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Exercise sheet 14
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Discrete Probability

1. **Extinction probability with large total progeny.** Prove Proposition D of the lecture: For a branching process with i.i.d. offspring having mean $E(X) > 1$, show that

$$P(k \le T < \infty) \le \frac{e^{-Ik}}{1 - e^{-I}},$$

where

$$I = \sup_{t \le 0} \big(t - \log E(e^{tX})\big) > 0.$$

What is I when X is Poisson distributed?

2. **Asymptotics for total progeny of Poisson branching process.** For a branching process with i.i.d. offspring $X \sim$Pois(λ) and total progeny T_λ, show that, as $n \to \infty$,

$$P(T_\lambda = n) = \frac{1}{\lambda\sqrt{2\pi n^3}} e^{-I_\lambda n}(1 + O(1/n)),$$

where I_λ is the large deviation rate function. In particular, when $\lambda = 1$,

$$P(T_1 = \lambda) = \frac{1}{\sqrt{2\pi n^3}}(1 + O(1/n)).$$

3. **Connectivity function.** Prove that, for $\lambda > 1$,

$$P(1 \longleftrightarrow 2) = \zeta_\lambda^2(1 + o(1)).$$

4. **Expected cluster size for supercritical case.** Prove that, for $\lambda > 1$,

$$\chi(\lambda) = \zeta_\lambda^2 n(1 + o(1)).$$

5. **Expected cluster size for critical case.** Prove Lemma 5 from the lecture: For $\theta < 0$ and $\lambda = 1 + \theta n^{-1/3}$, show that

$$\chi(\lambda) \le \frac{n^{1/3}}{|\theta|}.$$