Discrete Probability

Note: Parts of Exercise 1 return in different form on the next exercise sheet.

1. **The correlation length.** Fix $d \geq 2$ and set $e_1 = (1, 0, \dots, 0)$.

   (a) Prove that, for any $p \in [0, 1]$, $n, m \geq 0$,

   $$P_p\big(0 \longleftrightarrow (m+n)e_1\big) \geq P_p\big(0 \longleftrightarrow me_1\big) \cdot P_p\big(0 \longleftrightarrow ne_1\big).$$

   (b) Deduce that $\xi(p) = \big(\lim_{n\to\infty} -\frac{1}{n} \log P(0 \longleftrightarrow ne_1\big)^{-1}$ exists (the "correlation length"), and furthermore $P_p(0 \longleftrightarrow ne_1) \leq \exp\{-n/\xi(p)\}$.

   (c) Assuming $0 \longleftrightarrow \partial\Lambda_{n+m}$, show that there exists $x \in \partial\Lambda_n$ such that $\{0 \longleftrightarrow x\} \circ \{x \longleftrightarrow x + \partial\Lambda_m\}$. Deduce that for all n, m,

   $$P_p(0 \longleftrightarrow \partial\Lambda_{n+m}) \leq |\partial\Lambda_n|\, P_p(0 \longleftrightarrow \partial\Lambda_n)\, P_p(0 \longleftrightarrow \partial\Lambda_m).$$

   Further, show that $P_p(0 \longleftrightarrow \partial\Lambda_n) \geq \frac{e^{-n/\xi(p)}}{2^d d(2n+1)^{d-1}}$.

   (d) Show that for every $n \in \mathbb{N}$, $x \in \partial\Lambda_n$,

   $$\xi(p) \geq \frac{n}{-\log P_p\big(0 \overset{\Lambda_n}{\longleftrightarrow} x\big)}.$$

   Deduce that $\lim_{p\nearrow p_c} \xi(p) = \infty$, and show that $p \mapsto \xi(p)$ is continuous on $[0, p_c)$.

   (e) Prove that, for any $x \in \partial\Lambda_n$,

   $$P_p(0 \longleftrightarrow 2ne_1) \geq P_p(0 \longleftrightarrow x)^2.$$

   Deduce that

   $$P_p(0 \longleftrightarrow x) \geq \frac{c}{\|x\|_\infty^{2d(d-1)}} \exp\big\{ -\|x\|_\infty/\xi_p\big\}.$$

   (f) Finally, deduce that for any $x \in \mathbb{Z}^d$,

   $$P_{p_c}(0 \longleftrightarrow x) \geq \frac{c}{\|x\|_\infty^{2d(d-1)}}.$$

   This shows an algebraic lower bound for connection probabilities at p_c, and is in contrast to the exponential decay when $p < p_c$. Much more precise estimates are known for $x = ne_1$. These "Ornstein-Zernike-estimates" state that there exists $c = c(p) > 0$ such that

   $$P_p(0 \longleftrightarrow ne_1) = \frac{c}{n^{(d-1)/2}} \exp(n/\xi(p))\,(1 + o(1)).$$

2. **Percolation on the binary tree.** Denote by $\mathcal{T}$ the infinite tree where every vertex has exactly three neighbors. Aim of this exercise is to show that $p_c(\mathcal{T})$=1/2.

   (a) Show $p_c \geq 1/2$ using a path-counting argument.

   (b) Observe the following: For any *finite* connected subgraph G of $\mathcal{T}$, denote by e_G the number of edges inside G, and by b_G the number of boundary edges (an edge b is a *boundary edge* if one of its endpoints belongs to G, and the other does not). Then $e_G = b_G - 3$.
   Write $1 - \theta(p)$ as a sum over certain finite graphs and show that, for $p \in [0, 1]$,
   $$1 - \theta(p) = \left(\frac{1-p}{p}\right)^3 \big(1 - \theta(1-p)\big).$$

   (c) Show that, for $p > 1/2$,
   $$\theta(p) = 1 - \left(\frac{1-p}{p}\right)^3,$$
   and conclude that this holds as well for $p = 1/2$.

   (d) Finally, consider supercritical percolation on $\mathcal{T}$ with $1/2 < p < 1$. How many infinite components are there? Prove your answer.