Discrete Probability

1. Let $G = (V, E)$ be a graph with edge weights $w\colon E \to (0, \infty)$, and consider a random walk $(X_n)_n$ with transition matrix

$$P_{u,v} = \frac{w_{u,v}}{W_u}, \qquad \text{where} \quad W_u := \sum_{v' \in V} w_{u,v'}.$$

   (a) Find the invariant measure π on V and show that P is reversible w.r.t. π.

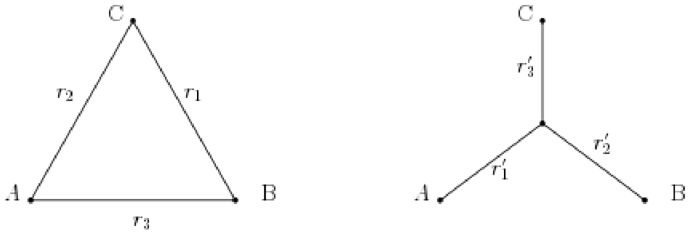
   (b) Let $U \subseteq V$, $W = V \setminus U$, and $s \in U$. Define

$$g(u) = \mathbb{P}(X_n = s \text{ for some } n < \tau_W \mid X_0 = u),$$

   where $\tau_W = \inf\{n \geq 0\colon X_n \in W\}$ is the first hitting time of the set W. Show that the function g is harmonic on $U \setminus \{s\}$.

2. Prove the series and parallel laws for electrical networks.

3. *Star–triangle transformation.* The triangle is replaced by a star in an electrical network, as illustrated below. Explain the sense in which the two networks are the same when the resistances are chosen such that $r_j r'_j = c$ for $j = 1, 2, 3$ and a suitable constant $c = c(r_1, r_2, r_3)$ to be determined.



4. Let $G = (V, E)$ be a finite connected graph with unit edge-weights. Show that the effective resistance between two distinct vertices s, t of the associated electrical network may be expressed as B/N, where B is the number of s/t bushes of G, and N is the number of its spanning trees.

5. *Existence of unit flows for weighted networks.* The aim of this exercise is to extend the proof of Theorem 2.4 to the setting of weighted graphs. Let $G = (V, E)$ be a finite connected network with strictly positive conductances $(w_e : e \in E)$. For any spanning tree T of G, define the *weight*

$$w(T) := \prod_{e \in T} w_e$$

and set

$$N^* = \sum_{T : T \text{ spann. tree of } G} w(T)$$

and accordingly $N^*(s, a, b, t)$ as a partial sum over those spanning trees, where the unique path from s to t passes the edge $\langle a, b \rangle$ in the direction from a to b. Show that

$$i_{a,b} = \frac{1}{N^*} \left(N^*(s, a, b, t) - N^*(s, b, a, t) \right), \qquad \langle a, b \rangle \in E$$

constitutes a unit flow through G from s to t satisfying Kirchhoff's laws.

6. *Uniqueness theorem.* Let $G = (V, E)$ be a finite or connected network with finite vertex-degrees, and let W be a finite subset of V, $W \neq V$. Let $f, g \colon V \to R$ be harmonic on W and equal on $V \setminus W$. Show, by the maximum principle or otherwise, that $f \equiv g$.