# Discrete Probability

1. Let $P$ be a transition matrix and $\mu, \nu$ two probability distributions on $\mathcal{S}$. Show that

$$\|\mu P - \nu P\|_{TV} \leq \|\mu - \nu\|_{TV}.$$

This implies that advancing a chain moves it closer to stationarity (w.r.t. $\|\cdot\|_{TV}$).

2. Let $\Psi$ be an irreducible transition matrix and $\pi$ a probability distribution on $\mathcal{S}$. Show that

$$p(x,y) := \begin{cases} \psi(x,y)\left[\frac{\pi(y)\psi(y,x)}{\pi(x)\psi(x,y)} \wedge 1\right] & \text{if } y \neq x; \\ 1 - \sum_{z:z\neq x} \psi(x,z)\left[\frac{\pi(z)\psi(z,x)}{\pi(x)\psi(x,z)} \wedge 1\right] & \text{if } y = x \end{cases}$$

defines a reversible Markov chain with stationary distribution $\pi$.

3. **(A generalized hard-core model)** We consider a generalization of the hard-core model which allows different "packing intensities" of 1's in the graph. To this end, we introduce a parameter $\lambda > 0$ ("fugacity") and change the probability measure $\mu_G$ into a probability measure $\mu_{G,\lambda}$ defined by

$$\mu_{G,\lambda}(\xi) = \begin{cases} \frac{\lambda^{n(\xi)}}{Z_{G,\lambda}} & \text{if } \xi \text{ is feasible} \\ 0 & \text{otherwise,} \end{cases}$$

where $n(\xi)$ is the number of 1's in $\xi$ and

$$Z_{G,\lambda} = \sum_{\xi \in \{0,1\}^V} \lambda^{n(\xi)} \mathbb{1}_{\{\xi \text{ is feasible}\}}$$

is a normalizing constant. The conditional probability that $v$ takes the value 1, given the values at all other vertices, equals $\frac{\lambda}{\lambda+1}$ if all neighbors are vacant, otherwise it is 0. The case $\lambda = 1$ has been treated in the lecture.

Construct an MCMC algorithm for this generalized hard-core model (and verify that it is irreducible and aperiodic chain).

4. **(Glauber dynamics for the Ising model)** Consider the Ising model on a finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$: for $\sigma \in \mathcal{S} = \{-1, +1\}^V$ let

$$\mu(\sigma) = \frac{1}{Z(\beta)} e^{-\beta H(\sigma)},$$

where $\beta > 0$ is a parameter (commonly known as "inverse temperature"),

$$H(\sigma) = -\sum_{\{v,w\}} \sigma(v)\sigma(w)$$

describes the energy of the configuration $\sigma$ and

$$Z(\beta) = \sum_{\sigma \in \mathcal{S}} e^{-\beta H(\sigma)}$$

is the *partition function*. Prove that the *Glauber dynamics* for the Ising model

$$P_{\sigma,\sigma'} = \frac{1}{V} \sum_{w \in \mathcal{V}} \frac{\exp\{\beta \sum_{v:\{w,v\}\in\mathcal{E}} \sigma'(w)\sigma'(v)\}}{\exp\{-\beta \sum_{v:\{w,v\}\in\mathcal{E}} \sigma'(v)\} + \exp\{\beta \sum_{v:\{w,v\}\in\mathcal{E}} \sigma'(v)\}} \mathbb{1}_{\{\sigma(v)=\sigma'(v) \text{ for all } v \neq w\}}$$

gives a reversible Markov chain with stationary distribution $\mu$.

5. **(An alternative proof of the convergence theorem for Markov chains)** Let $(X_n, Y_n)_n$ be two coupled Markov chains (with same irreducible and aperiodic transition matrix $P$) on $\mathcal{S}$ satisfying

$$\text{if } X_k = Y_k \text{ then } X_n = Y_n \text{ for all } n \geq k.$$

(a) Show that, if $X_0 \sim \mu$ and $Y_0 \sim \nu$, then

$$\|\mu P^n - \nu P^n\|_{TV} \leq \mathbb{P}(\tau_{\text{couple}} > n),$$

with

$$\tau_{\text{couple}} := \min\{k \mid X_n = Y_n \text{ for all } n \geq k\}.$$

(b) If in (a) we take $\nu = \pi$ (the stationary measure) then we obtain a bound on the difference between $\mu P^n$ and $\pi$. The only thing left to show is that $P(\tau_{\text{couple}} < \infty) = 1$. Show that if the two chains are taken independent of each other (until they meet), then they are assured to eventually meet.