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Riemannian Geometry

Problem Set 7

1. *Sectional curvature and Jacobi fields.* Let M be a two-dimensional Riemannian manifold. Let $p \in M$ and $V \subseteq T_pM$ be a neighborhood of the origin where $\exp_p$ is a diffeomorphism. Let $S_r(0) \subseteq V$ be a circle of radius r around the origin, and let L_r be the length of $\exp_p(S_r)$ in M. Prove that the sectional curvature at p is

$$K(p) = \lim_{r \to 0} \frac{3}{\pi} \frac{2\pi r - L_r}{r^3}.$$

Hint: Recall from the previous exercise sheet that

$$\|J(t)\| = t - \frac{1}{6} g(R(v,w)v,w)t^3 + o(t^3).$$

2. *Second variation and geodesics.* Let f be a differentiable function on a Riemannian manifold M. Define the *Hessian* of f as

$$\mathrm{Hess} f(X) = \nabla_X \, \mathrm{grad} f;$$

recall that $\mathrm{grad} f$ is the vector field on M defined by

$$g(\mathrm{grad} f, v) = df_p v$$

for $p \in M$ and $v \in T_pM$.

Show:

(a) $g(\mathrm{Hess} f(X), Y) = X(Yf) - (\nabla_X Y)f$
(b) $\mathrm{Hess} f$ is symmetric.
(c) In critical points, $\mathrm{Hess} f$ is independent of the connection.

3. *Totally geodesic submanifolds.* Let M^m be a Riemannian manifold. Let ${N_1}^{n_1}$ and ${N_2}^{n_2}$ be totally geodesic, compact submanifolds with $n_1 + n_2 \geq m$.

Show:

(a) There exists a length-minimizing geodesic c from N_1 to N_2, and c meets both submanifolds orthogonally.
(b) Let p and q be the endpoints of c. There exists a $v \in T_pN_1$ such that parallel transport of v along c lands in T_qN_2.
(c) If M has strictly positive sectional curvature, N_1 and N_2 intersect.
Hint: Argue by contradiction and use part (b) *to find a suitable variation of* c.