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Topology

Problem Set 3

1. (10 points) For a topological space X and some subspace $A \subset X$ we define the space X/A as the quotient space of X generated by the relations $x \sim y \Leftrightarrow x, y \in A$. Denote by $D^n = \{x \in \mathbb{R}^n \mid x_1^2 + \ldots + x_n^2 \leq 1\}$ the *unit ball* and by $S^{n-1} \subset D^n$ the unit sphere. Show that
   (a) D^n/S^{n-1} is homeomorphic to S^n.
   (b) $S^n \times [0,1]/(S^n \times \{0\})$ is homeomorphic to D^{n+1}.

2. (10 points) For two spaces with fixed base points $x_0 \in X$ and $y_0 \in Y$ we define the *one-point union* $X \vee Y$ as the quotient of the disjoint union $X \coprod Y$ by the relation $x_0 \sim y_0$. Let p be any point in T^2. Prove that $T^2 \backslash \{p\}$ is homotopy equivalent to $S^1 \vee S^1$ for any choice of basepoints of S^1.

   Hint: Consult the picture from sheet 1. Independence of p (resp. the basepoints) is easily justified if for any pair of points $x, y \in T^2$ (resp. in S^1) one finds a homeomorphism mapping x to y.

3. Let X be the space $S^2 \cup \{(x, y, 0) \mid x^2 + y^2 \leq 1\} \subset \mathbb{R}^3$ equipped with the subspace topology. Also let $Y = S^2 \vee S^2$ where the north pole $(0,0,1)$ of one sphere is glued to the south pole $(0,0,-1)$ of the other sphere (in fact one can show as above that the construction is independent of specific choices of points). Prove that X and Y are homotopy equivalent.

4. (10 points) Show that the following statements about a topological space X are equivalent:
   (i) $\pi(X, x_0)$ is the trivial group for any $x_0 \in X$.
   (ii) Every map $S^1 \to X$ is homotopic to a constant map.
   (iii) Every continuous map $S^1 \to X$ is the restriction of a continuous map $D^2 \to X$.

Please hand in your solutions on November 5 at the end of the lecture.