Discrete Probability

1. Let $X \sim \mathrm{BIN}(n, p)$ and $X^* \sim \mathrm{POI}(\lambda)$ with $\lambda = np$. Prove that there is a coupling of X and X^* with the property
$$P(X \neq X^*) \leq \frac{\lambda^2}{n}.$$

2. Prove that Theorem 4.3 implies that $|\mathcal{C}_{\max}|/\log n$ converges to $1/I_\lambda^*$ as $n \to \infty$ in probability whenever $\lambda < 1$.

3. If not done already: Prove Exercise 5 from Sheet 12.
   * Prove Cramér's upper bound using the exponential Chebyshev inequality.

4. Prove Theorem 4.2 from the lecture.

5. **Total progeny of a Poisson branching process.**
   (a) * Prove the following *Random walk hitting time theorem*:
   Consider a random walk $S_n := k + Y_1 + \cdots + Y_n$ ($k \in \mathbb{N}$ is the starting point) with i.i.d. integer-valued steps $(Y_i)_{i\geq 1}$ with the property $P(Y_i \geq -1) = 1$. Then
   $$P(S_n = 0 \text{ for the first time}) = \frac{k}{n} P(S_n = 0), \qquad n \in \mathbb{N}.$$
   (b) For a branching process with i.i.d. offspring distributed like an $\mathbb{N}$-valued random variable X,
   $$P(T = n) = \frac{1}{n} P(X_1 + \cdots + X_n = n - 1), \qquad n \in \mathbb{N},$$
   where $(X_n)_{n\geq 1}$ are i.i.d. copies of X and T is the total progeny.
   (c) Derive a formula for $P(T = n)$ if the offspring is Poisson(λ)-distributed.

6. **A binomial number of binomial trials.** Show that if $N \sim \mathrm{BIN}(n, p)$ and, conditionally on N, $M \sim \mathrm{BIN}(N, q)$, then $M \sim \mathrm{BIN}(n, pq)$.

7. Let X be an $\mathbb{N}_0$-valued random variable, and denote
$$p_k = P(X = k), \qquad k \in \mathbb{N}_0.$$
Prove that the the *conjugate pair*
$$p'_k := \eta^{k-1} p_k, \qquad k \in \mathbb{N},$$
is a probability distribution, where $\eta = G_X(\eta)$ is the extinction probability of a branching process with offspring distribution $(p_k)_{k\geq 0}$.

The importance of the last exercise is linked to the *duality principle for branching processes*: The branching process with offspring distribution $(p_k)_{k\in\mathbb{N}_0}$ conditioned on extinction, has the same distribution as the branching process with offspring distribution $(p'_k)_{k\in\mathbb{N}_0}$.