Discrete Probability

1. Prove
$$\frac{2|k|^2}{\pi^2} \le \sum_{j=1}^{d} \big(1-\cos(k_j)\big) \le \frac{|k|^2}{2}$$
for every $k=(k_1,\dots,k_d)\in[-\pi,\pi)^d$.

2. Prove rigorously that
$$\chi(p) \ge \frac{1}{2d(p_c-p)},$$
i.e., $\gamma \le 1$. To this end, consider the functions $\tau_p^n(x,y) := P_p(x \longleftrightarrow y \text{ in } \Lambda_n)$ (with $\Lambda_n = \{-n,\dots,n\}^d$) and
$$\chi^n(p) := \max_{x\in\Lambda_n} \sum_{y\in\Lambda_n} \tau_p^n(x,y).$$
Prove that the functions $p \mapsto 1/\chi^n(p)$ are equicontinuous and $\chi^n(p)\to\chi(p)$ for every $p\in[0,1]$. Deduce that $p\mapsto 1/\chi(p)$ is continuous and, in particular, $\chi(p_c)=0$.

3. For $p<p_c$ and $k\in[-\pi,\pi)^d$, prove that
$$\hat\tau_p(k)\ge 0.$$
For this purpose, it is most convenient to view $\tau_p(x,y)$ ($=\tau_p(y-x)$) as an operator and prove that it is of positive type, that is,
$$\sum_{x,y\in\mathbb{Z}^d} \bar f(x)\tau(x,y)f(y) \ge 0$$
for any summable function $f\colon \mathbb{Z}^d\to\mathbb{C}$ (where $\bar f(x)$ is the complex conjugate of $f(x)$). Then the claim follows from Bochner's theorem.

4. Let $p<p_c$. Prove that the triangle diagram is maximal at the origin:
$$\Delta_p(0) = \max_{x\in\mathbb{Z}^d} \Delta_p(x).$$
More generally, show that $\tau_p^{*s}(x)\le\tau_p^{*s}(0)$ and $(D*D*\tau_p^{*s})(x)\le(D*D*\tau_p^{*s})(0)$ for every $x\in\mathbb{Z}^d$ and $s\ge1$, where $D(x)=\mathbb{1}_{\{|x|=1\}}$. Conclude that these bounds also hold for $p=p_c$.