Prof. Dr. Sebastian Hensel
Anna Ribelles Pérez

Riemannian Geometry

Problem Set 1

1. Recall that we defined the (flat) torus T^n in three different ways:
   (a) as a subset of $\mathbb{C}$:
   $$T^n = \{(z^1, \dots, z^n) \in \mathbb{C}^n \ : \ |z^i| = 1 \ \forall i\}$$
   (b) as the product manifold of n copies of S^1, with the metric induced by the standard embedding of the unit circle in $\mathbb{R}^2$
   (c) as the quotient of $\mathbb{R}^n$ by the action of a lattice $\mathbb{Z}^n$ given by translation.

   Show that these definitions are equivalent (up to scaling).

2. Consider the following group actions.
   (a) $\mathbb{R} \setminus \{0\}$ acting on $\mathbb{R}$ by multiplication
   (b) $\{+\mathrm{Id}, -\mathrm{Id}\} \cong \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{R}^n$ or S^n by reflection along a hyperplane.
   (c) $\mathbb{Z}/m\mathbb{Z}$ acting on R^2, where $[k] \in \mathbb{Z}/m\mathbb{Z}$ acts by rotation by $\frac{2\pi k}{m}$.

   Explain why the Quotient Manifold Theorem does not apply in each of the cases.

3. Find a necessary and sufficient condition for a vector field X on $\mathbb{R}^n$ to descend to a vector field on T^n.

4. Consider a tiling of the Euclidean plane by regular hexagons, and the surface obtained by taking the quotient of the translation action.
   (a) Show that the quotient is in fact diffeomorphic to the standard torus.
   (b) (*) Are both induced metrics the same (up to scaling)?

5. Compute the Levi-Civita connection on $\mathbb{H}^n$.