, x_n)$ such that $T \models \forall x_1 \dots \forall x_n(\phi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n))$.

Quantifier elimination is an immensely helpful property in a first order theory. If the underlying language is reasonably simple, it enables us to understand the definable sets in a model of the theory, that is, the solution sets of first order formulas. Typically, if we wish to show that a first order theory has some nice property (e.g. that it is stable, or simple, or NIP, or ...) then a first step would be to prove a quantifier-elimination theorem, possibly first by adding some relation symbols to the language, interpreted in models of the theory by certain key formulas with quantifiers.

Proposition 2.6 *Let M be an ω-categorical structure over a relational language. Then M is homogeneous if and only if* $\mathrm{Th}(M)$ *has QE.*

Proof. Suppose first that $\mathrm{Th}(M)$ has QE. Let $\bar{a}, \bar{b} \in M^n$ and $f : \bar{a} \to \bar{b}$ be an isomorphism. By the Ryll-Nardzewski Theorem, there is a formula $\phi(\bar{x})$ isolating $\mathrm{tp}^M(\bar{a})$. By QE, we may suppose that ϕ is quantifier-free. Thus, as isomorphisms preserve quantifier-free formulas, $M \models \phi(\bar{b})$, so $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$, and so by Remark 2.4 there is an automorphism of M extending f.

In the other direction, assume that M is homogeneous, and let $\phi(x_1, \dots, x_n)$ be a formula. Let $p_1, \dots, p_k$ be the n-types, and suppose that of these, $p_1, \dots, p_l$ contain ϕ. Let ψ_i isolate p_i for each i. Up to T-equivalence, there are finitely many formulas in $x_1, \dots, x_n$, and in particular finitely many quantifier-free ones. If $\bar{a}, \bar{b} \in M^n$ satisfy the same quantifier-free formulas then they lie in the same $\mathrm{Aut}(M)$-orbit so satisfy all the same formulas. It follows easily that we may suppose each ψ_i is quantifier-free. But now ϕ is equivalent to the quantifier-free formula $\bigvee_{i=1}^{l} \psi_i$. $\Box$

3 Automorphism groups of homogeneous structures

Let $S := \mathrm{Sym}(\mathbb{N})$ denote the symmetric group on $\mathbb{N}$, that is the group of all permutations of $\mathbb{N}$. There is a natural topology on S: a basic open set is determined by a bijection f between two finite subsets of $\mathbb{N}$, and has the form $O_f := \{g \in S : g \text{ extends } f\}$. Thus, if $U := \mathrm{dom}(f)$, then O_f is a left coset of $S_{(U)} := \{g \in S : g|_U = \mathrm{id}|_U\}$ (the *pointwise stabiliser* of U). (Here, we write the group action on the left, so if a group G acts on the set X, then $g(x)$ is the image of $x \in X$ under $g \in G$. In general, for $A \subset X$ we write $G_{(A)}$ for the pointwise stabiliser in G of A.) This topology makes S into a topological group: that is, the multiplication map $S \times S \to S$ and inverse map $S \to S$ are both continuous. The topology is Hausdorff.

Proposition 3.1 *(i) A subgroup $G \leq S$ is closed (as a subset of S) if and only if there is a first order structure M with domain $\mathbb{N}$ such that $G = \mathrm{Aut}(M)$.*

(ii) If G is a closed subgroup of S, and $H \leq G$, then H is dense in G if and only if G and H have, for each $n \in \mathbb{N}^{>0}$, the same orbits in their induced actions on $\mathbb{N}^n$.

Proof. Exercise. In (i), to construct M, we introduce a relation symbol of arity n for each orbit of G on $\mathbb{N}^n$. $\square$

The above topology on S is metrisable. There are many ways to do this, but for example, we can define a metric d on S, putting $d(f, g) = \frac{1}{n+1}$ if n is least such that either $f(n) \neq g(n)$ or $f^{-1}(n) \neq g^{-1}(n)$. With this metric, any closed subgroup G of S is a complete separable metric space. In particular, such G is a *Polish group*, that is, a topological group such that the topology comes from a complete metric space which is *separable*, that is, has a countable dense subset.

Exercise 3.2 Show that a closed subgroup G of S is compact iff G has no infinite orbits on $\mathbb{N}$, and is locally compact iff, for some finite $A \subset \mathbb{N}$, $G_{(A)}$ has no infinite orbits on $\mathbb{N}$. Deduce that if M is homogeneous over a finite relational language (or just ω-categorical), then G is not locally compact.

We now turn to automorphism groups of homogeneous and ω-categorical structures. By Theorem 2.2 $(i) \Leftrightarrow (vi)$, any ω-categorical structure has a very rich automorphism group. In particular, it has size $2^{\aleph_0}$. (In fact, if G is any closed permutation group on a countably infinite set X such that for any finite $A \subset X$ we have $|G_{(A)}| > 1$, then $|G| = 2^{\aleph_0}$; this can be seen either directly, building a tree of height ω whose nodes are labelled by finite restrictions of elements of G and whose branches give elements of G, or by a topological argument.) For homogeneous structures this is not so in general. For example, let G be *any* closed subgroup of $\mathrm{Sym}(\mathbb{N})$ (for example the trivial group). Then there is a homogeneous structure M with domain $\mathbb{N}$ such that $G = \mathrm{Aut}(M)$: for each G-orbit on M^n, introduce an n-ary relation symbol interpreted by that orbit.

However, Fraïssé's theorem gives a powerful way of constructing interesting examples of large (closed) permutation groups. Just as a quick example, it is known (via the classification of finite simple groups) that any 6-transitive permutation group on a finite set X has the form $\mathrm{Sym}(X)$ or $\mathrm{Alt}(X)$. However, in the infinite case we have the following example.

Example 3.3 Let $k \geq 2$ be any positive integer, and let L be a language with a single k-ary relation R. Let $\mathcal{C}$ be the collection of all finite L-structures which are symmetric and irreflexive in the sense of Definition 4.4 below. Then $\mathcal{C}$ is an amalgamation class, so has a Fraïssé limit M. Any two $(k-1)$-tuples of distinct elements of M are isomorphic, so $\mathrm{Aut}(M)$ is $(k-1)$-transitive on M. However, there are distinct $a_1, \ldots, a_k, b_1, \ldots, b_k \in M$ with $M \models R\bar{a} \wedge \neg R\bar{b}$, and there is no $g \in \mathrm{Aut}(M)$ with $g(\bar{a}) = \bar{b}$, so $\mathrm{Aut}(M)$ is not k-transitive.

Example 3.4 If M is ω-categorical and $G = \mathrm{Aut}(M)$, then for each $k \in \mathbb{N}^{>0}$ G has a finite number $n_k(G)$ of orbits on the collection of *unordered* k-subsets of M. Many interesting combinatorial sequences arise as $(n_k(G))$ for some such automorphism group G, often through Fraïssé's Theorem – see examples in [4]. Just as one example, we can find a homogeneous structure M such that $n_k(\mathrm{Aut}(M)) = k!$ as follows. Let L be a language with two binary relation symbols $<_1$ and $<_2$, and consider the set $\mathcal{C}$ of all finite L-structures in which each of $<_1$ and $<_2$ is a total order of the domain (but with no assumed connection between the two orderings). Then $\mathcal{C}$ is an amalgamation class, and if M is the Fraïssé limit we find $n_k(\mathrm{Aut}(M)) = k!$ for each $k > 0$.

For ω-categorical structures, one may to some extent translate between the languages of model theory and permutation groups. For example, by Remark 2.4, if M is ω-categorical and $A \subset M$ is finite, and $\bar{b}, \bar{b}' \in M^n$, then $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{b}'/A)$ if and only if $\bar{b}$ and $\bar{b}'$ lie in the same orbit in the action of $\mathrm{Aut}(M)_{(A)}$ on M^n. And $X \subset M^n$ is A-definable (that is, the solution set of a formula with parameters from A) if and only if it is a union of $\mathrm{Aut}(M)_{(A)}$-orbits on M^n.

Given these translations, it is natural to ask to what extend an ω-categorical structure M can be *reconstructed* from $\mathrm{Aut}(M)$. For a precise question, we need to specify both how much information is given about $\mathrm{Aut}(M)$ (is it given as a permutation group, or a topological group, or an abstract group?) and how precisely we aim to recover M.

We can only hope to recover M up to the level of knowing the $\emptyset$-definable sets (of n-tuples); for example, the random graph and its complement, which are graph-theoretically very different, have the same automorphism group (as a permutation group). And it is immediate that if M and M' are ω-categorical structures with the same domain and $\mathrm{Aut}(M) = \mathrm{Aut}(M')$ (as permutation groups), then M and M' have the same $\emptyset$-definable sets, as these are just the unions of orbits of n-tuples.

Consider the following two structures, M and M'. Here M is a pure countable infinite set, in the empty language, and M' is a graph (in a language with a binary relation R), whose domain is the collection of 2-element subsets of M, with vertex $\{a_1, a_2\}$ joined to vertex $\{b_1, b_2\}$ iff $|\{a_1, a_2\} \cap \{b_1, b_2\}| = 1$. Clearly the automorphism group of M is $G := \mathrm{Sym}(M)$, and G acts as a group of automorphisms of M'. In fact, it is easily checked that $G = \mathrm{Aut}(M')$, essentially because M is interpretable in M' in the sense of Definition 3.5 below. These two actions of G are not isomorphic as permutation groups, for G is k-transitive for all k on M but not 2-transitive on M'. However, it turns out that the two actions give the same topology on G. For example, if $A \subset M$ is finite, pick distinct $b, c \in M \setminus A$, and let A' be the set of 2-element subsets of $A \cup \{b, c\}$, so $A' \subset M'$. Then $\mathrm{Aut}(M')_{(A')}$ (a basic open subgroup of $\mathrm{Aut}(M')$) is an open subgroup of the basic open subgroup $\mathrm{Aut}(M)_{(A)}$ of $\mathrm{Aut}(M)$. Conversely, if F' is a finite subset of M', let $F = \bigcup F'$ (a finite subset of M). Then $\mathrm{Aut}(M)_{(F)}$ is an open subgroup of $\mathrm{Aut}(M)$ properly contained in $\mathrm{Aut}(M')_{(F')}$.

What is the model-theoretic relationship between M and M'? For this, we need the following fundamental model-theoretic notion.

Definition 3.5 Let M and M' be first order structures over languages L and L' respectively. Then M' is *interpretable* in M if there is an $\emptyset$-definable subset X of M^n for some n, an $\emptyset$-definable equivalence relation E on M^n, and some bijection $f : M' \to M^n/E$ such that for each t and relation R of arity t on M', the following holds, where $\pi : X \to M^n/E$ is the natural map:

$$\{(\bar{x}_1, \dots, \bar{x}_t) \in M^{nt} : M' \models R(f^{-1} \circ \pi(\bar{x}_1) \dots, f^{-1} \circ \pi(\bar{x}_t))\}$$

is an $\emptyset$-definable subset of M^{nt} (and likewise for functions and constants of M').

We say that M and M' are *mutually interpretable* if each is interpretable in the other. In this case, there is an isomorphic copy M^* of M living in M (living in the copy of M' which lives in M, by composing the two interpretations). Likewise, there is an isomorphic copy M'^* of M' living in M'. We say that M and M' are *bi-interpretable* if the resulting isomorphisms $M \to M^*$ and $M' \to M'^*$ are $\emptyset$-definable (in M and M' respectively).

We now have the following theorem of Ahlbrandt and Ziegler [2] (proof omitted here). We say that two topological groups G, G' are *isomorphic as topological groups* if there is a group isomorphism $G \to G'$ which is also a homeomorphism.

Theorem 3.6 *Let M and M' be ω-categorical structures. Then M and M' are bi-interpretable iff* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(M')$ *are isomorphic as topological groups.*

This raises a further reconstruction question. Suppose M is ω-categorical, and $G = \mathrm{Aut}(M)$. To what extent is the topology on G canonical, i.e. to what extent can we recover the topology from the abstract group structure, and so recover M up to bi-interpretability?

Observe here that basic open subgroups all have *small index*, that is, index less than $2^{\aleph_0}$. Indeed, if $A \subset M$ is finite, and $\bar{a}$ is an enumeration of A (so

$G_{(A)} = G_{\bar{a}}$), then $\bar{a} \in M^n$ for some n, and as M^n is countable, so is $|G : G_{\bar{a}}|$, by the 'Orbit-Stabiliser Theorem' in elementary group theory. Hence, every open subgroup has countable index (as it contains a basic open subgroup). So we might hope to characterise the topology from the abstract group structure, by saying that the open subgroups are exactly those of countable index.

Definition 3.7 An ω-categorical structure M has the *small index property* (SIP) if every subgroup of $\mathrm{Aut}(M)$ of index $< 2^{\aleph_0}$ is open.

Theorem 3.8 *Let M and M' be ω-categorical structures, and suppose that M has the small index property and that* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(M')$ *are isomorphic as abstract groups. Then M and M' are bi-interpretable.*

Proof. Let $\alpha : \mathrm{Aut}(M) \to \mathrm{Aut}(M')$ be an isomorphism. By Theorem 3.6, it suffices to show that α is a homeomorphism. It is well-known that any continuous isomorphism between Polish groups is a homeomorphism. So we must show that α is continuous, that is, that α-preimages of open subgroups of $\mathrm{Aut}(M')$ are open in $\mathrm{Aut}(M)$. So let $H \leq \mathrm{Aut}(M')$ be open. Then $|\mathrm{Aut}(M') : H| \leq \aleph_0$, so as this is an abstract group property so respected by α, $|\mathrm{Aut}(M) : \alpha^{-1}(H)| \leq \aleph_0$. Thus, as M has the small index property, $\alpha^{-1}(H)$ is open in $\mathrm{Aut}(M)$, as required. □

The small index property was originally proved for some rather special ω-categorical structures, such as a pure set (Dixon, Neumann, and Thomas) and $(\mathbb{Q}, <)$ (Truss). Then a powerful approach emerged in the paper [10], where (SIP) was proved for ω-categorical ω-stable structures and for the random graph. We sketch this approach, in a simplified form.

If X is a complete metric space, then a subset Y of X is *comeagre* if it contains the intersection of countably many dense open subsets of X. A subset Y of X is *meagre* if $X \setminus Y$ is comeagre. The Baire Category Theorem states that if X is a complete metric space, then any comeagre subset of X is dense in X, so in particular non-empty. Thus, the intersection of any two comeagre sets is non-empty, so the collection of all comeagre subsets of X forms a filter of the power set of X. So comeagreness is a notion of largeness.

Definition 3.9 Let G be a Polish group. Then G has *ample homogeneous generic automorphisms* if, for every $n \in \mathbb{N}^{>0}$, in the action of G diagonally by conjugation on G^n, there is an orbit which is comeagre as a subset of G^n.

The existence of ample homogeneous generic automorphisms has many consequences.

Theorem 3.10 *Let M be an ω-categorical structure, and suppose that $G =$* $\mathrm{Aut}(M)$ *has ample homogeneous generic automorphisms. Then*

(i) [10] M has the small index property.

(ii) [10] G has uncountable cofinality*, that is, if $(G_i : i \in \mathbb{N})$ is an increasing sequence of proper subgroups of G, then $\bigcup_{i\in\mathbb{N}} G_i \neq G$.*

(iii) [12] G has the Bergman property*, that is, if $E \subset G$ with $\langle E \rangle = G$ and $1 \in E = E^{-1}$, then there is $k \in \mathbb{N}$ such that $G = E^k = \{e_1 \dots e_k : e_i \in E\}$.*

This theorem yields interesting fixed point theorems, pointed out in [12]. If M is ω-categorical and $G := \mathrm{Aut}(M)$ has ample homogeneous generic automorphisms, then G has Serre's property (FA), that is, if G acts on a combinatorial tree (a connected graph without circuits) without inversions (i.e. no element of G reverses an edge) then G has a global fixed point, that is, there is a vertex $v \in T$ such that $g(v) = v$ for all $g \in G$. Also, G has property (FH), that is, any action of G by isometries on a real Hilbert space has a global fixed point. See also the discussion of extreme amenability in Section 5.

How can we show that $\mathrm{Aut}(M)$ has ample homogeneous generic automorphisms? We have the following criterion, from [10].

Theorem 3.11 *Let M be ω-categorical, and $G := \mathrm{Aut}(M)$. Suppose that the following two conditions hold.*

(i) (extension property) For any finite $A \subset M$ and finite partial elementary maps $e_1, \dots, e_n$ between subsets of A, there is finite $B \subset M$ containing A and $f_1, \dots, f_n \in G$ such that each f_i fixes B setwise and extends e_i.

(ii) (amalgamation property for partial automorphisms) For any finite $A, B, C \subset M$ with $A \subseteq B \cap C$, there is $g \in G_{(A)}$, such that if f_1, f_2 are permutations of $g(B)$ and C respectively which are induced by $\mathrm{Aut}(M)$ and fix A setwise and agree on A, then $f_1 \cup f_2$ is an elementary map on $g(B) \cup C$.

Then G has ample homogeneous generic automorphisms.

For homogeneous structures M, condition (i) holds if the following completely natural condition holds for $\mathrm{Age}(M)$. We say that a class $\mathcal{C}$ of finite relational structures has the *extension property for partial automorphisms* (EP) if, for any $A \in \mathcal{C}$ and partial isomorphisms $e_1, \dots, e_n$ between substructures of A, there is $B \in \mathcal{C}$ containing A such that each e_i extends to an automorphism of B.

Condition (ii) obviously holds for the random graph: just choose a copy B' of B (isomorphic over A) so that $B \cap B' = C$ and there are no edges between $B \setminus A$ and $B' \setminus A$. More generally, it holds if $\mathcal{C}$ is a *free amalgamation class.* That is, given $A, B_1, B_2 \in \mathcal{C}$ and $f_1 : A \to B_1$, $f_2 : A \to B_2$, there is an amalgam D with corresponding embeddings $g_i : B_i \to D$ making the diagram commute, so that in addition, no tuple which meets both $g_1(B) \setminus g_1(f_1(A))$ and $g_2(B) \setminus g_2(f_2(A))$ satisfies any relation.

We end this section with a discussion of (EP). A very strong result on (EP) was obtained by Herwig and Lascar [9]. First, we introduce the notion of *weak substructure*, which is the analogue of the graph-theorist's notion of *subgraph* (as distinct from *induced subgraph*, which corresponds to the model theorist's notion of *substructure*). Here, it is convenient to distinguish between a structure $\mathcal{A}$ and its domain A. Given an n-ary relation symbol R and a structure $\mathcal{A}$ with domain A, we write R^A for $\{\bar{x} \in A^n : A \models R\bar{x}\}$. Now if L is a relational language and $\mathcal{A}$, $\mathcal{B}$ are L-structures, we say $\mathcal{A}$ is a *weak substructure* of $\mathcal{B}$ if $A \subseteq B$, and, for every n and relation symbol R of L of arity n, $R^{\mathcal{A}} \subseteq R^{\mathcal{B}}$. A class $\mathcal{K}$ of L-structures is called *monotone* if it is closed under weak substructures.

Theorem 3.12 *[9] Let $\mathcal{C}$ be a monotone free amalgamation class over a finite relational language L. Then $\mathcal{C}$ has (EP), so the automorphism group of the Fraïssé limit of $\mathcal{C}$ has ample homogeneous generic automorphisms.*

The proof of Theorem 3.12 is intricate, and it has close connections to a variety of other topics. It is a considerable refinement of an earlier proof, due to Hrushovski, of (EP) for the class of all finite graphs. I sketch below a short proof, from [9], that the class of finite graphs has (EP). In fact, the text below is taken almost verbatim from [3]. It proves a slight strengthening of (EP), which ensures that if Γ is the random graph, then $\mathrm{Aut}(\Gamma)$ has a dense *locally finite* subgraph. Solecki has shown that the corresponding strengthening of Theorem 3.12 also holds, so the Fraïssé limits of free homogeneous structures also admit a locally finite group of automorphisms which is dense in the full automorphism group.

Lemma 3.13 *Let Δ be a finite graph, and $G := \mathrm{Aut}(\Delta)$. Then the following hold.*

(i) For any set T of vertices of Δ, there is a finite graph Δ_T such that Δ is an induced subgraph of Δ_T, there is an embedding $\phi : G \to \mathrm{Aut}(\Delta_T)$ such that $\phi(g)$ extends g for each $g \in G$, and Δ_T has a vertex γ with $N(\gamma) \cap \Delta = T$.

(ii) There is a finite graph Δ' such that Δ is an induced subgraph of Δ', every partial isomorphism between subgraphs of Δ extends to an automorphism of Δ', and there is a monomorphism $\phi : G \to \mathrm{Aut}(\Delta')$ such that $\phi(g)$ extends g for all $g \in G$.

Proof. (i) Let $\{T_1 := T, T_2, \ldots, T_m\}$ be the orbit containing the element T, in the action of G on the power set $\mathcal{P}(\Delta)$. Let Δ_T have vertex set $\Delta \cup \{\delta_1, \ldots, \delta_m\}$, where $\delta_1, \ldots, \delta_m$ are distinct vertices not in Δ. For adjacency on Δ_T, we specify for each k that $N(\delta_k) = T_k$. (So there are no edges within $\{\delta_1, \ldots, \delta_m\}$; note that if $T_1 = \emptyset$ then $m = 1$.) Define $\phi : G \to \mathrm{Aut}(\Delta_T)$ as follows: for each $g \in G$ and $k \in \{1, \ldots, m\}$, if $T_k^g = T_{k'}$ then let $\delta_k^g = \delta_{k'}$; also $\phi(g)|_\Delta := g$. This is a group embedding.

(ii) Let $n_1, \ldots, n_r$ be the valencies of vertices in Δ, with $n_1 < \ldots < n_r$. Put $n := n_r$. For $k = 1, \ldots, r$, let

$$\Sigma_k := \{\gamma \in \Delta : \gamma \text{ has valency } n_k\}.$$

For each $k = 1, \ldots, r-1$, let Λ_k be a set of size $n - n_k$, chosen so that $\Delta, \Lambda_1, \ldots \Lambda_{r-1}$ are disjoint. Let Δ'' be the graph with vertex set $\Delta \cup \Lambda_1 \cup$

$\ldots \cup \Lambda_{r-1}$, extending Δ, so that for $k = 1, \ldots, r-1$, the vertices of Λ_k are adjacent to all vertices of Σ_k and to no other vertices. Then every vertex of Δ has exactly n neighbours in Δ''.

Let Δ' be the graph whose vertices are the n-element sets of edges from Δ'', with two Δ'-vertices $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ adjacent if and only if $|\{e_1, \ldots, e_n\} \cap \{e'_1, \ldots, e'_n\}| \geq 1$. There is a graph embedding $\Delta \to \Delta'$ given by $\gamma \mapsto \{e_1, \ldots, e_n\}$, where $e_1, \ldots, e_n$ are the edges in Δ'' with γ as an end point (Exercise). For convenience, we identify such γ with $\{e_1, \ldots, e_n\}$, so Δ is a subgraph of Δ'. To define $\psi : \mathrm{Aut}(\Delta) \to \mathrm{Aut}(\Delta')$, observe that we may regard $\mathrm{Aut}(\Delta)$ as a subgroup of $\mathrm{Aut}(\Delta'')$ by letting each element of $\mathrm{Aut}(\Delta)$ fix each vertex in $\Lambda_1 \cup \ldots \cup \Lambda_{r-1}$. Clearly, there is a natural homomorphism $\mathrm{Aut}(\Delta'') \to \mathrm{Aut}(\Delta')$, and this induces a homomorphism $\psi : \mathrm{Aut}(\Delta) \to \mathrm{Aut}(\Delta')$ which will be injective.

To complete the proof, we show if $\bar{\alpha}$ and $\bar{\beta}$ are tuples of Δ such that the map $\bar{\alpha} \mapsto \bar{\beta}$ is a graph isomorphism, then there is $g \in \mathrm{Aut}(\Delta')$ mapping $\bar{\alpha}$ to $\bar{\beta}$. To see this, let E be the edge set of Δ''. Any permutation of E induces an automorphism of Δ, and we must choose a permutation g^* of E so that the induced automorphism g of Δ maps $\bar{\alpha}$ to $\bar{\beta}$. If e is an edge between two elements α_r, α_s of $\bar{\alpha}$, then e^{g^*} is the edge between β_r and β_s. For other α_t from $\bar{\alpha}$, there is a bijection between the set of elements of E with α_t as one endpoint, and the other endpoint outside $\bar{\alpha}$, and the corresponding subset of E consisting of edges between β_t and vertices outside $\bar{\beta}$. This exists as the sets have the same cardinality. We may choose the permutation g^* of E so that it simultaneously extends all such bijections. For any such g^*, the induced automorphism g of Δ' will map $\bar{\alpha}$ to $\bar{\beta}$.

4 Structural Ramsey theory

We turn in this section to some topics in Ramsey theory closely connected to homogeneous structures. Much of the presentation here is taken from the excellent survey [18]. Applications to topological dynamics will then be discussed in the final section.

Below, if X is a (finite) set and k a positive integer, we write $\binom{X}{k}$ for the set of k-element subsets of A. More generally, if X and A are finite structures, then $\binom{X}{A}$ denotes the collection of all substructures of X which are isomorphic to A.

Recall first the primordial version of Ramsey's Theorem.

Theorem 4.1 (Finite Ramsey Theorem) *Let n, k, t be positive integers. Then there is a positive integer $N = N(n, k, t)$ such that if X is a set of size N and $\binom{X}{k} = S_1 \cup \ldots \cup S_t$ is a partition of the set of k-subsets of X into t parts, then there is a subset Y of X of size n all of whose k-subsets lie in the same S_i.*

We view the partition $\binom{X}{k} = S_1 \cup \ldots \cup S_t$ as a *colouring* of $\binom{X}{k}$, and say that Y is a *monochromatic* subset of X. We express the conclusion of the above theorem with the standard arrow notation $N \to (n)^k_t$.

I describe here versions of the finite Ramsey theorem where numbers are replaced by finite structures (here assumed to be relational). This is work of a number of mathematicians, but Nesetril and Rödl are central to its development, with a key result also proved independently by Abramson and Harrington, and related contributions by Prömel.

Definition 4.2 (i) Let A, B, C be finite relational structures in a fixed relational language L, and let t be a positive integer. We write $C \to (B)^A_t$ if, for every partition $\binom{C}{A} = S_1 \cup \ldots \cup S_t$ there is a substructure B' of C isomorphic to B such that, for some $i \in \{1, \ldots, t\}$, $\binom{B}{A} \subseteq S_i$.

(ii) Let $\mathcal{K}$ be a class of finite L-structures, closed under isomorphism. If $A \in \mathcal{K}$, then $\mathcal{K}$ has the *A-Ramsey property* if for every $B \in \mathcal{K}$ and positive integer t, there is $C \in \mathcal{K}$ such that $C \to (B)^A_t$. More generally, $\mathcal{K}$ is a *Ramsey class* if $\mathcal{K}$ has the A-Ramsey property for every $A \in \mathcal{K}$.

Observe that the original finite Ramsey theorem is just the statement that the collection of all finite sets (in the empty language) is a Ramsey class. Many other versions of Ramsey's theorem can be deduced from theorems about Ramsey classes.

Apart from the above observation that finite sets are Ramsey classes, positive results about Ramsey classes really have to be about rigid structures, and are therefore usually taken to be about totally ordered structures. For example, we have the following, from [18, Theorem 5.1]. The proof depends on the 'Ordering Lemma', which is false in general for classes of ordered structures – this is why positive results about Ramsey structures are generally about classes of ordered structures.

Proposition 4.3 *Let $\mathcal{K}$ be the class of finite graphs, $A \in \mathcal{K}$, and suppose that $\mathcal{K}$ has the A-Ramsey property. Then A is a complete graph or a null graph (i.e. an independent set).*

Sketch Proof, from [18]. Suppose that $A \in \mathcal{K}$ is not complete or null. Then there are two isomorphic copies A_1 and A_2 of A equipped with total orderings $<_1$ and $<_2$ of their vertex sets, such that $(A_1, <_1)$ and $(A_2, <_2)$ are non-isomorphic. Now let A' be the disjoint union of A_1 and A_2 and $<$ a total order of A' extending the $<_i$. It can be shown (the 'Ordering Lemma 5.2' of [18]) that there is a graph B such that for any total order $\prec$ of the vertex set of B, there is an embedding $(A', <) \to (B, \prec)$.

Now suppose for a contradiction that there is graph C such that $C \to (B)^A_2$. Let $\prec$ be any ordering of C, and define a partition $\binom{C}{A} = S_1 \cup S_2$, putting, for $A^* \in \binom{C}{A}$, $A^* \in S_1$ if $(A^*, \prec|_{A^*}) \cong (A_1, <_1)$, and $A^* \in S_2$ otherwise. Then there is no monochromatic substructure of C isomorphic to B. Indeed, the order $\prec$ on C induces an ordering of B, and there is an embedding $(A', <) \to (B, \prec)$.

The copy of A_1 in A' has image in B coloured S_1, and the copy of A_2 has image coloured S_2. $\Box$

We now formulate some positive results about Ramsey classes. There is a body of such results, stemming from many papers, such as [1, 15, 16, 17, 18, 19]. We just try to get across a few of the ideas.

Definition 4.4 Let L is a relational language, and $L' = L \cup \{<\}$. The binary relation symbol $<$ will always be interpreted by a total order of the domain. An L-structure A is *symmetric* if whenever $A \models R\bar{x}$ and $\bar{x}'$ is obtained from $\bar{x}$ by permuting the entries x_i, then $A \models R\bar{x}'$. We say A is *irreflexive* if, whenever $A \models R\bar{x}$, all the entries of $\bar{x}$ are distinct. We let $\mathcal{K}$ be the class of all finite symmetric irreflexive L-structures, and $\mathcal{K}'$ be the class of expansions of members of $\mathcal{K}$ to L', with $<$ interpreted by a total order.

Theorem 4.5 (Nesetril, Rödl) . *$\mathcal{K}'$ is a Ramsey class.*

We sketch the proof, from [18]. The proof technique below appears to be very flexible and adaptable. First, if $\mathcal{A}$ is a member of $\mathcal{K}'$, an *a-partite system* is a pair $\mathcal{A}' = (\mathcal{A}, (X_i)_{i=1}^a)$ (viewed as a first order structure, so with the X_i viewed as unary relations outside L', or better, as the equivalence classes of an equivalence relation on the domain, corresponding to a symbol outside L') such that $X_1 < ... < X_a$ and for each relation symbol R of L, tuple $\bar{x}$ with $\mathcal{A} \models R\bar{x}$, and $i = 1, \ldots, a$, at most one of the x_j lies in X_i. We say $\mathcal{A}'$ is a *transversal* if $|X_i| = 1$ for each i. The a-partite system $\mathcal{A}'$ is a *subsystem* of the b-partite system $\mathcal{B}' = (\mathcal{B}, (Y_i)_{i=1}^b)$ if there is an order-preserving injection $f : \{1, \ldots, a\} \to \{1, \ldots, b\}$, $X_i \subseteq Y_{f(i)}$ for each $i = 1, \ldots, a$, and $\mathcal{A}$ is a substructure of $\mathcal{B}$ (this is why we prefer to view the partition as given by an equivalence relation rather than by unary predicates). Also, given a subset Y of $X_1 \cup \ldots \cup X_a$, the *trace* of Y is $\{i : Y \cap X_i \neq \emptyset\}$.

Lemma 4.6 *Let t be a positive integer and let $\mathcal{A}', \mathcal{B}'$ be a-partite systems, with $\mathcal{A}'$ a transversal. Then there is an a-partite system $\mathcal{C}'$ with $\mathcal{C}' \to (\mathcal{B}')_t^{\mathcal{A}'}$.*

Sketch proof. We omit this. The proof uses the Hales-Jewett Theorem.

Sketch proof of Theorem 4.5. Fix $A, B \in \mathcal{K}$, and a positive integer t. We may suppose that $\mathcal{A}$ is a transversal a-partite system, and $\mathcal{B}$ is a transversal b-partite system, so $|A| = a$ and $|B| = b$. Let $B = \{y_1, \ldots, y_b\}$. We aim to construct $\mathcal{C}$ with $\mathcal{C} \to (\mathcal{B})_t^{\mathcal{A}}$. Let $p = \mathrm{Min}\{n : n \to (b)_t^a\}$ (with respect to the 'primordial Ramsey Theorem 4.1) and $q = \binom{p}{a}$, and let $S^1, \ldots, S^q$ enumerate the a-subsets of $\{1, \ldots, p\}$. We build inductively a sequence of p-partite systems $\mathcal{C}^0, \ldots, \mathcal{C}^q$, and will arrange that $\mathcal{C}$ can be taken to be $\mathcal{C}^q$.

The p-partite system $\mathcal{C}^0$ is chosen with parts $C_1^0 < \ldots < C_p^0$ so that for any $1 \leq i_1 < \ldots < i_b \leq p$, the induced b-partite system on $C_{i_1}^0, \ldots, C_{i_b}^0$ contains a copy of B. We may build $\mathcal{C}_0$ by taking many disjoint copies of $\mathcal{B}$.

Suppose some $\mathcal{C}^k$, with parts $\mathcal{C}^k_1, \dots, \mathcal{C}^k_p$ has been constructed. Let D^{k+1} be the a-partite system which is the substructure of C^k induced on the parts $\mathcal{C}^k_i$ such that $i \in S^{k+1}$. By Lemma 4.6, there is an a-partite system E^{k+1} such that

$$E^{k+1} \to (D^{k+1})^A_t. \quad (*)$$

Each copy of D^{k+1} in E^{k+1} lies in some copy of $\mathcal{C}^k$, and we may glue these copies of $\mathcal{C}^k$ together, making identifications only where forced, so that distinct copies of C^k intersect in a subset of E^{k+1}. The resulting p-partite system, obtained from this amalgamation is called $\mathcal{C}^{k+1}$. Note that here we are using the amalgamation property of $\mathcal{K}'$, to ensure that each $\mathcal{C}^k$ lies in this class. Finally, let $\mathcal{C} := \mathcal{C}^q$.

We claim that $\mathcal{C} \to (B)^A_t$. To see this, we argue by downwards induction on k, finding successively copies of $\mathcal{C}^{q-1}, \mathcal{C}^{q-2}, \dots, \mathcal{C}^0$, denoted $\mathcal{C}^{q-1,*}, \mathcal{C}^{q-2,*}, \dots \mathcal{C}^{0,*}$ such that, for each k, all copies of $\mathcal{A}$ in $\mathcal{C}^k$ with trace S^k have the same colour. It follows that in $\mathcal{C}^{0,*}$, the colour of a copy of A depends only on its trace – call this the *colour* of the trace (which is an a-subset of $\{1, \dots, p\}$). Let the parts of $\mathcal{C}^{0,*}$ be denoted by $\mathcal{C}^{0,*}_1, \dots \mathcal{C}^{0,*}_p$. Now as $p \to (b)^a_t$, there is a subset of $\{1, \dots, p\}$ of size b, say $\{i_1, \dots, i_b\}$, such that all a-subsets of $\{i_1, \dots, i_b\}$ have the same colour. By our assumption on $\mathcal{C}^0$, the induced b-partite system on $\mathcal{C}^{0,*}_{1_1} \cup \dots \cup \mathcal{C}^{0,*}_{i_b}$ contains a copy of B, and it follows that all copies of A in B have the same colour. $\square$

Below, if $\mathcal{C}$ is a class of finite L-structures, then $L' = L \cup \{<\}$ and $(\mathcal{C}, <)$ is the class of all L'-structures $(A, <)$ such that $A \in \mathcal{C}$ and $<$ is a total order of A. The following theorem in fact has a slightly more general version. The class $\mathcal{C}'$ should consist of members of $\mathcal{C}$ equipped with 'admissible orderings'.

Theorem 4.7 *Let $\mathcal{C}$ be a monotone class of finite structures over the relational language L, and suppose that $\mathcal{C}$ is closed under isomorphism and has (JEP). Then the following are equivalent.*

(i) The class $(\mathcal{C}, <)$ is a Ramsey class.

(ii) $\mathcal{C}$ is a free amalgamation class.

Note that a monotone class $\mathcal{C}$ satisfying (ii) above will have the form $\mathrm{Forb}_{\mathcal{K}}(\mathcal{F})$ for some set $\mathcal{F}$ of finite irreducible L-structures. Examples of such $\mathcal{C}$ include the class of all finite k-hypergraphs (for any fixed k), the class of all K_n-free graphs (for any fixed integer $n \geq 3$), and, for any set S of finite tournaments, the class of all finite digraphs not embedding any member of S.

In this theorem, an idea of a proof of $(ii) \Rightarrow (i)$ is given by the proof above of Theorem 4.5 sketched above. For an idea of $(i) \Rightarrow (ii)$ note the following, taken from [17, Theorem 1.2]. It yields that $(\mathcal{C}, <)$ above is an amalgamation class, and using the 'richness' condition mentioned in [17, Theorem 1.2(ii)], one also gets that, assuming (i) in Theorem 4.7, $\mathcal{C}$ (in the language L) is also an amalgamation class, and in fact has disjoint amalgamation.

Proposition 4.8 *Let $\mathcal{C}$ be an age of ordered structures over a finite relational language, and suppose that $\mathcal{C}$ is a Ramsey class. Then $\mathcal{C}$ has the amalgamation property.*

Proof (from [17]). Consider $A, B_1, B_2 \in \mathcal{C}$ with embeddings $f_i : A \to B_i$ (for $i = 1, 2$). By (JEP), there is $E \in \mathcal{C}$ and embeddings $g_i : B_i \to E$.

As $\mathcal{C}$ is a Ramsey class, there is $D \in \mathcal{C}$ with $D \to (E)^A_2$. We now partition $\binom{D}{A}$ into parts S_1, S_2 as follows. Note that as the members of $\mathcal{C}$ are rigid (for they are ordered) if $A' \in \binom{D}{A}$ then there is a *unique* isomorphism $A \to A'$. For $A' \in \binom{D}{A}$, put $A' \in S_1$ if there is an embedding $h : E \to D$, such that $A' = h \circ g_1 \circ f_1(A)$. Otherwise, $A \in S_2$.

By the Ramsey property, there is $E' \in \binom{D}{E}$ and an isomorphism $h' : E \to E'$ such that $X = \{h' \circ f(A) : f : A \to E \text{ an embedding}\}$ is a subset of either S_1 or S_2. In fact, $X \subseteq S_1$, since $A' := h' \circ g_1 \circ f_1(A) \in S_1$. Now let $A^* := h' \circ g_2 \circ f_2(A)$. Then also $A^* \leq E'$, so $A^* \in S_1$. Thus, there is $h : E \to D$ such that $A^* = h \circ g_1 \circ f_1(A)$. Putting $g_1^* = h \circ g_1$ and $g_2^* = h' \circ g_2$, we see that D is an amalgam of B_1 and B_2 via the maps g_1^* and g_2^*. $\square$

5 More on Polish groups

We aim here to give a brief account of material in [11], which uses the Ramsey theory of the last section to obtain remarkable results in Polish groups. The motivation comes from topological dynamics.

We consider a Hausdorff topological group G, and its G-flows, that is, its continuous actions on compact Hausdorff spaces X. A G-flow is *minimal* if every orbit is dense. We claim that every G-flow contains a minimal subflow. For given a G-flow X_0, choose a G-orbit O_0 in X_0, and let X_1 be the closure of O_0 in X_0. Then X_1 is compact and G-invariant so is a G-flow, so we may iterate to form a chain of G-flows $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$. Continuing transfinitely, and using compactness of X_0 and Zorn's lemma, we find that X_0 contains a minimal G-flow.

In fact, given the group G, there is a universal minimal G-flow $M(G)$, such that for any minimal G-flow Y, there is a homomorphism (i.e. continuous map respecting the G-actions) $M(G) \to Y$. The purpose of [11] is to study $M(G)$ for various G.

Definition 5.1 The topological group G is *extremely amenable* if $|M(G)| = 1$.

Note that extreme amenability of G is equivalent to the condition that every G-flow has a fixed point. There has been interest in extreme amenability for a long time – at least since 1966. By a theorem of Veech, any *locally compact* G has a free G-flow (meaning that for any $g \in G$ and $x \in X$, $g(x) = x \Rightarrow g = 1$) and so cannot be extremely amenable. Examples of extremely amenable groups emerged in work of Herer and Christensen in 1975, and many examples are now known.

One theorem in [11] is the following. An *order class* is a class of relational structures over a language including a symbol $<$, interpreted by a total order on each member of the class.

Theorem 5.2 *Let* $G \leq \mathrm{Sym}(\mathbb{N})$ *be closed, with the induced topology. Then the following are equivalent.*

(i) G *is extremely amenable*

(ii) G *(in the given action on* $\mathbb{N}$*) is the automorphism group of the Fraïssé limit of a Fraïssé order class with the Ramsey property.*

As a first observation, let $G \leq \mathrm{Sym}(\mathbb{N})$ be closed, and consider the set X of all total orders on $\mathbb{N}$. Each total order is a set of ordered pairs, that is an element of the power set of $\mathbb{N}^2$, and this power set may be identified with $Y := \{0,1\}^{\mathbb{N}^2}$. Thus, X may be viewed as a subset of $\{0,1\}^{\mathbb{N}^2}$. If we give $\{0,1\}$ the discrete topology and Y the product topology, then by Tychonoff's theorem Y is compact, and easily X is a closed subset of Y, so is also compact. Now the action of G on $\mathbb{N}$ induces a continuous action on X. In particular, we have the following, since if G is extremely amenable then G fixes an element of X.

Lemma 5.3 *Any extremely amenable closed subgroup of* $\mathrm{Sym}(\mathbb{N})$ *preserves a total ordering of* $\mathbb{N}$*.*

We now define a version of the Ramsey property in terms of permutation groups. Let $G \leq \mathrm{Sym}(\mathbb{N})$ be closed. Then a *G-type* is a set of the form $\{g(F) : g \in G\}$, where $F \subset \mathbb{N}$ is finite and non-empty. If ρ, σ are G-types we write $\rho \leq \sigma$ if $\forall F \in \rho \exists F' \in \sigma (F \subseteq F')$. If $\rho \leq \sigma$ are G-types, and $F \in \sigma$, put

$$\binom{F}{\rho} := \{F' \subseteq F : F' \in \rho\}.$$

Then, if $\rho \leq \sigma \leq \tau$ are G-types, $k \in \mathbb{N}^{>1}$, write $\tau \to (\sigma)^{\rho}_{k}$ if, for every $F \in \tau$ and colouring $c : \binom{F}{\rho} \to \{1, \ldots, k\}$, there is $F_0 \in \binom{F}{\sigma}$ which is monochromatic, i.e. such that c restricted to $\binom{F_0}{\rho}$ is monochromatic. Now G has the *Ramsey property* if for any G-types $\rho \leq \sigma$ and any $k \in \mathbb{N}^{>1}$ there is a G-type τ with $\tau \to (\sigma)^{\rho}_{k}$.

We aim next to prove

Proposition 5.4 *Any extremely amenable closed subgroup* G *of* $\mathrm{Sym}(\mathbb{N})$ *has the Ramsey property.*

For this, we first prove the following lemma.

Lemma 5.5 *Let G be an extremely amenable closed subgroup of* Sym($\mathbb{N}$)*, let $\rho \leq \sigma$ be G-types, let $k \in \mathbb{N}^{>1}$ and let $c : \rho \to \{1, \ldots, k\}$ be a colouring. Then there is $F \in \sigma$ such that c is constant on $\binom{F}{\rho}$.*

Proof. Put $Y = \{1, \ldots, k\}^{\rho}$, a compact set (as usual, with the Tychonoff topology induced from the discrete topology). Then G acts (continuously) on Y: if $p \in Y$ and $x \in \rho$, put $g \cdot p(x) = p(g^{-1}(x))$. Now $c \in Y$, so we may let $X := \overline{G.c}$, the closure of the G-orbit on Y containing c. Thus, by extreme amenability, G fixes some $\gamma \in Y$. Since G is transitive on ρ, it follows that $\gamma : \rho \to \{1, \ldots, k\}$ is a constant function, say taking value $i \in \{1, \ldots, k\}$.

Now let $F_0 \in \sigma$, and let $A := \{F' \in \rho : F' \subseteq F_0\}$, a finite subset of ρ. Since $\gamma \in \overline{Gc}$, there is $g \in G$ such that $g^{-1} \cdot c|A = \gamma|A$; that is, $c(g(F')) = \gamma(F') = i$ for all $F' \in A$. So it suffices to put $F := g(F_0)$. □

Proof of Proposition 5.4. By an induction on k, it suffices to prove the Ramsey property with $k = 2$. So let ρ, σ be G-types with $\rho \leq \sigma$. Suppose for a contradiction that there is no G-type $\tau \geq \sigma$ with $\tau \to (\sigma)^{\rho}_{2}$. Let $F_0 \in \sigma$ be fixed. Then for every finite $E \supset F_0$ there is a function $c_R : \binom{E}{\rho} \to \{1, 2\}$ which is not constant on any $F \in \binom{E}{\sigma}$. Let I be the set of all finite non-empty subsets of $\mathbb{N}$, and pick an ultrafilter $\mathcal{U}$ on I with the property that for every finite $F \subset \mathbb{N}$, $\{E \in I : F \subseteq E\} \in \mathcal{U}$.

We define a colouring $c : \rho \to \{1, 2\}$ as follows. For each $D \in \rho$, either $\{E \supset D \cup F_0 : c_E(D) = 1\} \in \mathcal{U}$, or $\{E \supset D \cup F_0 : c_E(D) = 2\} \in \mathcal{U}$. In the first case put $c(D) = 1$, and in the second, put $c(D) = 2$.

By Lemma 5.5, there is $F \in \sigma$ such that c is constant on $\binom{F}{\rho}$, say with value i. It follows that if $D \in \binom{F}{\rho}$, then $A_D := \{E \supseteq F \cup F_0 : c_E(D) = c(D) = i\} \in \mathcal{U}$. Then also $Z := \bigcap_{D \in \binom{F}{\rho}} A_D \in \mathcal{U}$, so is non-empty. Pick $E \in Z$. Then $E \supseteq F_0 \cup F$ is finite, and for each $D \in \binom{F}{\rho}$, $c_E(D) = c(D) = i$. Thus c_E is constant on $F \in \binom{E}{\sigma}$, a contradiction. □

Finally, we prove part of Theorem 5.2.

Proof of Theorem 5.2 (i) ⇒ (ii). Assume that G is extremely amenable. By introducing relation symbols for orbits on n-tuples, we may suppose that there is a homogeneous structure M with domain $\mathbb{N}$ such that $G = \mathrm{Aut}(M)$. By Lemma 5.3, G preserves a total order $<$ on M, and we may suppose $<$ interprets a relation symbol of the language. Now by homogeneity, G-types correspond exactly to isomorphism types, so the Ramsey property for Age(M) follows from Proposition 5.4.

(ii) ⇒ (i). We omit the details. Rather easily, condition (ii) implies the following condition (*), and the key is to show that (*) implies extreme amenability. For this, see [11, 4.1,4.2].

(*) Let $H \leq G$ be an open subgroup of G, and let $c : G/H \to \{1, \ldots, k\}$ be a k-colouring of the set of left cosets of H in G. Let $A \subset G/H$ be finite. Then there is $g \in G$ and $i \in \{1, \ldots, k\}$ such that for each coset $aH \in A$, $c(gaH) = i$. $\square$

Corollary 5.6 *For any of the following homogeneous structures M, let $\mathcal{C}$ be the age of M (in a language L), let $L' = L \cup \{<\}$, let $\mathcal{C}'$ be the class all L'-structures $(A, <)$ such that $A \in \mathcal{C}$ and $<$ is a total order on A. Let M' be the Fraïssé limit of $\mathcal{C}'$. Then* $\mathrm{Aut}(M')$ *is extremely amenable.*

(i) The random graph.

(ii) The universal homogeneous K_n-free graphs, for any $n \geq 3$.

(iii) The Henson digraphs given by a clas of finite tournaments (see Example 1.9(4)).

(iv) For any $k \geq 3$, the universal homogeneous k-hypergraph.

Proof. By Theorem 4.7, $\mathcal{C}'$ is a Ramsey class. The result follows now from Theorem 5.2. $\square$

Many other examples of extremely amenable groups are given in [11]. Also, for some closed subgroups of $\mathrm{Sym}(\mathbb{N})$ which are *not* extremely amenable, the universal minimal G-flow $M(G)$ is described. For example, if Γ is the random graph and $G = \mathrm{Aut}(\Gamma)$, then $M(G)$ is the space of all linear orderings of Γ. On the other hand, if G is the automorphism group of the homogeneous graph Δ which is the disjoint union of $\aleph_0$ copies of the complete graph on $\aleph_0$ vertices, then $M(G)$ is the collection of all orderings of Δ for which each of the (maximal) complete graphs is convex in the ordering.

We sketch some ideas behind this, taken from [11, Section 7]. First, let L be a relational language, $<$ a binary relation symbol not in L, and $L' := L \cup \{<\}$. There is a combinatorial notion of a Fraïssé order class being *reasonable.* It is equivalent to the class of reducts to L of members of $\mathcal{C}'$ being an amalgamation class and having Fraïssé limit the L-reduct of the Fraïssé limit of $\mathcal{C}'$. For example, the class of all finite totally ordered K_3-free graphs is a Fraïssé order class.

Continuing this notation (so with $\mathcal{C}$ the class of L-reducts of a reasonable Fraïssé order class $\mathcal{C}'$), we say that $\mathcal{C}'$ has the *ordering property* if the following holds: for every $A \in \mathcal{C}$, there is $B \in \mathcal{C}$ such that for every linear ordering $<_A$ on A and $<_B$ on B, if $A' := (A, <_A) \in \mathcal{C}'$ and $B' := (B, <_B) \in \mathcal{C}'$, then A' is isomorphic to a substructure of B'.

Now suppose $\mathcal{C}'$ is a reasonable Fraïssé order class over L' with Fraïssé limit M', let M be the reduct of M' to L (so $M' = (M, <)$), and put $G = \mathrm{Aut}(M)$. Let X be the compact space of all linear orderings of M (a G-flow). Let $X_{\mathcal{C}}$ be the closure in X of the G-orbit which contains the ordering $<$ of M.

Theorem 5.7 (Theorems 7.4 and 7.5 of [11]) *Adopt the above assumptions and notation.*

(a) The following are equivalent.

(i) $X_{\mathcal{C}}$ *is a minimal G-flow.*

(ii) $\mathcal{C}'$ *has the ordering property.*

(b) If $\mathcal{C}'$ *is a Ramsey class with the ordering property, then* $X_{\mathcal{C}}$ *is the universal minimal G-flow.*

Example 5.8 (Sections 6 and 7 of [11]) Let L have a single binary relation symbol E. In the language L', let OEQ be the class of all finite L'-structures such that E is an equivalence relation and $<$ is a total order. Then OEQ does not have the Ramsey or ordering properties.

To see that OEQ does not have the Ramsey property, let $A = \{a, b\}$ with $\neg Eab$ and $a < b$, and let $B = \{a, b, c\}$ with $a < b < c$ and with E-classes $\{a, c\}$ and $\{b\}$. There is no $C \in OEQ$ such that $C \to (B)^A_2$. For given any C, order the E-classes according to the order of the least elements. Colour a copy $\{a', b'\}$ of A red if a' is in a lower class than b', and green otherwise. Then any copy of B in C realises both colours.

To see that OEQ does not have the ordering property, let A be as above. Then for any structure $B \in OEQ$ such that each E-class is convex, there is no embedding $A \to B$.

However, it can be shown that the class $COEQ$ of finite L' structures such that E is an equivalence relation, $<$ is a total order, and each equivalence class is convex, does have the Ramsey and ordering properties. Thus, the automorphism group of its Fraïssé limit is extremely amenable, and we have a description of the universal minimal G-flow of the automorphism group of an equivalence relation with $\aleph_0$-classes, all countably infinite (so the group is the wreath product of two infinite symmetric groups).

The above material on G-flows is one important application of the previous material on structural Ramsey theory. I mention very briefly some others.

1. The original motivation of Ramsey's Theorem, in Ramsey's paper, was to show that for any first order sentence σ of the form $\exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m \phi$, where ϕ is quantifier-free, there is an algorithm which decides whether σ holds in every finite structure. An analogous result for a class of second order formulas, with a 0-1 law, was proved by Kolaitis and Vardi in 1987, using aspects of the Nesetril-Rödl Ramsey theory.

2. Schmerl used structural Ramsey theory to show that a very large class of countable recursively saturated structures are generated by an indiscernible sequence.

3. For homogeneous structures M such as the random graph, the universal homogeneous K_n-free graph, and higher arity analogues, if $G = \mathrm{Aut}(M)$ then it is very natural to ask for a classification of the closed subgroups H of $\mathrm{Sym}(M)$ which contain G. This is equivalent to asking for a classification of the structures with domain M which are definable without parameters in M, where we identify two structures if they have the same automorphism group, i.e. have the same $\emptyset$-definable sets. Thomas (in [22] and other papers) used structural Ramsey theory

to obtain such classifications. Motivated by questions on constraint satisfaction, Bodirsky and Pinsker have recently extended some of these results.

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