Discrete Probability

1. Show that any measurable event can be approximated by events depending on finitely many edges in the following sense: for any $A \in \mathcal{F}$ (the sigma-algebra of events), there exists as sequence of events (B_n) such that B_n depends only on the state of edges in a box of radius n, and $P_p(A \Delta B_n) \overset{n\to\infty}{\longrightarrow} 0$.

2. Show that $p_c(\mathbb{Z}) = 1$. Show that $p_c(\mathbb{Z} \times \{0, \dots, n\}) = 1$ for any $n \in \mathbb{N}$.

3. (a) Verify that the Harris-/FKG-inequallity holds for decreasing events, too.

   (b) If A and B are increasing events, then
   $$\{A \circ B\} = \{\omega \mid \exists K \subseteq \mathcal{E} \colon \omega_K \in \mathcal{A}, \omega_{\mathcal{E}\setminus K} \in B\}$$
   where $(\omega_K)_e = \begin{cases} \omega_e & \text{if } e \in K, \\ 0 & \text{otherwise.} \end{cases}$

   (c) Show that $(A \circ B) \circ C = A \circ (B \circ C)$.

4. *Proof of the BK-inequality for increasing events.* Consider two increasing events A and B that depend on the edges in $E = \{e_1, \dots, e_n\}$ only. Consider the duplicated set $\{e_1, \dots, e_n, e_1', \dots, e_n'\}$ and the product measure $\mathbb{P}$ where each coordinate is a Bernoulli(p) variable. For $j \le n$, set $\omega^j = (\omega_{e_1}, \dots, \omega_{e_{j-1}}, \omega_{e_j'}, \dots, \omega_{e_n'})$ and
   $$\tilde{A}^j = \{\tilde{\omega} \colon \omega^j \in A\} \text{ and } \tilde{B} = \{\tilde{\omega} \colon \omega^{n+1} \in B\}, \qquad j \in \{1, \dots, n+1\}.$$

   (a) Show that $\mathbb{P}(\tilde{A}^1 \circ \tilde{B}) = P_p(A)\, P_p(B)$ and $\mathbb{P}(\tilde{A}^{n+1} \circ \tilde{B}) = P_p(A \circ B)$.

   (b) Show next that $j \mapsto \mathbb{P}(\tilde{A}^j \circ \tilde{B})$ is decreasing in j by constructing a measure-preserving injection $\tilde{\omega} \mapsto s(\tilde{\omega})$ from $\tilde{A}^{j+1} \circ \tilde{B}$ to $\tilde{A}^j \circ \tilde{B}$. Let $\tilde{\omega}$ be a configuration in $\tilde{A}^{j+1} \circ \tilde{B}$. Let K and J be "witnesses" of $\tilde{A}^{j+1}$ and $\tilde{B}$, respectively, for $\tilde{\omega}$. If there exists K not containing e_j, then simply set $s(\omega) = \omega$. Otherwise (K contains e_j) obtain $s(\tilde{\omega})$ by exchanging e_j and e_j'. Check that s is one-to-one and that $\mathbb{P}(\tilde{\omega}) = P_p(s(\tilde{\omega}))$.

   (c) Deduce the BK-inequality.

5. Denote by $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$ the box of radius n around the origin. Suppose that the edges of this box are open with probability $p \in [0, 1]$ and closed otherwise, independently of each other. We interpret the box as a porous stone that is put in water, where only the open edges transport water. Denote by X_n the fraction of vertices of the box that get wet. Show
   $$E_p(X_n) \to \theta(p) \qquad \text{as } n \to \infty.$$