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Exercise sheet 2
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Discrete Probability

1. A transition matrix P is *symmetric* if $P(x,y) = P(y,x)$ for all $x, y \in S$.
   Show that if P is symmetric, then the uniform distribution on S is stationary for P.

2. Prove Lemma 1.8 from the lecture: Let P be a transition matrix.
   (a) Show that all eigenvalues λ satisfy $|\lambda| \le 1$.
   (b) Show that all real eigenvalues of the *lazy chain* $(P+I)/2$ are nonnegative.
   (c) Let P be irreducible. Show that $\mathcal{T}(x) := \{n \in \mathbb{N}_0 \mid P^n(x,x) > 0\} \subset 2\mathbb{Z}$ iff -1 is an eigenvalue of P.
   (d) More generally, let P be irreducible and $z \in \mathbb{C}$ a kth root of unity. Show that $\mathcal{T}(x) \subset k\mathbb{Z}$ if and only if z is an eigenvalue of P.

3. Prove Lemma 1.10 from the lecture: For any $n \in \mathbb{N}$,
   $$\|P^n(x,\cdot) - \pi(\cdot)\|_{TV} \le \max_{y \in S}\left(1 - \frac{P^n(x,y)}{\pi(y)}\right), \qquad x \in S,$$
   and thus $d(n) \le s(n)$ (where the *seperation distance* $s(n)$ is defined as the maximum of the right hand side).

4. Prove Lemma 1.11 from the lecture: Let P be a transition matrix of an irreducible and reversible Markov chain. Show that for a function $f\colon S \to \mathbb{R}$,
   $$\mathbb{V}_\pi(P^n f) \le \lambda_\star^{2n}\, \mathbb{V}_\pi(f),$$
   where $\lambda_\star := \max\{|\lambda| \colon \lambda \text{ is eigenvalue of } P, \lambda \neq 1\}$.

5. Complete the proof of Lemma 1.13 from the lecture by showing that the measure $q \in \mathcal{M}_1(S \times S)$ is a *coupling* of the measures $\mu, \nu \in \mathcal{M}_1(S)$, where
   $$q(x,x) = \mu(x) \wedge \nu(x), \qquad x \in S,$$
   and for $x \neq y$,
   $$q(x,y) = \begin{cases} 0 & \text{if } q(x,x) = \mu(x) \text{ or } q(y,y) = \nu(y); \\ \frac{(\mu(x)-\nu(x))\,(\nu(y)-\mu(y))}{1-\sum_z q(z,z)} & \text{elsewhere.} \end{cases}$$

6. Prove Theorem 1.6 from the lecture ("Ergodic theorem for Markov chains"):
Let $(X_n)_{n\in\mathbb{N}_0}$ be an irreducible Markov chain on the state space S starting in $x \in S$ with stationary distribution π, let $f\colon S \to \mathbb{R}$ be an $\mathbb{R}$-valued function and

$$S_n := \sum_{k=0}^{n-1} f(X_k)\,.$$

Define $\tau^+_{x,0} := 0$ and, recursively,

$$\tau^+_{x,k} := \min\{n > \tau^+_{x,k-1} \mid X_n = x\}\,.$$

(a) Show that $\lim_{n\to\infty} \frac{S_{\tau^+_{x,n}}}{n} = \mathbb{E}_\pi(f)\mathbb{E}(\tau^+_{x,1})$.

(b) Show that $\lim_{n\to\infty} \frac{\tau^+_{x,n}}{n} = \mathbb{E}(\tau^+_{x,1})$ a.s.

(c) Conclude that $\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}_\pi(f)$ a.s.. What would be the generalization to arbitrary starting distributions?