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Riemannian Geometry

Problem Set 5

1. *Triangles in the hyperbolic plane.* Any three distinct points A, B, C, on $\mathbb{H}^2$ which do not all lie on the same bi-infinite geodesic define a unique *geodesic triangle* $\Delta_{a,b,c}$, where a, b, c are geodesic segments connecting B and C, A and C, and A and B, respectively.

   Show: Every geodesic triangle in $\mathbb{H}^2$ has an incircle of radius $r < \delta = \frac{\ln(3)}{2}$.

   Hints: First, given a point $M \in \mathbb{H}^2$, why do all the points at a fixed distance from M form a (Euclidean) circle? Second, consider the properties of ideal triangles, *i.e. those defined by three points on $\partial_\infty \mathbb{H}^2$ and the three bi-infinite geodesics connecting them. To argue with minimal computation, feel free to jump between the half-plane and disk models.*

2. *Octagons in the hyperbolic plane.* In this exercise, we will construct a hyperbolic metric by gluing sides of a hyperbolic octagon. We will work in the Poincaré disk model.

   (a) Let $0 < r < 1$. The eight points $v_k = re^{\frac{\pi}{4}ik}$ define a regular octagon whose interior angles depend on r. We want to find r such that these angles are $\frac{\pi}{4}$.

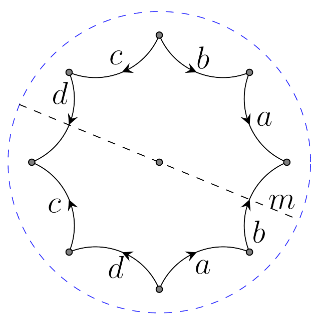
   i. Show that the circle in $\mathbb{C}$ containing the geodesic between v_0 and v_1 has midpoint
   $$z_0 = \frac{r^2+1}{2r} + i\frac{r^2+1}{2r}\tan\left(\frac{\pi}{8}\right).$$
   Hint: z_0 lies on the bisector of the segment between $v_0 = r$ and $v_0^* = 1/r$.

   ii. Let $b = \frac{r^2+1}{2r}$. Show that $\measuredangle v_0 z_0 b = \frac{\pi}{8}$ if and only if the interior angles of the octagon equal $\frac{\pi}{4}$.

   iii. Show that the interior angles of the octagon equal $\frac{\pi}{4}$ if and only if
   $$\tan\left(\frac{\pi}{8}\right) = \frac{|b - v_0|}{|z_0 - b|}$$

   iv. Conclude that $r = \sqrt[4]{\frac{1}{2}}$.

   (b) Construct a quotient of this octagon which is locally isometric to $\mathbb{H}^2$ by identifying its edges in the following way:



Hint: To construct charts around points on the edges, you can consider hyperbolic reflections along geodesic segments and geodesics such as m.

3. *Some properties of the curvature tensor.* Show:

   (a) $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,T) = 0$

   (b) $R(X,Y,Z,W) = -R(X,Y,W,Z)$

   (c) $R(X,Y,Z,W) = R(Z,W,X,Y)$.