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Riemannian Geometry

Problem Set 8

1. *Curvature on* $\mathbb{C}P^n$. The aim of this exercise is to understand the curvature of $\mathbb{C}P^n$ using Jacobi fields.

   Recall the definition of complex projective space $\mathbb{C}P^n$ as quotient of the unit sphere $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ by the action of the unit circle $S^1 \times S^{2n+1} \to S^{2n+1}, (u, z) \mapsto uz$, together with the Fubini-Study metric, which makes the projection $S^{2n+1} \to \mathbb{C}P^n$ a Riemannian submersion (see lecture notes, Example 1.8).

   Recall also the definition of a *horizontal vector field*: given a Riemannian submersion $p : (\tilde{M}, \tilde{g}) \to (M, g)$, $\tilde{m} \in \tilde{M}$, a vector field X on $\tilde{M}$ is horizontal if $X(\tilde{m})$ is in the orthogonal complement $H_{\tilde{m}}$ of the fiber $(d_{\tilde{m}}p)^{-1}(0)$.

   Let c be a geodesic parametrized by arclength on $\mathbb{C}P^n$, $c(0) = m, c'(0) = v$. Let u be a unit tangent vector in $T_m\mathbb{C}P^n$ orthogonal to v. We aim to compute the sectional curvature of the plane generated by u and v.

   Let $\tilde{m}$ be a preimage of m, and $\tilde{u}$ and $\tilde{v}$ be horizontal vectors in $T_{\tilde{m}}S^{2n+1}$. The geodesic $\tilde{c}(t) = \cos t \cdot \tilde{m} + \sin t \cdot \tilde{v}$ is a horizontal lift of c. Define the variation

   $$\tilde{h}(s,t) = \cos t \cdot \tilde{m} + \sin t(\cos s \cdot \tilde{v} + \sin s \cdot \tilde{u})$$

   with associated Jacobi field

   $$\tilde{J}(t) = \sin t \cdot \tilde{U}(t),$$

   where $\tilde{U}$ is the parallel vector field along $\tilde{c}$ with $\tilde{U}(0) = \tilde{u}$.

   Then $\tilde{h}$ descends to a geodesic variation h of c, with associated Jacobi field

   $$J(t) = (d_{c(t)}p \circ \tilde{J})(t) = \sin t((d_{c(t)}p \circ \tilde{U})(t)).$$

   Let $U(t) = (d_{c(t)}p \circ \tilde{U})(t)$. Show:

   (a) If $\tilde{u}$ is orthogonal to $i\tilde{v}$, then

   $$R(v, u)v = u.$$

   (b) If $\tilde{u} = \pm i\tilde{v}$, then

   $$R(v, u)v = 4u$$

   (c) In general,

   $$u = \cos\alpha \cdot u_0 + \sin\alpha \cdot Iv$$

   with u_0 orthogonal to iv and I the isomorphism on $T_m\mathbb{C}P^n$ induced by multiplication by i. Then the sectional curvature of the plane generated by u and v is

   $$K(u, v) = 1 + 3\sin^2\alpha.$$

   Hint: Recall, or re-prove, that for $X, Y \in \Gamma(\mathbb{C}P^n)$ *with horizontal lifts* $\tilde{X}, \tilde{Y}$, *the Levi-Civita connections satisfy*

   $$\nabla_X Y(p(\tilde{m})) = (d_{\tilde{m}}p \circ \tilde{\nabla}_{\tilde{X}}\tilde{Y})(\tilde{m}).$$