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Topology

Problem Set 8

1. (10 points) Let $A = \langle S \mid R\rangle$ and $B = \langle S' \mid R'\rangle$. Also let C be a group with generating set $E \subset C$ and let $f_A: C \to A$ and $f_B: C \to B$ be homomorphisms. For any $c \in C$, let s_c and s'_c be words in $F\langle S \cup S'\rangle$ representing $f_A(c)$ and $f_B(C)$. Show that the group

$$G := \langle S \cup S' \mid R \cup R' \cup \{s_c^{-1} s'_c \mid c \in E\}\rangle$$

together with certain maps $A \to G$, $B \to G$ has the universal property of $A *_C B$. Prove that a group with this property is unique up to isomorphism.

2. (10 points)
   (a) Let $A = (a_{ij}) \in \mathrm{GL}(2, \mathbb{Z})$ be an invertible matrix with integer entries. Show that the map

   $$f_A: T^2 \to T^2 \qquad (v, w) \mapsto (v^{a_{11}} w^{a_{12}}, v^{a_{21}} w^{a_{22}})$$

   is a homeomorphism, where the product is complex multiplication.
   (b) We define M_A to be the space obtained by glueing two disjoint copies of $S^1 \times D^2$ along the map f_A, i.e. we identify $x \in T^2 \subset S^1 \times D^2$ in the first copy with $f_A(x)$ in the second copy. Prove that $\pi_1(M_A) \cong \mathbb{Z}_{a_{12}}$.

3. (10 points) Consider $T^2 = I^2/\sim$ as on exercise sheet 1 and the small open disc $D \subset T^2$ corresponding to the open ball around $(\frac{1}{2}, \frac{1}{2}) \in I^2$ with radius $\frac{1}{4}$. Let Σ_2 be the *surface of genus 2* which is defined as the quotient of two copies of $T^2 \backslash D$ by identifying the boundaries $\partial D \subset T^2 \backslash D$ of D in the two copies via the identity. Show that

$$\pi_1(\Sigma_2) \cong \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1}\rangle.$$

4. (10 points) We call a covering $\tilde{X} \to X$ *Abelian* if it is path-connected, normal, and its deck transformation group is Abelian. Show that if X is path-connected, locally path-connected, and semilocally simply-connected, then there is a universal Abelian covering space, i.e. an Abelian covering space of X, which is a covering of any other Abelian covering space of X. Prove that it is unique up to isomorphism and explicitly construct the universal Abelian cover of $S^1 \vee S^1$.

Please hand in your solutions on December 10 at the end of the lecture.