Discrete Probability

Critical exponents for percolation on the binary tree. The purpose of this exercise sheet is to calculate that on the 3-regular tree $\mathcal{T}$ (a.k.a. binary tree) we have that $p_c = p_T = 1/2$, and the *critical exponents* take on their mean-field values $\beta = \gamma = \rho = 1$ and $\delta = \Delta = 2$, where

$$\begin{aligned} \chi(p) &\simeq (p_c - p)^{-\gamma} && \text{as } p \nearrow p_c, \\ \theta(p) &\simeq (p - p_c)^{\beta} && \text{as } p \searrow p_c, \\ P_{p_c}(|\mathcal{C}(0)| \geq n) &\simeq n^{-1/\delta}, && \text{as } n \to \infty, \\ P_{p_c}(\exists x \in \mathcal{C}(0) : \operatorname{dist}(0,x) = n) &\simeq n^{-1/\rho}, && \text{as } n \to \infty, \end{aligned}$$

and the *gap exponent* $\Delta > 0$ is defined by,

$$\frac{E_p\big[|\mathcal{C}(0)|^{k+1}\big]}{E_p\big[|\mathcal{C}(0)|^{k}\big]} \simeq (p_c - p)^{-\Delta} \qquad \text{as } p \nearrow p_c \text{ for } k = 1, 2, 3, \ldots$$

with the unwritten assumption that Δ is independent of k.

1. Recall from Sheet 8, Exercise 2 that the binary tree, $p_c = 1/2$ and

$$\theta(p) = \begin{cases} 0 & \text{if } p < 1/2, \\ 1 - \left(\frac{1-p}{p}\right)^3 & \text{if } p \geq 1/2. \end{cases}$$

   Derive that $\beta = 1$ for $\mathcal{T}$.

2. Now we address the critical exponent γ. For $x \neq o$, we write $\mathcal{C}_{\text{BP}}(x)$ for the forward cluster of x in $\mathcal{T}$, i.e., those vertices $y \in \mathcal{T}$ that are connected to x and for which the unique path from x to y only moves away from the root o. Then,

$$|\mathcal{C}(o)| = 1 + \sum_{e \sim o} I_{o,e} |\mathcal{C}_{\text{BP}}(e)|,$$

   where the sum is over all neighbors e of o, $(I_{0,e})_{e\sim o}$ are three independent Bernoullli(p)-variables, and $(|\mathcal{C}_{\text{BP}}(e)|)_{e\sim o}$ is an i.i.d. sequence independent of $(I_{o,e})_{e\sim o}$.

   Derive the identity $E_p|\mathcal{C}_{\text{BP}}(x)| = 1 + 2pE_p|\mathcal{C}_{\text{BP}}(x)|$ and conclude

$$E_p|\mathcal{C}_{\text{BP}}(x)| = \frac{1}{1-2p}$$

   for $p < 1/2$. Derive further an expression for $\mathbb{E}_p(|\mathcal{C}(o)|)$ and verify that $p_T = 1/2$. Conlcude $\rho = 1$.

3. Next we address the arm exponent ρ. Define

$$\theta_n = P_{p_c}\big(\exists v \in \mathcal{C}_{\mathrm{BP}}(x) \text{ such that } \operatorname{dist}(x,v) = n\big)$$

and proof the recursion relation

$$1 - \theta_n = (1 - p_c\theta_{n-1})^2.$$

Show that $\theta(n) = 4/n\big(1 + O(1/n)\big)$ and conclude that $\rho = 1$.

4. Calculate $E\big(|\mathcal{C}_{\mathrm{BP}}(x)|^k\big)$ and derive $\Delta = 2$.

Remark. You may verify that the same critical exponents are true for the r-regular tree $\mathcal{T}_r$, where $p_c(\mathcal{T}_r) = p_T(\mathcal{T}_r) = 1/(r-1)$.