Discrete Probability

1. **Correlation length.** Let $\xi(p)$ denote the correlation length. Using the relevant bounds on $P_p\big(0 \longleftrightarrow (n,0,\dots,0)\big)$ and $P_p\big(0 \longleftrightarrow \partial\Lambda_n\big)$ proven in the lecture, show the following:

   (a) For any $x \in \mathbb{Z}^d$ and $p \le p_c$, there is $c > 0$ such that
   $$\frac{1}{c\|x\|_\infty^{4d(d-1)}} \exp\big\{-\|x\|_1/\xi_p\big\} \le P_p(0 \longleftrightarrow x) \le \exp\big\{-\|x\|_\infty/\xi_p\big\}.$$

   (b) Find an algebraic lower bound on $P_{p_c}\big(0 \longleftrightarrow \partial\Lambda_n\big)$ from the proof of Theorem 3.9, and deduce that $\xi(p_c) = \infty$.

   (c) Show continuity of $p \mapsto \xi(p)$ by verifying that it is both upper and lower semicontinuous on $(0, p_c)$ (as decreasing resp. increasing limit of continuous functions).

   (d) Verify that $p \mapsto \xi(p)$ is increasing on $[0, p_c]$. Furthermore, show that it is *strictly* increasing on $[0, p_c]$.

2. **An explicit value.** For bond percolation on the square lattice $\mathbb{Z}^2$, show that
   $$\mathbb{P}_{1/2}\big((0,0) \leftrightarrow (1,0)\big) = \frac{3}{4}.$$

3. **One-arm probability in two dimensions.** Consider site percolation on the triangular lattice $\mathbb{T}$, and let $\Lambda(n)$ denote the ball of radius n (in graph distance) centered at the origin. Use the RSW theorem to show that
   $$cn^{-\alpha} \le P_{1/2}\big(0 \longleftrightarrow \partial\Lambda_n\big) \le Cn^{-\beta}, \qquad n \ge 1,$$
   for suitable constants $c, C, \alpha, \beta > 0$.