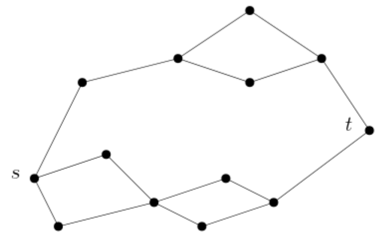
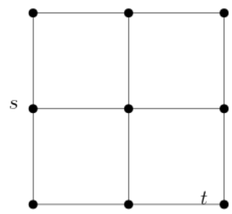
Discrete Probability

1. The following graphs are given, all edges have unit resistance.





Calculate the effective resistance between s and t.

2. Let G be a countably infinite but locally finite connected graph with nominated vertex 0. Let H be a connected subgraph of G containing 0. Show that simple random walk, starting at 0, is recurrent on H whenever it is recurrent on G, but the converse need not hold.

3. *Nash-Williams Criterion.* For a graph $G = (V, E)$ with distinguished vertex 0, we say that $\Pi \subset E$ *seperates 0 from* ∞ if every infinite simple path from 0 must include an edge in Π ('Π is a cutset'). Prove that the following statement ist true:

   If $(\Pi_n)_n$ is a sequence of pairwise disjoint finite cutsets in a locally finite network G, each of which separates 0 from ∞, then

$$R_{\text{eff}} \geq \sum_n \left(\sum_{e \in \Pi_n} r_e^{-1} \right)^{-1}.$$

   In particular, G is recurrent if the right hand side is infinite.

4. *Yet another proof of Pólya's theorem.* Let $G \colon \mathbb{Z}^d \to [0, \infty]$ be the Green's function of simple random walk $(X_n)_n$ on the d-dimensional lattice $\mathbb{L}^d$. Prove that $G(0) < \infty$ if and only if $d \geq 3$ using Fourier transformation (a.k.a. characteristic functions). To this end, observe that if X is $\mathbb{Z}^d$ valued, then the characteristic function $\varphi_X(k) := \mathbb{E}(e^{ikX})$ is $(2\pi)^d$-periodic.

   (a) Derive first the Fourier transform of X_1 (random walk after 1 step).

   (b) Deduce the Fourier transform of $X_n = X_1 * X_1 * \cdots * X_1$.

   (c) Calculate $G(0) = \sum_{n=0}^{\infty} \mathbb{P}(X_n = 0)$ via Fourier inversion by integrating over k in the domain $[-\pi, \pi)^d$, and verify that the result is finite iff $d \geq 3$.

5. Show that the simple random walk on $\mathbb{Z}$ with drift (going one step to the right or left with probabilities $p \neq \frac{1}{2}$ and $1-p$) is transient.

6. (a) Show that recurrent Markov chains on a countable state space have a stationary (not necessarily normalizable) measure.

   (b) Show that an irreducible Markov chain on a countable state space has a stationary distribution iff it is positive recurrent.

Suggestion: We have now completed the treatment of random walks and electrical networks. I suggest that you read Sections 2.1–2.5 in the book of Lyons and Peres, who present the material in a different yet enlighting manner.
The book can be found online on `http://mypage.iu.edu/~rdlyons/prbtree/prbtree.html`.