Discrete Probability

1. Show the following:

   (a) If A and B are increasing events, then $A \circ B$ is again increasing.

   (b) If A is increasing and B is decreasing, then $A \circ B = A \cap B$.

2. Give an alternative proof of Russo's formula using the increasing coupling of the percolation measures with parameter p and $p + \varepsilon$ (where ε converges to 0).

3. Show that $p \mapsto \theta(p)$ is strictly increasing whenever $p > p_c$.

4. **(Disjoint occurence)** Let $X_1, \dots, X_n$ be i.i.d. Bernoulli(p)-distributed random variables. For integers k and ℓ, denote by A and B the events

$$A = \left\{ \sum_{i=1}^{n} X_i \geq k \right\} \quad \text{and} \quad B = \left\{ \sum_{i=1}^{n} X_i \geq \ell \right\}.$$

   (a) What is the event $A \circ B$ in this example?

   (b) In this setting, prove directly $P(A \circ B) \leq P(A)\, P(B)$ (without using the BK-inequality).

5. **(Some useful Analysis.)** A real function f is called *upper semicontinuous* if for all x from the domain, and all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(y) \leq f(x) + \varepsilon$ whenever $|x - y| \leq \delta$. Furthermore, f is *lower semicontinuous* if $-f$ is upper semicontinuous.

   Observe that f is continuous if and only if it is both upper *and* lower semicontinuous.

   Prove the following:

   (a) If f is the decreasing limit of continuous functions (i.e., the functions $f_1, f_2, f_3, \dots$ are continuous, $f_i(x) \geq f_{i+1}(x)$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for all x from the domain and $i \in \mathbb{N}$), then f is upper semicontinuous.

   (b) If f is upper (*resp. lower*) semicontinuous and weakly increasing (*resp. decreasing*), then it is continuous from the right.

6. **(Continuity)** Prove that $p \mapsto P_p(x \leftrightarrow y)$ is a continuous function on $[0, 1]$, for all $x, y \in \mathbb{Z}^d$.