MAGIC-050 Set theory, lecture 5

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Burali-Forti 'paradox': there is no set of all ordinals.

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This unique ordinal is called *the order type* of the well ordering.

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Suppose A and B are sets, $R \subseteq B \times B$ is a well order on B and $f : A \to B$ is a bijection. Then the *induced order* $<^* \subseteq A \times A$, defined by $x <^* y$ if and only if $f(n)Rf(y)$, is a well order on A.

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Are all infinite sets countable? No, but let's first see some countabe sets.

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(Schroeder-Bernstein) Suppose that there is an injection from A to B and an injection from B to A. Then there is a bijection between A and B.