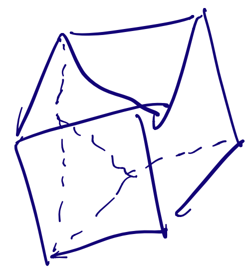
GR bla bla.

Polygon



$$\#C - \# E + \# F = 2$$

genus = #holes

or actually $\chi = 2 - 2g$

Claim: This is a discrete version of Gauß-~~Bonnet~~-Thm.:

$$\frac{1}{2\pi}\int_\Sigma R = \chi$$

idea: curvature only in corners of polygon.

deficit angle

$$R = \sum_c \left(2\pi - \sum_{i \text{ at } c} \theta_i\right) \delta_c$$

$$\int R = \sum_c \left(2\pi - \sum_{i \text{ at } c} \theta_i\right)$$

$$= 2\pi \#C - \sum_i \theta_i$$

$$= 2\pi \#C - \sum_{\text{face } F} \sum_{i \text{ at } F} \theta_i$$

$$= 2\pi \#C - \sum_F (\#(\text{edge at } F) - 2)\pi$$

$$= 2\pi \#C - 2 \cdot \#\text{edge}\, \pi + 2\pi \#F$$

(double counting

$$= 2\pi(\#C - \#E + \#F).$$

The RHS can also be generalized. It turns out that

$$\chi = \sum_{r=0}^{2} (-1)^r \dim H^r(M)$$

$$= \text{"} \dim \operatorname{Ker} d - \dim \operatorname{Ker} \overset{d^+}{\overset{\shortparallel}{\star d \star}} \text{"}$$

$$= \operatorname{index} d$$

This is only the tip of the ice berg: Many "index theorems" were discovered in the 1960/70s where an integral over a 1-polynomial of the curvature of some bundle is computed by an index of some (covariantized) Dirac operator.

These are related to "anomalies" of quantum field theories, the failure of a quantum theory to have the symmetries of the classical theory.

A 4D analogue is

$$\frac{1}{8\pi^2}\int \mathrm{tr}(F\wedge F) = \text{index } \cancel{D}$$

"instanton number"

In particular all these are in $\mathbb{Z}$ and topological invariants.

NB $F\wedge F = d(A\wedge F)$

Let's do something else.

Start with a cpt 3-manifold M (no metric). Maxwell's equations required a metric (action is $\frac{1}{e}\int F\wedge * F$, $*$ contains metric) but in this case, one can also write down a 'topological theory' with action

$$I_M = \frac{1}{K}\int_M A\wedge F$$

This is gauge invariant under $\tilde{A} = A + d\Lambda$

as

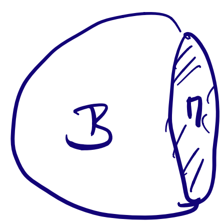
$$\tilde{I} = \frac{1}{k}\int_M (d\Lambda)\wedge F \quad + I$$

$$= \frac{1}{k}\int_M \Big(\underbrace{d(\Lambda F)}_{0 \text{ Stokes}} - \underbrace{\Lambda dF}_{0}\Big) + I$$

but in general A with $F = dA$ exists only locally, so what are we supposed to integrate?

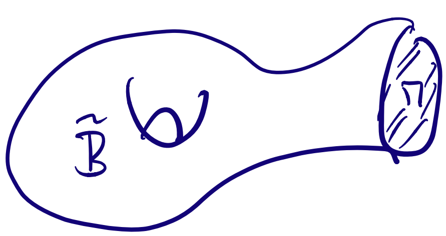
Witten tells us: Take instead a 4-manifold B with $\partial B = M$ and rather define

$$I'_M = \int_B F\wedge F$$



Then, thanks to Stokes $I' = I$.

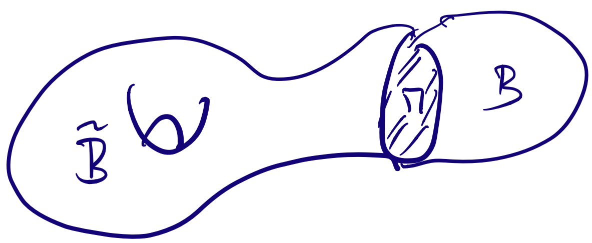
But this seems to depend on an arbitrary choice of B. What about $\tilde{B}$ with $\partial\tilde{B} = M$ as well



?

$$\tilde{I}_M' - I_M' = \frac{1}{\varkappa}\int_{B\cup\tilde{B}} F\wedge F$$

cpt and no boundary!



$$\in \frac{8\pi^2}{\varkappa}\mathbb{Z} \qquad \text{by index thm.}$$

⇒ path integral with integrand

$$e^{iI_M'} \quad \text{is well defined}$$

as long as $\varkappa \in 4\pi\mathbb{Z}$.

⇝ quantization of coupling const.

Witten received a Fields medal for pointing out that

$$\int \mathcal{D}A \; e^{iI_M'(A)} \qquad \text{(made proper sense of)}$$

yields invariants of M (and knots in it).
String theory provides many more connections of gauge theories & geometry