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Topology

Problem Set 6

1. (10 points) Let X be path-connected, locally path-connected, and semilocally simply-connected.
   (a) Let $p\colon \tilde{X} \to X$ be a covering and $U \subset X$ be a path-connected open neighbourhood of a point x for which the inclusion induces the trivial map $\pi_1(U,x) \to \pi_1(X,x)$. Show that U is *evenly covered* i.e. $p^{-1}(U)$ is a disjoint union of open sets which map homeomorphically to U under p.
   (b) Let $p_1\colon \tilde{X}_1 \to \tilde{X}_2$ and $p_2\colon \tilde{X}_2 \to X$ be covering maps. Show that $p_2 \circ p_1$ is a covering map.
   (c) Let $\tilde{X} \to X$ be the universal covering. Show that $\tilde{X}$ is a covering of any other path-connected covering space over X.

2. (10 points) Let X and Y be spaces with basepoints $x \in X$ and $y \in Y$ such that x and y are deformation retracts of respective open neighbourhoods in X and Y i.e. (in the case of x) there exists a neighbourhood $U \subset X$ of x and a homotopy $H\colon U \times I \to U$ such that $H(p,0) = p$, $H(p,1) = x$, and $H(x,t) = x$ for all $p \in U$, $t \in I$. Show that $\pi_1(X \vee Y, \{x,y\})$ is generated as a group by the images of

$$i_*^X\colon \pi_1(X,x) \to \pi_1(X \vee Y, \{x,y\}) \quad \text{and} \quad i_*^Y\colon \pi_1(Y,y) \to \pi_1(X \vee Y, \{x,y\})$$

   where i^X and i^Y are the obvious maps from X and Y to $X \vee Y$.

3. (20 points) Recall that the real projective plane $\mathbb{R}P^2$ is given as the quotient of S^2 obtained by identifying x and $-x$ for any $x \in S^2$.
   (a) Find the universal covering of $\mathbb{R}P^2$ and compute its group of deck transformations.
   (b) Find the universal covering of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ and show that its deck transformation group is generated by two elements of order 2 (i.e. whose square is zero).
   (c) Show that there is a subgroup $H \subset G = \pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2)$ such that $H \cong \mathbb{Z}$ and $G/H \cong \mathbb{Z}_2$.

Please hand in your solutions on November 26 at the end of the lecture.