Mathematisches Institut der Universität München
Markus Heydenreich

Exercise sheet 1
22 Oct 2018

Discrete Probability

We are considering a Markov chain with transition matrix P on a *finite* state space S.

1. Let P be the transition matrix of random walk on the n-cycle, where n is odd. Find the smallest value of r such that $P^r(x, y) > 0$ for all states x and y.

2. Show: If P is an irreducible transition matrix, then $\text{period}(x) = \text{period}(y)$ for all $x, y \in S$.

3. (a) Show that any nonempty set of relatively prime natural numbers that is closed under addition contains all but finitely many natural numbers.
   (b) Let P be an irreducible aperiodic transition matrix on S. Show that there is an $N \in \mathbb{N}$ such that $P^n(x, y) > 0$ for all $x, y \in S$ and all $n \geq N$.

4. Let π be a stationary distribution of an irreducible Markov chain on S.
   Show that $\pi(x) > 0$ for all $x \in S$ without using the explicit formula for π.

5. A Markov chain $(X_n)_{n\in\mathbb{N}_0}$on S with transition matrix P and initial distribution π is called *reversible* if the so-called "detailed balance equations"
$$\pi(x)P(x, y) = \pi(y)P(y, x)\,.$$
   is satisfied for all $x, y \in S$.
   (a) Show that a distribution satisfying the detailed balance equations is stationary.
   (b) Show that, for any $n \in \mathbb{N}$, $(X_0, \ldots, X_n)$ and $(X_n, \ldots, X_0)$ have the same distribution in this case.
   (c) The time reversal $(\hat{X}_n)$ of a general irreducible Markov chain (X_n) with transition matrix P and stationary distribution π is defined via the transition matrix
$$\hat{P}(x, y) := \frac{\pi(y)P(y, x)}{\pi(x)}\,.$$
   Show that π is stationary for $\hat{P}$ and, for any $n \in \mathbb{N}$, $(\hat{X}_0, \ldots, \hat{X}_n)$ is distributed like $(X_n, \ldots, X_0)$.

6. Let P be a transition matrix and μ an initial distribution. Give an alternative proof of existence of a stationary distribution by defining
$$\nu_n := \frac{1}{n}\left(\mu + \mu P + \cdots + \mu P^{n-1}\right)$$
   and considering the pointwise limit of a suitable subsequence.

7. There are Kinder eggs with n different sorts of toys. Each time you buy one, you get each sort with equal probability. What is the expected number you have to buy in order to get all the toys?