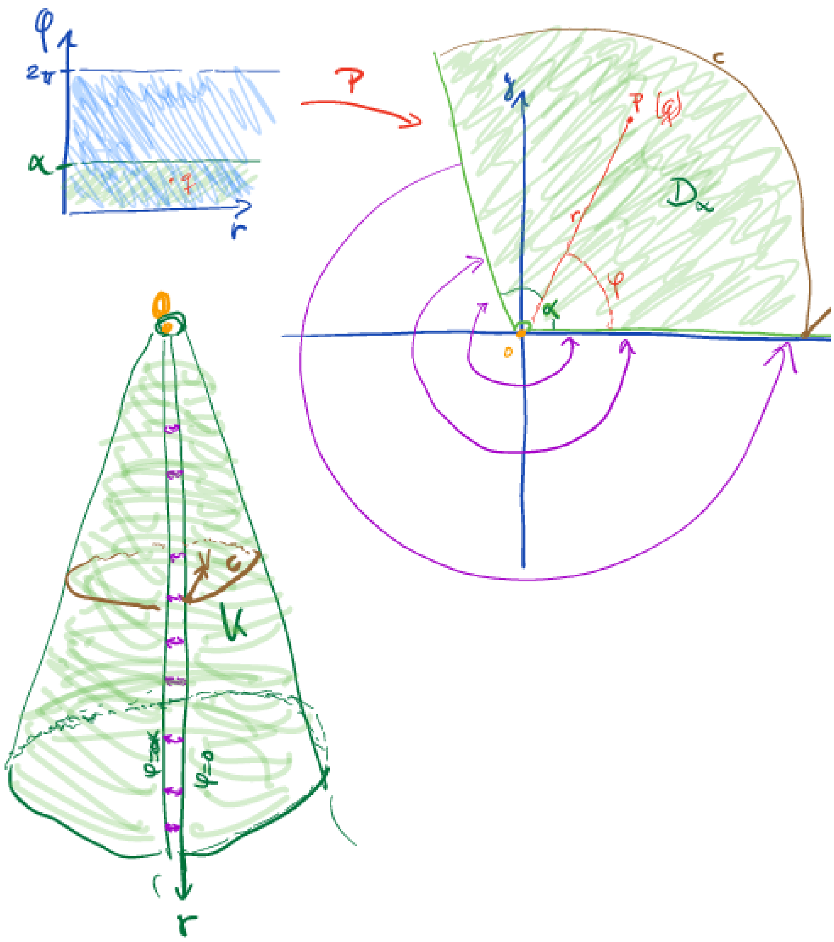
Name: ______________________________

Problem 2. Cone [1+1+2+2+2+1+1+3+2 points]

Consider $\mathbb{R}^2 \setminus \{(0,0)\}$ with polar coordinates $P\colon \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}^2$ with $(r,\varphi) \mapsto (r\cos\varphi, r\sin\varphi)$.



Rotations R_β by $\beta \in [-2\pi, 2\pi]$ act on $\mathbb{R}^2 \setminus \{(0,0)\}$ in the usual way and can be defined via $R_\beta(P(r,\varphi)) = P(r, \varphi+\beta)$ (even though P is not injective, you don't need to show that this is well defined).

For $0 < \alpha < 2\pi$ and $\epsilon > 0$, define the thickened wedge $D_{\alpha,\epsilon} := P(\mathbb{R}_{>0} \times (-\epsilon, \alpha+\epsilon))$. For $p_1, p_2 \in D_{\alpha,\epsilon}$ define an equivalence relation $p_1 \sim p_2 :\Leftrightarrow p_1 = R_\alpha(p_2)$ or $p_2 = R_\alpha(p_1)$. Denote the quotient by $K_0 = D_\alpha/\sim$ and equip it with the quotient topology. We denote the equivalence class of $p \in D_{\alpha,\epsilon}$ by $[p] \in K_0$.

In this exercise, we will equip K_0 with the structure of a smooth manifold, and study a specific Riemannian metric on it.

(a) Show that K_0 is Hausdorff.

(b) Denote the rotated (unthickened) wedges by $D^\pm := R_{\pm\epsilon/2}(D_{\alpha,0})$. Define two charts $\phi_\pm\colon [D^\pm] \to D^\pm$ via $\phi_\pm([p]) = p$. Show that the chart transitions are smooth.

(c) Define $K = K_0 \cup \{0\}$ by extending the map P continiously to include the origin for $r = 0$. Add a third chart covering 0 that turns K into a smooth manifold. In particular, verify that the chart transitions are smooth.

(d) On $[D^+]$ define a metric g using the identification via ϕ_+ with $D^+_{\alpha,0}$ and using the standard euclidean metric on $\mathbb{R}^2$, i.e.

$$(\phi_+^{-1})^* g = dx \otimes dx + dy \otimes dy.$$

Compute this metric in the chart provided by the polar coordinates map P.

(e) Show that a rotation R_β is a local isometry, i.e. that $(R_\beta^* g)_{[p]} = g_{[p]}$ for points $[p] \in [D^+]$ such that $R_\beta(p) \in D^+$.

(f) Use this to show that there is a unique extension of g to all of K_0.

(g) Let ∇ be the Levi-Civita connection for this metric. Show that the Riemann tensor vanishes on K_0. (*Hint: This can be done without computation.*)

(h) Compute the parallel transport of a vector along the closed path $c\colon [0,\alpha] \to K$ with $c(\varphi) = P(r,\varphi)$ for some fixed $r > 0$ as well as for a path $\tilde{c}\colon [r_0, r_1] \subset \mathbb{R}_{>0} \to K$ with $\tilde{c}(\rho) = P(\rho, \varphi_0)$ for some fixed φ_0.

(i) Show that the metric cannot be extended to all of K (*Hint: this can be done by showing that the Riemann curvature tensor of a hypothetical metric would not be compatible with what we concluded about parallel transport*).

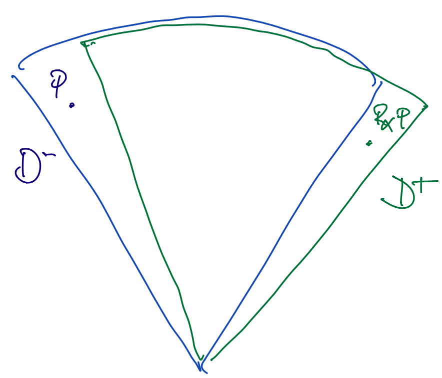
Problem 2 Example Solution

a) K_0 Hausdorff: $[p], [q] \in K_0$ $[p] \neq [q]$.

These two have at most four pre-images that are pairwise distinct.
Let δ be the minimum of pairwise distances and distance to the boundary.
Then images of balls with radius $\delta/3$ do the job.

b) Chart transitions:

NB: Even though $\phi_\pm([p]) = p$ we do not necessarily have $\phi_+([p]) = \phi_-([p])$!



$[D^+] \cap [D^-]$ has two disconnected components: One coming from $D^+ \cap D^-$ where the transition is the identity (smooth)

and one where the transition

$$D^- \setminus D^+ \longrightarrow D^+ \setminus D^-$$

is a rotation by R_α, which is also smooth.

c) In order to get an <u>open</u> coordinate neighbourhood, we need to stretch the angle via

$$[D_{\alpha,0}] \xrightarrow{[\;]^{-1}} D_{\alpha,0} \xrightarrow{\mathbb{P}^{-1}} \mathbb{R}_{>0} \times (0,\alpha) \xrightarrow{(id,\frac{2\pi}{\alpha}\cdot)} \mathbb{R}_{>0} \times (0,2\pi)$$

This can then be extended by continuity to $\mathbb{R}^2$ which provides a chart around the origin. All maps involved are smooth.

d) $$(\varphi_+^{-1})^* g = dx \otimes dx + dy \otimes dy$$
$$= (\cos\varphi\, dr - r\sin\varphi\, d\varphi)^{\otimes 2}$$
$$+ (\sin\varphi\, dr + r\cos\varphi\, d\varphi)^{\otimes 2}$$
$$= (\cos^2\varphi + \sin^2\varphi)\, dr \otimes dr$$
$$+ 2(-\cos\varphi\, r\sin\varphi + \sin\varphi\, r\cos\varphi)\, dr \otimes d\varphi$$
$$+ r^2(\sin^2\varphi + \cos^2\varphi)\, d\varphi \otimes d\varphi$$
$$= dr \otimes dr + r^2\, d\varphi \otimes d\varphi$$

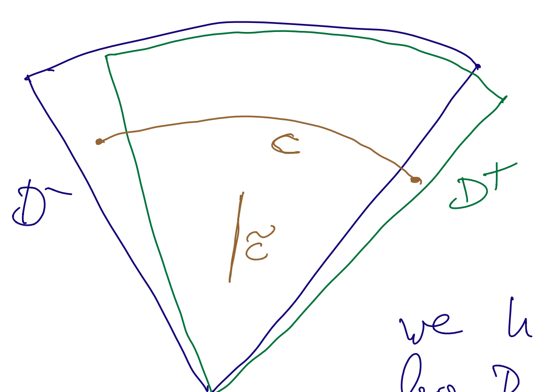
e) Since $d(\varphi+\beta) = d\varphi$, we have

$$(\phi_+^{-1})^* R_\beta^* g = (R_\beta \circ \phi_+^{-1})^* g$$
$$= dr \otimes dr + r^2\, d(\varphi+\beta) \otimes d(\varphi+\beta)$$
$$= dr \otimes dr + r^2\, d\varphi \otimes d\varphi$$
$$= (\phi_+^{-1})^* g.$$

f) $K_0 \setminus [D^+]$ can be rotated by $R_{\alpha/2}$ to $[D^+]$ which is a smooth map and we can thus extend g as $R_{\alpha/2}^* g$. This is smooth by e).

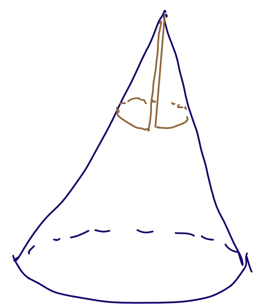
g) In our charts, the metric is the flat metric of $\mathbb{R}^2$ and thus $\nabla = D$ and the curvature vanishes (which can be checked locally).

h) This can be computed in polar coordinates but this is cumbersome. Easier: Cartesian coordinates of $\mathbb{R}^2$ where parallel transport is literally parallel, nothing rotates.



But for c, to connect the beginning in $D^- \setminus D^+$ to the end in $D^+ \setminus D^-$, we have to apply a rotation by R_α which also rotates along vector by an angle α (independent of r!)

i) If there were an extension, the vanishing curvature would have to be extended by 0. But that would imply (by theorem from last lecture) that the parallel transport around the (degenerate) square



would have to vanish for $r \to 0$ in contradiction to h).

NB: Arguing that the Christoffel-symbols diverge as $\frac{1}{r}$ does not work as this is a coordinate singularity. To see this take $\alpha = 2\pi$ where the cone is just flat $\mathbb{R}^2$ but Christoffels still diverge like $1/r$.

Problem 1. Cohomology [2+4+4+5 points]

In this exercise we consider throughout the manifold $M = S^1 \times S^2$.
You may use *without proof* that the manifolds $\mathbb{R}^n, (0,1)^n$ and D^n (the open unit ball) and a hemi-sphere are all diffeomorphic. Similarly, you may use *without proof* that the two-sphere S^2 is the union of two balls U_+, U_- which intersect in a region diffeomorphic to $S^1 \times (-1,1)$.

(a) Show that $H^0_{\mathrm{dR}}(M) \cong H^3_{\mathrm{dR}}(M) \cong \mathbb{R}$.
(b) Show that $H^1_{\mathrm{dR}}(M)$ is not trivial.
(c) Show that $H^1_{\mathrm{dR}}((0,1) \times S^2) = 0$. *Hint: Cover the manifold with two balls.*
(d) Show that $H^1_{\mathrm{dR}}(M) \cong \mathbb{R}$ (*Hint: observe that $M = U_1 \cup U_2$, where each U_i is diffeomorphic to $(0,1) \times S^2$, and so that $U_1 \cap U_2$ has two connected components C_1, C_2. Use the previous part to find antiderivatives on the U_i and compare them on the C_j.*)

Solution

a) We know from class that $\dim H^0_{dR}(M)$ is the number of connected components of M. Since $S^1 \times S^2$ is connected, this shows $\dim H^0_{dR}(M) = \mathbb{R}$.

The manifold $S^1 \times S^2$ is oriented, connected and compact, and has dimension 3. Hence, a theorem from class implies $H^3_{dR}(M) = \mathbb{R}$.

b) Consider the map

$$p : S^1 \times S^2 \to S^1$$

obtained by projecting to the first factor, and let θ be a closed 1–form on S^1 with $\int_{S^1} \theta = 1$ (we know that this exists from class).

Let $\omega = p^*\theta$. This is a closed one-form on M, since $d(p^*\theta) = p^* d\theta = 0$.

Now, let $q \in S^2$ be any point. Then

$$\int_{S^1 \times p} \omega = \int_{S^1 \times p} p^*\theta = \int_{S^1} \theta = 1.$$

Thus, ω is not exact, and therefore $0 \neq [\omega] \in H^1_{dR}(M)$.

c) As was done in class, write

$$S^2 = U_1 \cup U_2, \quad U_1 \cap U_2 \simeq S^1 \times (0,1)$$

where U_1, U_2 are diffeomorphic to the open unit disk.

Let ω be any 1–form on $(0,1) \times S^2$. Since $(0,1) \times U_i$ is diffeomorphic to a three-ball, there are functions $f_i : U_i \to \mathbb{R}$ so that $d\omega|_{(0,1)\times U_i} = f_i$ by the Poincare lemma.

The function $f_1 - f_2$ is defined on $U_1 \cap U_2 \simeq S^1 \times (0,1)$ and has vanishing derivative there. Hence, since $U_1 \cap U_2$ is connected, the function $f_1 - f_2$ is constant with value $c \in \mathbb{R}$.

Defining

$$f(x) = \begin{cases} f_1(x) \text{ if } x \in U_1 \\ f_2(x) + c \text{ if } x \in U_2 \end{cases}$$

gives then a function with $df = \omega$. This shows that any closed 1-form is exact, and thus $H^1_{dR}(M) = 0$.

d) We use the notation as in the hint. Let ω be a closed 1–form, and let $f_i : U_i \to \mathbb{R}$ be functions so that $df_i = \omega$. These exist, since $\omega|_{U_i}$ is a closed 1-form on $U_i \simeq (0,1) \times S^2$ and by part c) any such form is exact. Furthermore, since the U_i are connected, the f_i are determined up to the addition of a constant.

Let $p \in C_1$ and $q \in C_2$ be any two points. Consider the number

$$D(\omega) = (f_1(p) - f_2(p)) - (f_1(q) - f_2(q)).$$

Observe that this does not depend on the choice of the f_i, since the number is unchanged when we replace f_i by $f_i + c$.

Next, observe that $D(\omega)$ is linear in ω, and clearly $D(df) = 0$. Hence, D defines a linear function on $H^1_{dR}(M)$.

Furthermore, observe that ω is exact if $D(\omega) = 0$. Namely, observe that $f_1 - f_2$ is constant on both C_1 and C_2 (since these sets are connected and the derivative of $f_1 - f_2$ vanishes there), and so $D(\omega) = 0$ implies that $f_1 - f_2$ has the same value c on C_1 and C_2. Then, as in c), the function

$$f(x) = \begin{cases} f_1(x) \text{ if } x \in U_1 \\ f_2(x) + c \text{ if } x \in U_2 \end{cases}$$

gives then a function with $df = \omega$.

Together, these show that D defines an injective map $H^1_{dR}(M) \to \mathbb{R}$. Hence, $H^1_{dR}(M)$ has dimension at most 1, and by part b), it therefore has dimension exactly one.