THE SET OF UNIQUELY ERGODIC IETS IS PATH-CONNECTED

JON CHAIKA AND SEBASTIAN HENSEL

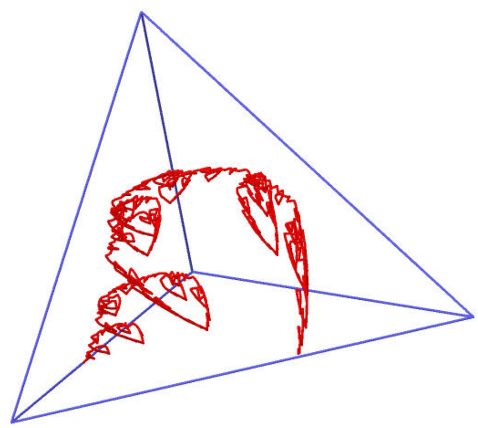

FIGURE 1. A path of uniquely ergodic 4-IETs

1. INTRODUCTION

An interval exchange transformation (from now on abbreviated to *IET*) is a piece-wise isometric map of an interval to itself that rearranges sub-intervals according to a permutation π (see Section 2 for formal definitions). While simple to define, interval exchange transformations have deep and interesting dynamical properties as well as connections to foliations on Riemann surfaces, and therefore have been the subject of intense study over the last years. See [Z06], [Y10] or [V06] for good surveys.

In this article we are concerned not with the dynamics of a single IET, but the set of all IETs with a given permutation π of n symbols. This set carries a natural topology and is homeomorphic to an Euclidean simplex Δ_π of dimension $n-1$. It is known that unless the permutation has obvious combinatorial obstructions most IETs are uniquely ergodic:

Theorem. *(Masur* [M82]*, Veech* [V82]*) Given an irreducible permutation almost every IET with that permutation is uniquely ergodic with respect to Lebesgue measure.*

J. Chaika was supported in part by NSF grant DMS 1330550.

On the other hand, it is known that the set of non-uniquely ergodic IETs is also fairly large in a geometric sense:

Theorem. *(Masur-Smillie* [MS91]*) The set of minimal and not uniquely ergodic IETs with a non-degenerate permutation on n intervals has Hausdorff dimension greater than $n-1$.*

The Hausdorff dimension of minimal and not uniquely ergodic 4-IETs has been computed by the J. Athreya and the first named author.

Theorem (Athreya-Chaika [AC])**.** *The set of minimal and not uniquely ergodic IETs with a non-degenerate permutation on 4 intervals has Hausdorff dimension greater than $\frac{5}{2}$.*

In this article we investigate the set of uniquely ergodic IETs from a topological point of view. More precisely, the main result of this paper is

Theorem 1.1. *Let $n \geq 4$ and let π be any non-degenerate permutation on n symbols. Then the set of uniquely ergodic unit length IETs with permutation π is path connected.*

For the formal definition of non-degenerate permutations see Section 2. Intuitively, non-degenerate means that no induced map on a sub-interval has less than n singularities. In the case of $n = 2$, the space of 2-IETs is equal to the space of rotations; where such an IET is uniquely ergodic if and only if the rotation angle is irrational, which is clearly a disconnected subset.

3-IETs can be thought of as the induced map (see Definition 2.3) of a rotation by $\frac{1-L_1}{1+L_2}$ on $[0, \frac{1}{1+L_2})$ (compare [KS67], [K75]) and they are uniquely ergodic if and only if $\frac{1-L_1}{1+L_2}$ is irrational. So a set $\frac{1-L_1}{1+L_2} = y \in \mathbb{Q}$ gives a curve (or plane if one does not normalize the lengths of an IET) of non-minimal (and so not uniquely ergodic) IETs that disconnect the space of 3-IETs.

Thus the bound on n in Theorem 1.1 is optimal.

Outline of proof. The proof of Theorem 1.1 begins with a reduction to the case of $n = 4$. Namely, the space of n-IETs contains many copies of the space of 4-IETs by setting the length of enough intervals to be 0 – and in these subspaces we know that the uniquely ergodic IETs are path-connected. Studying the combinatorics of a non-degenerate permutation, we can show that the union U of all these subspaces is path-connected. Using a limiting argument we can then show that each uniquely ergodic IET can be connected by a path to a point in U.

In the case of $n = 4$ we are able to give an explicit inductive procedure that constructs paths in the space of uniquely ergodic IETs. Unique ergodicity of an IET T can be detected by the Rauzy induction of T. Thus ideally one would want to construct paths c where Rauzy induction has desirable properties for each $c(t)$.

A key problem however is that Rauzy induction is undefined on a countable set of codimension 1 planes in the simplex. Continuous paths necessarily cross these "fail planes", and in general most points on such a plane do not correspond to uniquely ergodic IETs.

This suggests the following naive strategy: start by joining two IETs S, T by a straight line segment $S \to T$. If this line intersects the plane where Rauzy induction fails, select a uniquely ergodic point R on that plane, and replace the line segment by a concatenation $S \to R \to T$ of two segments. Now both $S \to R$ and $R \to T$ share the same first Rauzy induction step. We can continue this process iteratively, always replacing straight segments by concatenations after some number of Rauzy steps.

There are two main issues with this approach: why does the sequence of approximate paths converge, and why are all points on it uniquely ergodic?

We deal with both of these issues simultaneously by taking care how to choose the intermediate IETs on the fail planes. More precisely we want that limit points which do not eventually follow the Rauzy expansion of one of the points on a fail plane to have infinitely many matrices which define contracting maps on the simplex in their Rauzy expansion. This will yield both unique ergodicity and continuity at these limit points. For the points on the fail planes, unique ergodicity is clear from the construction, but continuity needs to be proved by a different argument.

Section 2 of the article describes some necessary background on IETs and Rauzy induction, while Section 3 contains some basic results and notation on matrices that is used throughout.

Section 4 describes the basic mechanism used to construct the paths in the case $n = 4$ (depending on an explicit construction done in Section 8) – the necessary results to show continuity at points on "fail planes" is developed in Section 5, the results for other points is developed in Section 6. Finally, Section 7 contains the proof of Theorem 1.1 for $n = 4$. In Section 9 we extends the results to n-IETs.

Questions. The work in this article suggests several possible directions of further research. One possibility is to ask further topological question about sets of specific IETs. In particular, we have

Question 1 (Mladen Bestvina)**.** In the case of n-IETs for $n \geq 5$, does the set of uniquely ergodic n-IETs satisfy higher connectivity properties?

Question 2. Is the set of minimal n-IETs path connected?

On the other hand, one could try to generalize the question of the topology of the set of uniquely ergodic transformations from IETs to other settings. One example is the following

Question 3. Is there 1-parameter diagonal flow on $SL_4(\mathbb{R})/SL_4(\mathbb{Z})$ so that the set of points whose orbit under this flow is bounded path connected?

Another example is a question that inspired this work. The space of measured foliations on a Riemann surface of genus g is homeomorphic to a sphere of dimension $6g - 7$. The set of uniquely ergodic laminations has full measure [M82], and one can ask

Question 4. For a surface of genus $g \geq 2$, is the set of uniquely ergodic laminations path-connected?

There is a closely related space $\mathcal{EL}$ (the space of ending laminations) which has been studied extensively. Ending lamination space is known to be connected and locally path-connected [G09], [LS09], and even has been identified explicitly in some low-complexity cases [HP11], [G11]. However, the set of uniquely ergodic foliations is only a subspace of ending lamination space, and so these results do not yield any direct information about the topology of the set of uniquely ergodic laminations. Somewhat in the line of our results is a result by Leininger-Schleimer which ensures the existence of spheres in the set of uniquely ergodic laminations [LS11]. To the authors' knowledge Question 4 is still open.

Once could also try to adapt the methods of the current paper to the setting of surface foliations by building flat surfaces from IETs. The direct analog of the case $n = 4$ (which all our work is inductively based on) would concern foliations in the stratum $\mathcal{H}(2)$ in genus 2. However, in this stratum every saddle connection defines a simple closed curve, and thus arational foliations are not path connected. Indeed in every Rauzy class there are permutations where the IETs corresponding to arational foliations are not path connected. However, one can still ask the following question:

Question 5. For a suitable permutation π, is the set of all IETs corresponding to arational uniquely ergodic foliations path-connected?

Acknowledgments: We would like to thank Jayadev Athreya and Howard Masur for inspiring discussions and interest in the project. Furthermore, we express our gratitude for the numerous and detailed comments by the anonymous referee, which helped to improve the paper greatly.

2. Interval Exchange Transformations and Rauzy Induction

In this section we fix the notation for interval exchange transformations (*IETs*) that we will use, and also collect some well known results that we will use throughout. We refer the reader to [V06] for a detailed treatment.

We denote a permutation π on the symbols $1, \dots, n$ by the list containing the image of the symbols under the inverse permutation π^{-1}. That is, the identity permutation on four symbols is denoted by (1234), the transposition of the first two symbols is (2134). The permutation which maps every symbol to the next one (cyclically) would be (4123). The reason for this convention will be clear later.

Definition 2.1. *Given $\hat{L} = (l_1, l_2, ..., l_d)$ where $l_i \geq 0$, we obtain d sub-intervals of the interval $\left[0, \sum_{i=1}^{d} l_i\right)$:*

$$I_1 = [0, l_1), I_2 = [l_1, l_1 + l_2), ..., I_d = [l_1 + ... + l_{d-1}, l_1 + ... + l_{d-1} + l_d).$$

*Given a permutation π on the set $\{1, 2, ..., d\}$, we obtain a d-*Interval Exchange Transformation *(IET)*

$$T\colon \left[0, \sum_{i=1}^{d} l_i\right) \to \left[0, \sum_{i=1}^{d} l_i\right)$$

which exchanges the intervals I_i according to π. That is, if $x \in I_j$ then

$$T(x) = x - \sum_{k<j} l_k + \sum_{\pi(k')<\pi(j)} l_{k'}.$$

If such a T is given, we denote its (unnormalized) length vector by $\hat{L}(T)$ and its permutation by $\pi(T)$.

With our convention, $\pi(T)$ is then described by the order of intervals from left to right after applying the IET. If T is an IET, we denote by

$$L(T) = \hat{L}(T)/|\hat{L}(T)|_1$$

the (normalized) length vector of T. We say that a IET is *normalized* if it is defined on the unit interval. Every IET can be rescaled to a normalized one, and we will usually identify IETs which just differ by such a rescaling without explicit mention. When we speak about a length vector we mean the normalized one, unless specified explicitly.

Intuitively, a permutation on n symbols is degenerate if any IET with that permutation has either fewer than $n-1$ discontinuities or an induced map of (see below for the definition) it has fewer than $n-1$ discontinuities. The technical conditions are as follows ([V78, Section 3])

Definition 2.2. *A permutation π on n symbols is* degenerate *if there is some $j < n$ so that one of the following holds.*

(1) $\pi(j+1) = \pi(j) + 1$
(2) $\pi(j) = n$, $\pi(j+1) = 1$ *and* $\pi(1) = \pi(n) + 1$
(3) $\pi(j+1) = 1$ *and* $\pi(1) = \pi(j) + 1$
(4) $\pi(j+1) = \pi(n) + 1$ *and* $\pi(j) = n$

Definition 2.3. *Let $T : [0,1) \to [0,1)$ be measurable and Lebesgue measure preserving and $A \subset [0,1)$. The* induced map of T on A *is $T_A : A \to A$ given by*

$$T_A(x) = T^{n(x)}(x), \qquad n(x) = \min\{n > 0 : T^n(x) \in A\}.$$

Recall that the Poincaré recurrence theorem guarantees that T_A is defined almost everywhere in A.

Next, we briefly describe the most important points of *Rauzy induction*, the renormalization method we will use throughout to study IETs. Our treatment of Rauzy induction will be the same as in [V82, Section 7].

Let T be a n-IET with permutation π. Let δ_+ be the rightmost discontinuity of T and δ_- be the rightmost discontinuity of T^{-1}. Let $\delta_{max} = \max\{\delta_+, \delta_-\}$. Consider the induced map of T on $[0, \delta_{\max})$ denoted $T|_{[0,\delta_{\max})}$. The result is again an IET, perhaps with a different permutation. We can renormalize it so that it is once again a n-IET on $[0,1)$. That is, let $R(T)(x) = \frac{1}{\delta_{\max}} T|_{[0,\delta_{\max})}(x\delta_{\max})$. This is the Rauzy induction of T.

If $\delta_+ \neq \delta_-$ then the permutation and length vector of the IET $R(T)$ can be combinatorially determined.

Namely, if $\delta_{max} = \delta_+$ we say the first step in Rauzy induction is a. In this case the permutation of $R(T)$ is given by

$$\pi'(j) = \begin{cases} \pi(j) & j \leq \pi^{-1}(n) \\ \pi(n) & j = \pi^{-1}(n) + 1 \\ \pi(j-1) & \text{otherwise} \end{cases}.$$

We keep track of what has happened to the interval lengths under Rauzy induction by a matrix $M(T,1)$ where

$$M(T,1)[ij] = \begin{cases} \delta_{i,j} & j \le \pi^{-1}(n) \\ \delta_{i,j-1} & j > \pi^{-1}(n) \text{ and } i \neq n \\ \delta_{\pi^{-1}(n)+1,j} & i = n \end{cases}.$$

If $\delta_{max} = \delta_-$ we say the first step in Rauzy induction is b. In this case the permutation of $R(T)$ is given by

$$\pi'(j) = \begin{cases} \pi(j) & \pi(j) \le \pi(n) \\ \pi(j)+1 & \pi(n) < \pi(j) < n \\ \pi(n)+1 & \pi(j) = n \end{cases}.$$

Again, we keep track of what has happened to the interval lengths under Rauzy induction by a matrix

$$M(T,1)[ij] = \begin{cases} 1 & i = n \text{ and } j = \pi^{-1}(n) \\ \delta_{i,j} & \text{otherwise} \end{cases}.$$

In the case where $\delta_+ = \delta_-$ both the formulas in case a and case b allow to identify $R(T)$ with an IET. We write $R_a(T)$ and $R_b(T)$ for these choices.

The change in permutation under Rauzy induction can be depicted in a *Rauzy diagram.* Below is the diagram for $n = 4$.

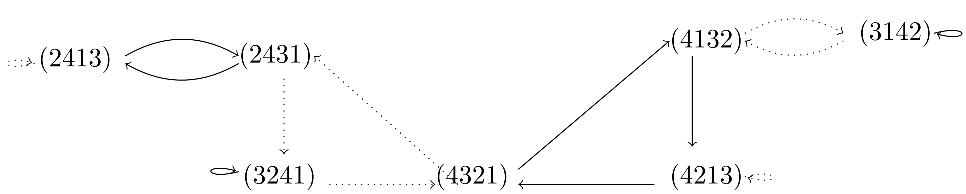

FIGURE 2. The Rauzy diagram for $n = 4$.

The matrices described above depend on whether the step is a or b and the permutation $\pi(T)$. The following well known lemmas which are immediate calculations help motivate the definition of $M(T,1)$.

Lemma 2.4. *If $R(T) = S$ then the length vector $L(T)$ is a scalar multiple of $M(T,1)L(S)$.*

We will denote the positive orthant by

$$\mathbb{R}^d_+ = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d | x_i > 0 \,\text{for all}\, i\right\}$$

and the open Euclidean simplex by

$$\mathring{\Delta}_d = \left\{ (x_1,\ldots,x_d) \in \mathbb{R}^d_+, \sum_{i=1}^{d} x_i = 1 \right\}.$$

We let $M\Delta = M\mathbb{R}^+_d \cap \mathring{\Delta}_d$.

Lemma 2.5. *Let T be some IET. An IET S with permutation $\pi(S) = \pi(T)$ and whose length vector $L(S)$ is contained in $M(T, 1)\Delta$ has the same first step of Rauzy induction as T.*

We define the n^{th} matrix of Rauzy induction by

$$M(T, n) = M(T, n-1)M(R^{n-1}(T), 1).$$

It follows from Lemma 2.5 that for an IET with length vector in $M(T, n)\Delta$ and permutation π the first n steps of Rauzy induction agree with T.

The set of all normalized n-IETs with a given permutation π can be naturally identified with a simplex which we denote by Δ_π. Iterated Rauzy induction defines a partition of Δ_π into smaller simplices in the following way. The subset of Δ_π corresponding to IETs on which Rauzy induction is undefined is a codimension-1 simplex embedded in Δ_π. In each of the complementary full-measure simplices, the set where Rauzy induction is defined once but not twice again is a union of two codimension-1 simplices. We denote by $\mathcal{P}_k$ the full-measure partition of Δ_π into the (open) simplices on which Rauzy induction is defined for the first k steps. A simplex in $\mathcal{P}_k$ consists of all the IETs which follow the same first k steps in the Rauzy diagram under Rauzy induction.

The projective linear map defined by the matrix $M(T, k)$ maps the standard simplex $\Delta_{\pi'}$ (where π' is the permutation corresponding to the IET $R^k(T)$) to the simplex in $\mathcal{P}_k$ which contains T.

We will also need a criterion that ensures unique ergodicity of IETs. A *Rauzy path* is a finite or infinite path in a Rauzy diagram $\mathcal{R}$. Associated to a Rauzy path is the product of matrices describing the change on lengths of the intervals which we will call the *Rauzy matrix* of the path. The following is well-known.

Theorem 2.6. *(Veech* [V78, page 225]*) Suppose that T is an IET where $R^n(T)$ is defined for all $n \geq 1$, and such that $\bigcap_{n\geq 1} M(T, n)\Delta = \{T\}$. Then T is uniquely ergodic.*

To check that the prerequisite of Theorem 2.6 is satisfied, it suffices to check that the angle between the columns of $M(T, n)$ converges to 0, since $M(T, n)\Delta$ consists of convex combinations of the columns of $M(T, n)$.

2.1. **Shadows of Rauzy paths.** We need to make the following consideration for our proof to hold for all uniquely ergodic 4-IETs, as opposed to just uniquely ergodic 4-IETs that have all powers of Rauzy induction defined.

Let T be a minimal d-IET with the length of an interval equal to 0. Let $\hat{T}$ be the minimal $d-1$-IET given by "forgetting the interval of T with length 0." For example if $L(T) = (\alpha, 0, 1-\alpha)$ and $\pi(T) = (321)$ then $L(\hat{T}) = (\alpha, 1-\alpha)$ and $\pi(\hat{T}) = (21)$. Assume that $R^n(\hat{T})$ is defined for all n. Let T_ϵ be the d-IET with $\pi(T_\epsilon) = \pi(T)$ and

$$L(T_\epsilon) = \begin{cases} L(T) & \text{if } L(T) \neq 0 \\ \epsilon & \text{else} \end{cases}.$$

For all $\ell > 0$ there exists n (possibly 2 different n) and $\sigma_n \in \{1, ..., d\}$ so that

- $\lim_{\epsilon\to 0} L(R^n T_\epsilon)_i = L(R^\ell \hat{T})_{i-c_{i,\sigma_n}}$ for all i

- For any $i, j \neq \sigma_n$ we have $\pi(R^n(T_\epsilon))(i) > \pi(R^n(T_\epsilon))(j)$ iff $\pi(R^\ell(\hat{T}))(i - c_{i,\sigma_n}) > \pi(R^\ell(\hat{T}))(j - c_{j,\sigma_n})$ for all small enough ϵ

where $c_{k,\ell} = \begin{cases} 0 & \text{if } i < \ell \\ 1 & \text{if } i > \ell \\ \text{undefined} & \text{if } i = \ell \end{cases}$.

Given n, for all ϵ small enough ϵ we have $M(T_\epsilon, n)$ and $M(\hat{T}, \ell)$ are related in the following way: after deleting the σ_n^{th} column of $M(T_\epsilon, n)$ one is left with a $d \times (d-1)$ matrix with a row of all zeros, so that when it is deleted one has $M(T, \ell)$.

We prove the result by induction on ℓ. We assume that this is true for ℓ with corresponding number n. We say j is in *critical position* if $j \in \{d, \pi^{-1}d\}$. The proof breaks into two parts, when σ_n is not in critical position in which case the corresponding number for $\ell+1$ is $n+1$ and when σ_n is in critical position in which case the corresponding number for $\ell+1$ is $n+2$.

Case 1: σ_n is not in critical position. In this case it is straightforward for $\ell+1$ and the corresponding number is $n+1$ with

$$\sigma_{n+1} = \begin{cases} \sigma_n & \text{if } R(R^n T_\epsilon) \text{ is 'b' or } R(R^n T_\epsilon) \text{ is 'a' and } \sigma_n < \pi^{-1}(d) \\ \sigma_n + 1 & \text{else} \end{cases}.$$

Case 2: σ_n is in a critical position. Let $j = d$ if $\sigma_n = \pi^{-1}(d)$ and $\pi^{-1}(d)$ if $\sigma_n = d$. By Conclusion 1 (which we are inductively assuming), for all small enough ϵ we have that $L(T_\epsilon)_j > L(T_\epsilon)_{\sigma_n} = \epsilon$. Set

$$\sigma_{n+1} = \begin{cases} \pi^{-1}(d) + 1 & \text{if } \sigma_n = d \\ \sigma_n & \text{else} \end{cases}.$$

Conclusion 1 follows inductively. Indeed, except for one entry $L(R^{n+1}T_\epsilon)$ is a permutation of $L(R^n T_\epsilon)$ and the exceptional entry is $L(R^n T_\epsilon)_j - \epsilon$. Conclusion 2 is straightforward to check, as is the claim on the matrices (the deleted column should be σ_{n+1}). Moreover σ_{n+1} is not in the critical position so we may follow case 1 for $\ell+1$ which will correspond to $n+2$. To see that σ_{n+1} is not in critical position, if $\sigma_n = d$ then $\pi^{-1}(d) \neq d-1$ be the minimality of $R^\ell \hat{T}$ (which follows from our assumptions on $\hat{T}$). The case of $\sigma_n = \pi^{-1}(d)$ is similar.

This discussion has a bearing on minimal IETs with nonzero interval lengths that will have a failure of Rauzy induction. Indeed, assume T is a minimal d-IET that has R^k defined on it, but R^{k+1} is not defined on it. This means that the two intervals of $R^k T$ in critical position have the same length. One can formally continue Rauzy induction (by either R_a or R_b) one step and have an IET, S with the length of one entry 0. One also gets a matrix $M = M(T, k)M'$ where M' is the matrix given by the formal step or Rauzy induction. We may now forget the entry of S that has zero length and obtain a $(d-1)$-IET, $\hat{S}$. We may relate its path under Rauzy induction to S_ϵ as in the previous discussion. To recover an approximation to T we just multiply the matrices and lengths we obtain for S by M. This procedure can be successively iterated if there is a subsequent failure of Rauzy induction for $\hat{S}$.

3. Matrix Terminology and Combining Matrices

In this section we will collect some notation and terminology on (4×4)–matrices with nonnegative integral entries.

If A is such a matrix, then we will denote by $C_i(A)$ the i-th column of A. We will often call i the *index* of the column in such a context. If P is some property a column of a matrix (or a vector in general) may have, we will say that the *index i or column i has P* if $C_i(A)$ has P, when the matrix A is clear from the context.

We use the 1-norm for vectors throughout, so $|C_i(A)|$ will denote the sum of the entries in the i-th column. We will often need to consider the column of A with the largest sum of entries. We denote this by $C_{max}(A)$. Should there be several columns which all have the largest entry sum, we adopt the convention that $C_{max}(A)$ is the column with smallest index realising this maximum.

Recall that in a matrix product AB, the column $C_i(AB)$ is a sum of the columns of A, with coefficients from $C_i(B)$:

$$C_i(AB) = \sum_{j=1}^{4} (C_i(B))_j C_j(A)$$

Thus, we say that B *adds column i to column j (or: column i is added to column j)* if the i-th entry of $C_j(B)$ is positive (usually, it will be the case that the j-th entry of $C_j(B)$ is also positive, to make the terminology completely justified, but we do not insist on this). Note that it is possible for a column to add to itself. Equivalently, we may say that *column j has column i added to it.* If there is any j so that column i is added to column j, then we simply say that *column i is added to another column.* Similarly, we simply say that *column i has a column added to it* if there is a corresponding j.

This terminology will be used in particular when considering powers of some matrix B.

Next, we introduce the central new notion of this section. Combining matrices will be generalisations of positive matrices, in the sense shown in Lemma 3.4 below.

Definition 3.1. *A matrix M is called* combining *if there are two groups of columns,* active *and* passive *ones. We require that that*

1. *There are at least 2 active columns.*
2. *At most one column of M is neither active nor passive. This column is called* idle *if it exists.*
3. *The only columns that are added to other columns are the active ones.*
4. *Every active column is added to all other active columns and each passive column has at least one active column added to it.*

A finite Rauzy path P is said to be combining *if the corresponding matrix is combining.*

Example 3.2. (1) Every positive matrix is combining so that every column is active.

(2) The matrix
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 2 \end{pmatrix}$$
is combining. Namely, columns 1 and 4 are active, while 2 and 3 are passive.

(3) The matrix
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 \end{pmatrix}$$
is **not** combining. Column 2 has Columns 1,3 and 4 added to it. Thus, all three of those would need to be active. However, column 4 does not have column 3 added to it, violating the definition of active.

(4) The matrix
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$
is combining, where columns 3 and 4 are active, 1 is passive and column 2 is idle.

Theorem. *(Perron-Frobenius) If M is an $n \times n$ matrix with all entries positive then there exists a unique largest (in absolute value) eigenvalue, called the Perron-Frobenius eigenvector. The corresponding eigenvector can be chosen to be positive (i.e. has all entries positive) and is called the Perron-Frobenius eigenvector. It is the only eigenvector with that property.*

One can observe that matrices as in the previous theorem act as a contraction in the so called Hilbert projective metric and we obtain the following result (see e.g. [V78, page 240]).

Proposition 3.3. *If M is an $n \times n$ matrix with all entries positive then any non-negative (non-zero) vector v has that $M^n v$ converges exponentially quickly to the direction of the Perron-Frobenius eigenvector.*

We conclude with a lemma generalizing the Perron-Frobenius theorem to the case of combining matrices. It will be used frequently in the sequel.

In its formulation, we will call the j–th entry $(C_i(A))_j$ of a column *active* (or passive, or idle), if the corresponding index j has this property (i.e. the column $C_j(A)$ is active, passive, or idle).

Lemma 3.4. *Let A be a combining matrix. Then there are numbers $E, \gamma' > 1$ with the following properties: let $n > 3$ be any integer, and let i, j be indices corresponding to active or passive columns.*

i) $|C_i(A^n)|/|C_j(A^n)| \leq E$.

ii) Any two of the active entries in columns i or j of A^n differ by a factor of at most E.

iii) We have

$$\sin \angle(C_i(A), C_j(A)) \leq \frac{E}{\gamma'^n}$$

The proof requires the following standard lemma.

Lemma 3.5 (Law of Sines)**.** *Let $v, w \in \mathbb{R}^n$,*

$$|\sin \angle(v+w, w)|| = \frac{\|v\|}{\|v+w\|} |\sin \angle(v, w)|.$$

Proof of Lemma 3.4. Suppose that $\tau \leq 4$ columns are active, and let $Y \subset \mathbb{R}^4$ be the τ-dimensional subspace which is preserved by A and so that the matrix $\hat{A}$ describing the action of A on Y is positive. Thus by the Perron-Frobenius theorem there is a unique positive eigenvector $\hat{w} \in Y$ of $\hat{A}$, which has positive eigenvalue μ that is the largest eigenvalue in absolute value. If $\tau < 4$, consider the action of $\hat{A}$ on the invariant subspace of dimension $\tau - 1$ which does not contain $\hat{w}$, $\tilde{A}$. There exists $\gamma < \mu$ and C so that $\|\tilde{A}^k\|_{op} \leq C\gamma^k$.

i) By the Proposition 3.3 it follows that the projection of the active columns to the τ dimensional subspace in the previous paragraph does not lie in the $\hat{A}$ invariant $\tau - 1$ dimensional subspace complemented to the Perron-Frobenius eigenvector. So the size of these columns grow proportionally to the Perron-Frobenius eigenvalue of $\hat{A}$. The passive columns get some multiples of active columns added to them. These active columns are growing exponentially (according to the Perron-Frobenius eigenvalue) and so the column is proportional to the last summand once n is large enough. For small n proportionality also holds simply by finiteness. If we consider a passive column v of A^n it has the form $\sum_{j=1}^{n-1} w_j$ where the w_j are active columns of A^j. Since the active columns are growing exponentially, w_{n-1} is proportional to the largest active column of A^n.

ii) This is true for powers of the τ-by-τ matrix $\hat{A}$. Indeed by Proposition 3.3 the ratio of the entries converges to the ratio of the entries of the Perron-Frobenius eigenvector. Arguing as in (i) the passive column(s) are sums of vectors with this property, so they inherit it.

iii) Arguing as before, by Proposition 3.3 the active columns of A converge exponentially fast to $\hat{w}$ under taking powers, and the angle between the active columns decreases as required.

Consider next a passive column of A^n. This column has the form $\sum_{i=0}^{n} v_i$, where v_0 is the initial passive column of A, and each v_i is a linear combination of active columns of A^i. In particular, each v_i has norm between $D_1\mu^i$ and $D_2\mu^i$ where μ is the Perron-Frobenius eigenvalue and D_1, D_2 depend on the matrix A.

Next, note that

$$\sin \angle\left(\sum_{i=0}^{n} v_i, \hat{w}\right) \leq \sin \angle\left(\sum_{i=0}^{n} v_i, \sum_{i=n/2}^{n} v_i\right) + \sin \angle\left(\sum_{i=n/2}^{n} v_i, \hat{w}\right)$$

since (absolute values of) angles satisfy the triangle inequality and $\sin(\theta+\phi) \leq \sin(\theta) + \sin(\phi)$ for all $0 \leq \theta, \phi$ with $\theta, \phi < \frac{\pi}{2}$.

Since $\frac{\sum_{i=1}^{\frac{n}{2}} D_2\mu^i}{\sum_{i=\frac{n}{2}}^{n} D_1\mu^i}$ decays exponentially in n, Lemma 3.5 implies that the first summand decays exponentially as well.

On the other hand, as $\sum_{i=n/2}^{n} v_i$ is a positive linear combination of the active columns of $A^{n/2}$, the second summand decays exponentially as well, by the previous comments.

□

4. Building blocks

This section sets up most of the novel terminology used to construct the paths necessary to prove the following

Theorem 4.1. *The set of uniquely ergodic* 4*-IETs with permutation in the Rauzy class of* (4321) *is path-connected.*

The proof will involve an explicit construction of paths by prescribing Rauzy inductions. The main tool in this construction is given by the following definition.

Definition 4.2. *A* building block b *is a triple of IETs* $b = (T_1, F, T_2)$ *such that*

(1) T_1, F, T_2 *are uniquely ergodic with the same underlying permutation* π.
(2) F *lies on the plane given by the failure of the first step of Rauzy induction.*

We call T_1, T_2 *the* endpoints *of the building block, and* F *the* midpoint*.*

The next definition lies at the core of our argument. Intuitively we would like to say that a building block $b' = (S_1, G, S_2)$ is left compatible with $b = (T_1, F, T_2)$ if the IETs T_1 and F share a number of common Rauzy steps, and the outcome is the pair S_1, S_2 of IETs. We formally need to phrase this slightly differently to avoid the problem that Rauzy induction is not well-defined for F.

Definition 4.3. *A building block* $b' = (S_1, G, S_2)$ *is* left compatible *with a building block* $b = (T_1, F, T_2)$ *if the following holds:*

(1) $S_1 = R^{k_1}(T_1)$.
(2) Every point on the straight line between T_1 *and* F *has Rauzy induction defined for* k_1 *steps, and it agrees with the one of* S_1.
(3) $L(F) = M(T_1, k_1)L(S_2)$

We then write $b \stackrel{L}{\to} b'$.

The Rauzy matrix associated to the compatability $b \stackrel{L}{\to} b'$ *is then defined to be the matrix* $M(T_1^{right}, k_1)$ *where* T_1^{right} *denotes the limit from the right..*

We define right compatible and $\stackrel{R}{\to}$ *similarly. We take* compatible *and* $\to$ *to mean left compatible or right compatible.*

Definition 4.4. *A* building block sequence *is a (finite or infinite) sequence* $b_i = (T_1^i, F^i, T_2^i)$ *of building blocks so that* b_{i+1} *is compatible with* b_i *for all* i*. We often write such a sequence as* $b_1 \to b_2 \to \dots$.

For a finite building block sequence $b_1 \to \dots \to b_n$ *we define the* associated Rauzy matrix $M(b_1 \to \dots \to b_n)$ *as the product* $M = M_1 \dots M_{n-1}$ *where* M_i *is the Rauzy matrix associated to the compatibility* $b_i \to b_{i+1}$.

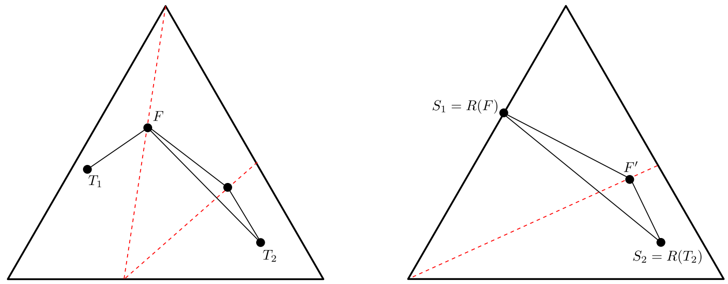

FIGURE 3. Building blocks. For clarity, the dimension is reduced from 3 to 2. The dotted lines represent fail planes of Rauzy induction.

The following definition of depth of a building block sequence is crucial for our construction.

Intuitively, we want to define the depth of a building block sequence as the number of times it swaps between consecutively always choosing right and always choosing left.

To state it formally, note that a building block sequence $b_1 \to \cdots \to b_n$ defines a "direction sequence" $d_1, \ldots, d_{n-1}$, where $d_i \in \{L, R\}$ and $d_i = L$ if and only if b_{i+1} is left compatible with b_i. We say that d_i *swaps direction at most s times* if there are indices $1 = t_0 < t_1 < \cdots < t_{s+1} = n$ so that d_i is constant for $t_j \le i < t_{j+1}$.

Definition 4.5. *Let $b_1 \to \cdots \to b_n$ be a building block sequence The* depth *of the building block sequence is the smallest s so that the associated direction sequence d_i swaps direction at most s times.*

As an important example, a building block sequence $b_1 \to \cdots \to b_n$ has depth 0 if either for all i, b_i is left compatible to b_{i-1} or for all i, b_i is right compatible to b_{i-1}.

Intuitively, the sequence $b_1 \to \cdots \to b_n$ has depth k if the compatibilities $b_i \to b_{i+1}$ swap k times between choosing L or R.

Definition 4.6. *i) A* building block loop *is a building block sequence $b_1 \to \cdots \to b_n$ such that the last element b_n is compatible with the first element b_1.*
ii) Such a loop is minimal, *if $b_i \neq b_j$ for all $1 \le i < j \le n$.*
iii) The depth *of a building block loop $b_1 \to \cdots \to b_n$ is defined to be the depth of the building block sequence $b_1 \to \cdots \to b_n \to b_1$.*
iv) If $b_1 \to \cdots \to b_n$ is a depth 0 building block loop, then one of the endpoints of b_1 is contained in $M(b_1 \to \cdots \to b_n \to b_1)\Delta$. We call that IET the endpoint of the building block loop.

We will frequently use product notation for building block sequences. That is, let P is a building block sequence $b_1 \to \cdots \to b_n$ and Q is a building block sequence $b_{n+1} \to \cdots \to b_m$. We say that P is compatible with Q if b_n is compatible with b_{n+1}. In that case we denote by $P \to Q$ the sequence

$$b_1 \to \cdots \to b_n \to b_{n+1} \to \cdots \to b_m$$

obtained by concatenating P and Q. Similarly, if P is a building block loop $b_1, \ldots, b_n$, the P^n is the sequence $P \to \cdots \to P$ obtained by concatenating P with itself n times.

Definition 4.7. *If P is a building block loop $b_1 \to \cdots \to b_n$, then we say that a building block sequence $c_1 \to c_2 \to c_3$ compatible with P* leaves the loop P *if the sequence $c_1 \to c_2 \to c_3$ does not contain P as an initial segment, and is not contained in P as an initial segment.*

In Section 8 we will show the following Proposition by an explicit construction.

Proposition 4.8. *There is a finite set $\mathcal{B}$ of building blocks with the following properties*

- **Transitivity:** *For every permutation π in the Rauzy class of (4321) there is a building block in the set $\mathcal{B}$ whose endpoints lie on different sides of the fail plane.*
- **Completeness:** *Every building block in $\mathcal{B}$ has a left and right compatible building block in the set $\mathcal{B}$.*
- **Combining Loops:** *If $b_1 \to \cdots \to b_k$ is a minimal depth 0 loop formed from building blocks in the set $\mathcal{B}$, then the matrix $M(b_1 \to \cdots \to b_k)$ is combining.*
- **Isolated Idle:** *Suppose P is a building block loop of depth 0 formed by building blocks in the set $\mathcal{B}$, so that $M(P)$ has an idle column i. Then, for any building block sequence $b_1 \to b_2 \to b_3$ which leaves P formed by building blocks in the set $\mathcal{B}$, each column of $M(P \to b_1 \to b_2 \to b_3)$ has at least two nonzero entries.*
- **Almost Positivity:** *There is a number $c \geq 0$ so that every building block sequence formed by building blocks in the set $\mathcal{B}$ of depth at least c has an initial segment $b_1 \to \cdots \to b_k$ with $k \leq c$ so that $M(b_1 \to \cdots \to b_k)$ is* Almost Positive*: i.e. it has $\tau > 1$ rows with all entries positive and the other rows are rows of the identity matrix. By this we mean that they contain exactly one entry 1, and all other entries are 0.*

In Sections 5 to 7 we will show how to use Proposition 4.8 to show Theorem 4.1. Hence, we fix once and for all a finite set $\mathcal{B}$ as given by Proposition 4.8. Any mention of building blocks will refer to elements of this set. We will also refer to the properties guaranteed by Proposition 4.8 by their name, without explicit mention of Proposition 4.8.

Following the sketch outlined in the introduction there is a general dichotomy of points on the paths: finite depth points eventually follow the Rauzy induction of a fixed uniquely ergodic IET (which is the left or right endpoint of a building block). Continuity at these points will be shown using the tools in Section 5. Infinite depth points on the other hand are those that do not eventually lie on a fail plane, and whose Rauzy induction does not eventually follow one of the endpoints. For these points, convergence, unique ergodicity and continuity all need to be checked, using techniques developed in Section 6. Finally, in Section 7 we collect the pieces and prove Theorem 4.1.

5. Finite Depth

Lemma 5.1. *Let $P = b_1 \to \cdots \to b_k$ be a minimal depth 0 loop. Denote by b the index of the column which is idle if it exists. We consider the building block sequence P^n. Then there exists a vector V such that*

$$\frac{C_i(M(P^n))}{|C_i(M(P^n))|} \to V$$

for all $i \neq b$.

Furthermore, the size of the columns $i \neq b$ grow exponentially, and the size of column b is constant.

Proof. This is immediate from Lemma 3.4 and **Combining Loops**. □

Lemma 5.2. *Let P be a minimal depth 0 loop, and let P_0 be a building block sequence compatible with P. We consider building block sequences of the form $P_k = P_0 \to P^{n_k} \to Q_k$ where Q_k is a building block sequence of length 3 which leaves the loop P and $\lim_{k\to\infty} n_k = \infty$.*

Let T_∞ be the IET corresponding to the endpoint of P defined by the loop. Let $\{T_i\}_{i\in\mathbb{N}}$ be IETs so that $T_k \in M(P_k)\Delta$ (i.e. the Rauzy induction of T_k agrees with the one defined by P_k)

Then the T_k converge (as a sequence of points in the simplex) to T_∞.

Proof. To begin, note that we may assume that P_0 is empty. Namely, the Rauzy induction corresponding by P_0 defines a continuous map $M(P_0)$ of the simplex which does not change the result.

We fix $m_k > 0$ so that the Rauzy expansion of the building block sequence P^{n_k-1} has m_k Rauzy steps. Further, let m'_k be the number of Rauzy steps in $P \to Q_k$. Thus, the initial $N_k = m_k + m'_k$ Rauzy steps of the interval exchange T_k agree with the Rauzy expansion defined by P_k.

Next note that the (unnormalized) length vector of T_k satisfies

$$\hat{L}(T_k) = M\left(P^{n_k} Q_k\right) \hat{L}\left(R^{N_k}(T_k)\right)$$

The length vector for T_k is $L(T_k) = \hat{L}(T_k)/|\hat{L}(T_k)|_1$.

Write $L\left(R^{m_k}(T_k)\right) = \sum_{i=1}^4 a_i e_i$. We have $\sum_{i=1}^4 a_i = 1$. We let v_i denote the i-th column of $M(P^{n_k-1})$. Note that in both of these shortcuts we are suppressing the dependence of the a_i and the v_i on k. We have

$$L(T_k) = \frac{\sum_{i=1}^4 a_i v_i}{\sum_{i=1}^4 a_i |v_i|_1}$$

Next, we claim that there is a number $\epsilon_0 > 0$ (depending only on the set of building blocks and not k) such that at least two of the a_i are at least ϵ_0. Namely, we have

$$(a_1, \ldots, a_4) = M(P \to Q_k)v/|M(P \to Q_k)v|_1$$

for some vector v with non-negative entries and $|v|_1 = 1$. Hence, at least one entry v_i of v has size at least $1/4$. By Isolated Idle, each column of $M(P \to Q_k)$ has at least two positive entries, and hence $M(P \to Q_k)e_i$ has at least two nonzero entries for each i. Thus, at least two entries of the (unnormalized) vector $M(P \to Q_k)v$

have size at least $1/4$. On the other hand, note that $M(P \to Q_k)$ is a matrix whose entries can be bounded by some constant which does not depend on k (since the set of building blocks is finite). Thus, $|M(P \to Q_k)v|_1$ can likewise be bounded by some constant independent of k. This implies the existence of ϵ_0 as claimed.

Now suppose that we are given some $\epsilon > 0$. We choose K so that for all $n > K$ the following hold (where V is the vector given by Lemma 5.1)

i) For all but at most one i, we have
$$|v_i - |v_i|_1 V|_1 < \epsilon |v_i|_1$$
We call the i where this fails the *bad* i and v_i the *bad column*. We call other indices $j \neq i$ and the corresponding columns v_j *good*. This property can be ensured by Lemma 5.1.

ii) If there is a bad column v_i, then
$$\frac{a_i v_i}{\sum_{j\text{good}} a_j |v_j|} < \epsilon.$$
This is possible since at least one coefficient a_j of a good column v_j is at least ϵ_0, and the sizes of the good columns grow exponentially, while the size of the bad column is uniformly bounded by Lemma 5.1.

Now, let $n > K$ be given.

We first consider the case in which there is no bad i for $M(P^N)$. In that case, we compute
$$|L(T_k) - V|_1 = \left| \frac{\sum_{i=1}^4 a_i v_i - \sum_{i=1}^4 a_i |v_i|_1 V}{\sum_{i=1}^4 a_i |v_i|_1} \right|_1 \leq \frac{\sum_{i=1}^4 a_i \left| v_i - |v_i|_1 V \right|_1}{\sum_{i=1}^4 a_i |v_i|_1}.$$
Thus by i)
$$|L(T_k) - V|_1 < \frac{\sum_{i=1}^4 a_i |v_i|_1 \epsilon}{\sum_{i=1}^4 a_i |v_i|_1} = \epsilon.$$
Now suppose that there is a bad i, which without loss of generality we may assume to be 1.

In this case, we compute
$$L(T_k) = \frac{\sum_{i=1}^4 a_i v_i}{\sum_{i=1}^4 a_i |v_i|_1} = \frac{a_1 v_1}{\sum_{i=1}^4 a_i |v_i|_1} + \frac{\sum_{i=2}^4 a_i v_i}{\sum_{i=1}^4 a_i |v_i|_1}.$$
By property ii), the first summand has size at most ϵ. Now note that for any numbers $K < D$ and c one has
$$\frac{K}{D} - \frac{K}{c+D} = \frac{cK}{D(c+D)} < c\frac{K}{D}\frac{1}{D} < c\frac{1}{D}.$$
We apply this estimate for K being each of the four entries of the vector $\sum_{i=2}^4 a_i v_i$ with $D = \sum_{i\text{good}} a_i |v_i|_1$ and $c = a_1 |v_1|_1$ to conclude that the second summand in the expression for $L(T_k)$ is within 4ϵ of
$$\frac{\sum_{i=2}^4 a_i v_i}{\sum_{i=2}^4 a_i |v_i|_1}.$$
Arguing as in the first case, this is within ϵ of V. In conclusion, we thus have $|L(T_k) - V|_1 < 6\epsilon$.

Thus, the normalized length vectors $L(T_k)$ converge to V. □

Corollary 5.3. Let P be an infinite Rauzy path so that at least 3 columns increase in size an unbounded amount[1] and all columns which do increase an unbounded amount converge projectively to some fixed vector V. Let $Q_1, ..., Q_k$ be Rauzy paths with positive associated Rauzy matrices.

Then for every $\epsilon > 0$ there is a $N > 0$ so that if $n > N$, and T is any IET whose initial n-steps of Rauzy induction agree with P and then are followed by some Q_i, then $L(T)$ is within ϵ of V.

Proof. By assumption of the Corollary, for any ϵ there is an N so that for all $n > N$ conditions i) and ii) stated in the proof of Lemma 5.2 hold for the column v_i of the Rauzy matrix $M(P_n)$ describing the first n steps of P (where V is now the vector given by the assumption).

Furthermore, as Q_i is assumed to be positive, $L(R^N(P_n))$ has at least two entry of size at least ϵ_0 (arguing as in the previous proof, with positivity of Q_i replacing the part of the argument involving Isolated Idle).

From here, one can finish the proof exactly as in the proof of Lemma 5.2 above. □

6. Infinite Depth

Next, we analyse infinite depth points. Here, the situation is much more involved. Convergence, continuity and unique ergodicity will all follow from the same mechanism, which we now describe.

Throughout, we consider an infinite building block sequence (b_i) of infinite depth. This in particular means that the sequence b_i does not end in an infinite power of a depth 0 loop.

We rewrite the sequence $b_1 \to b_2 \to \dots$ as a sequence of paths $B_1 \to B_2 \to \dots$, where each B_i is either a power of a minimal depth 0 loop, or a single building block, and the re-grouping is maximal with that property (that is: if B_i is a power of a minimal depth 0 loop then the sequence following B_i leaves the loop). By our assumption, each B_i is a finite building block sequence, and the sequence of B_i' is infinite.

Denote throughout this section by $A_i = M(B_i)$ the matrix corresponding to B_i.

By Proposition 4.8 these matrices satisfy the following:

Corollary 6.1. There are numbers $c, N > 0$ such that for all i the following holds:

- (P) The product $A_i \cdot ... \cdot A_{i+c}$ is positive or
- (R) There exists $c' \leq c$ so that $A_i \cdot .... \cdot A_{i+c'}$ has $\tau > 1$ rows with all entries positive and the other $(4 - \tau)$ rows are rows of the identity matrix.

Furthermore, each matrix A_i satisfies one of the following.

- (S) $||A_i|| < N$ or
- (C) A_i is the power of a combining matrix B with $\|B\| < N$.

[1]Note that all matrices of Rauzy induction have only non-negative entries and thus such an increase is automatically monotone.

The following lemma follows from the finiteness of the set of building blocks and property (C):

Lemma 6.2. *For all k there exists n so that if $\|A_i\| > n$ then A_i is at least the k^{th} power of a combining matrix.*

6.1. **Column size bound.** We now begin to study the infinite matrix product $\prod A_i$. The first intermediate goal will be the following theorem.

Theorem 6.3. *There is a number $K > 0$ so that for all n the ratio of the largest to the second smallest column of $(\prod_{i=1}^n A_i)$ is at most K.*

The proof requires several preliminary lemmas, beginning with the following elementary estimate.

Lemma 6.4. *Let $a, b, c, d > 0$ be four positive numbers, so that $b \leq Kd$. Then*

$$\frac{a+b}{c+d} \leq \frac{a}{c} \text{ or } \frac{a}{c} \leq K$$

Proof. Suppose the second estimate is not satisfied, that is $a > Kc$. Then

$$\frac{c}{a}b < \frac{c}{Kc}Kd = d$$

Furthermore,

$$\frac{a+b}{c+d} = \frac{a}{c}\frac{c+\frac{c}{a}b}{c+d} < \frac{a}{c}$$

This shows the estimate. $\square$

Lemma 6.5. *Suppose B is a (4×4)-matrix with non-negative entries. Let A be a matrix whose entries are bounded by $K \geq 1$.*

i) Suppose that A has all positive entries. Then

$$\frac{|C_i(BA)|}{|C_j(BA)|} \leq K$$

for all i, j.

ii) Suppose that A has only positive entries in at least two rows, and the other rows are equal to rows of the identity matrix. Then

$$\text{either } \max_{i,j} \frac{|C_i(BA)|}{|C_j(BA)|} \leq \frac{|C_i(B)|}{|C_j(B)|} \text{ or } \max_{i,j} \frac{|C_i(B)|}{|C_j(B)|} \leq K$$

Proof. i) The i-th column of BA is obtained by summing the columns of B according to the entries of the i-th column of A. In such a linear combination each column of B appears at least once, and at most K times by the assumption on A. Thus, the 1-norm of any column is at least equal to $|C_1(B)|+\ldots+|C_4(B)|$ and at most $K(|C_1(B)| + \ldots + |C_4(B)|)$. This shows i).

ii) Suppose for ease of notation that rows 3 and 4 of A are the ones which are not positive (if only one non-positive row exists then the argument works analogously). By the assumption, every column of BA is the sum some multiples of columns 1 and 2 of B and possibly one copy of column 3 of 4. There is a permutation π so that the entry $\pi(i)$ of row i is positive (as the matrix A is nonsingular). We thus have for all i, j:

$$|C_j(BA)| \geq |C_{\pi(j)}(B)| + (|C_1(B)| + |C_2(B)|)$$

and

$$|C_i(BA)| \le |C_{\pi(i)}(B)| + K(|C_1(B)| + |C_2(B)|)$$

Thus, we have

$$\frac{|C_i(BA)|}{|C_j(BA)|} \le \frac{|C_{\pi(i)}(B)| + K(|C_1(B)| + |C_2(B)|)}{|C_{\pi(j)}(B)| + (|C_1(B)| + |C_2(B)|)}$$

By Lemma 6.4 (applied for $b = d = |C_1(B)| + |C_2(B)|$ and $a = |C_{\pi(i)}(B)|, c = |C_{\pi(j)}(B)|$) the right hand side is between $\frac{|C_{\pi(i)}(B)|}{|C_{\pi(j)}(B)|}$ and K which implies the lemma.

□

Recall that an entry in a combining matrix A is *active*, if it is in row i and column i of A is active.

Definition 6.6. *If M is a matrix we denote by $C_u(M)$ denote the column of M whose norm is second smallest.*

Lemma 6.7. *There is are constants $\gamma < 1$, K with the following property. Suppose A is a matrix corresponding to a n-th power of a minimal depth 0 loop for $n \ge 3$ and B is a non-negative matrix. Then*

$$\frac{|C_{\max}(BA)|}{|C_u(BA)|} \le \max\left\{K, \frac{|C_{max}(B)|}{|C_u(B)|}\gamma^{n-2}\right\}$$

Proof. By finiteness of the set $\mathcal{B}$ of building blocks, it suffices to show the statement for each matrix A corresponding to a power of a minimal depth 0 loop, and then let K, γ be the largest of the resulting constants. We argue similarly to the proof of case (2) of Lemma 6.5. Again, for simplicity of notation, assume that 1 and 2 are the active columns, 3 is passive and 4 is passive or idle.

Let E be the constant from Lemma 3.4 applied to the matrix corresponding to the depth–0 loop detemined by A. Fix columns i, j. Let m be the smallest, and M be the size of largest of the active entries in columns i and j. By Lemma 3.4 we have $M \le Em$. Thus for any $i, j \ne 4$ (if column 4 is idle) or any i, j (if column 4 is passive) we have

$$|C_i(BA)| \ge |C_j(B)| + m(|C_1(B)| + |C_2(B)|)$$

and

$$|C_i(BA)| \le |C_j(B)| + Em(|C_1(B)| + |C_2(B)|).$$

Arguing exactly as in case ii) of Lemma 6.5 one shows that quotients between such i, j either the ratio improves, or is less than some fixed constant. Moreover, because the action of combining matrices on active columns gives a positive matrix, m increases exponentially with the number of loops taken and therefore we have that

$$\frac{|C_i(BA)|}{|C_j(BA)|} \le \max\left\{K', \frac{|C_a(B)|}{|C_b(B)|}\gamma'^{n-2}\right\} \tag{1}$$

where $i, j, a, b \ne 4$. This establishes the claim (with $K = K'$, $\gamma = \gamma'$) unless $C_{\max}(BA)$ is the idle column. Indeed, if $C_i(BA) = C_{\max}(BA)$ where i is active or passive then we have

$$\frac{|C_i(BA)|}{|C_u(BA)|} \le \max_{j \text{ active or passive}} \frac{|C_i(BA)|}{|C_j(BA)|} \le \max\left\{K', \max \frac{|C_a(B)|}{|C_b(B)|}\gamma'^{n-2}\right\}$$
$$\le \max\left\{K', \max \frac{|C_{\max}(B)|}{|C_u(B)|}\gamma'^{n-2}\right\}.$$

Hence, we are done unless column 4 of A is idle and column 4 of BA is not the smallest column. In that case, it is the same as the fourth column in B. By splitting A into smaller powers $A = A_0^2 A_0^k$, where A_0 is combining and $k \ge 0$ we see that multiplying B by A_0^2 already adds both active columns of B to all but the idle column. If one of the active columns was larger than the idle column, this implies that the idle column of A is the smallest of BA. In this case we may use the bound of K' from above since both C_u and $C_{\max}$ are active or passive.

Thus, we are left with the case that the idle column of B is larger than both of the active columns of B. But then for all i, in BA_0^{i+2} the size of the active columns grow exponentially in i and so the ratio of the idle column to C_u decays exponentially. This implies the lemma if $C_{\max}(BA)$ is the idle column. Otherwise it follows from equation 1 and the fact that $\frac{|C_{\max}(B)|}{|C_u(B)|} \ge \frac{|C_i(B)|}{|C_u(B)|}$ for all i. □

Corollary 6.8. For any M there exists N so that if $\|D\|_{op} \le M$ then

$$\frac{|C_{\max}(BDA^r)|}{|C_u(BDA^r)|} \le \max\left\{K, \frac{|C_{\max}(B)|}{|C_u(B)|}\right\}$$

for any A, a matrix given by a minimal depth zero loop and $r \ge N$. Here, K does not depend on M.

Proof. Observe that $\frac{|C_{\max}(BD)|}{|C_u(BD)|} \le \|D\| \frac{|C_{\max}(B)|}{|C_u(B)|}$. Let γ be as in Lemma 6.7. Choose N so that $\gamma^{N-2}M \le 1$ and the corollary follows from Lemma 6.7. □

Proof of Theorem 6.3. To show the theorem, we will first define a re-grouping $D_k = \prod_{i=i(k)}^{i(k+1)-1} A_i$ where $i(k)$ are chosen so that $i(k+1) - i(k) \le c$ and so that the products $\prod_{i=1}^{n} D_i$ have the claimed property. We will then use this to show that the theorem holds.

To define the D_i, we argue inductively. To begin with, note that the set of all paths $B_1 \to \cdots \to B_k$ so that no B_i is at least a third power of a minimal depth 0 building block loop and so that $k \le c$ is finite. Thus, the set of operator norms $\|M(B_1 \to \cdots \to B_k)\|$ of the corresponding matrices is bounded – let M_0 be such a bound.

Next, let N_1 be the number guaranteed by Corollary 6.8 applied to the bound $M = M_0$. In particular, this implies that if A is any matrix, and $B_1 \to \cdots \to B_k \to P^n$ where $B_1 \to \cdots \to B_k$ is as above, P is a minimal depth 0 loop and $n \ge N_1$, then

$$\frac{|C_{\max}(AM(b_1 \to \cdots \to b_k \to P^n))|}{|C_u(AM(b_1 \to \cdots \to b_k \to P^n))|} \le \max\left\{K, \frac{|C_{\max}(A)|}{|C_u(A)|}\right\}$$

Suppose now that N_i, M_i are defined. We define M_{i+1} to be a bound for the operator norm of matrices $M(B_1 \to \cdots \to B_k)$ defined by sequences with $k \le c$ and so that no B_i is a minimal depth 0 loop with power at least N_i. Then, define N_{i+1} to be the output of Corollary 6.8 applied to $M = M_{i+1}$.

Finally, let K_2 be the output of Lemma 6.5 applied to the bound M_c. We will now describe the grouping D_i which will satisfy the column bound for $\max(K, K_2)$.

Namely, suppose that D_i is defined for $i \le n$ with the desired properties, and A_N is the final matrix appearing in the product defining D_n. By Almost Positivity, for some number $k \le c$ the matrix $A_{N+1} \dots A_{N+k}$ is Almost Positive (i.e. satisfies the property from Almost Positivity). There are now two cases:

(1) $\|A_{N+1} \dots A_{N+k}\| \le M_c$. Then we put $D_{n+1} = A_{N+1} \dots A_{N+l}$ and note that by Lemma 6.5 we have
$$\frac{|C_{\max}(D_1 \dots D_n D_{n+1})|}{|C_u(D_1 \dots D_n D_{n+1})|} \le \max\left\{K_2, \frac{|C_{\max}(D_1 \dots D_n)|}{|C_u(D_1 \dots D_n)|}\right\}$$

(2) $\|A_{N+1} \dots A_{N+k}\| > M_c$. Then at least one of the $A_j, N+1 \le j \le N+k$ corresponds to at least a N_c–th power of a minimal depth 0 loop. Let r be the minimal such j. We put $D_{n+1} = A_{N+1} \dots A_{r-1} A_r$. Note that $\|A_{N+1} \dots A_{r-1}\| \le M_c$ by minimality of r. Thus, by our choice of N_c we have
$$\frac{|C_{\max}(D_1 \dots D_n D_{n+1})|}{|C_u(D_1 \dots D_n D_{n+1})|} \le \max\left\{K, \frac{|C_{\max}(D_1 \dots D_n)|}{|C_u(D_1 \dots D_n)|}\right\}$$

Thus, in any case we have
$$\frac{|C_{\max}(D_1 \dots D_n D_{n+1})|}{|C_u(D_1 \dots D_n D_{n+1})|} \le \max\left\{\max\{K, K_2\}, \frac{|C_{\max}(D_1 \dots D_n)|}{|C_u(D_1 \dots D_n)|}\right\}$$

This shows inductively the existence of the desired column size bound for products $\prod D_i$. The desired bound on the products $\prod A_i$ follows simply because by construction the difference between $\prod A_i$ and a suitably chosen $\prod D_j$ has operator norm at most M_c by construction. □

Corollary 6.9. For any ζ there is an L with the following property. Let $B = \prod_{i=1}^N A_i$, and suppose that A is at least the L–th power of a minimal depth 0 loop. Then the idle column of BA^L has size at most ζ times the size of any active or passive column of BA^L.

Proof. By Theorem 6.3 the quotient in size between the largest and second smallest column of B is at most K. If the idle column of B is not already the smallest, then one of the active columns is has size at least $\frac{1}{K}$ times the size of the idle column. After K of applications of A at this active column was added to all non-idle columns, and the idle column has become the smallest. Since every further application of A increases the size of every non-idle column, the desired L exists. □

6.2. **Angle Contraction.**

Lemma 6.10. *There are constants $K' > 0, \xi > 1$ with the following property. Let B be a matrix so that $\max\limits_{1 \le i,j \le 4} \angle(C_i(B), C_j(B))) \le \alpha$, $\max\left(\frac{|C_i(B)|}{|C_u(B)|}\right) \le D$ and let A be the matrix defined by a minimal depth* 0 *loop. Then*
$$\max_{i,j active\ or\ passive} \sin \angle(C_i(BA^k), C_j(BA^k)) < K' \frac{\sin(\alpha)}{\xi^k}.$$

Proof. Because there are only finitely many minimal depth 0 loops in our building blocks, it suffices to show the lemma for a single one.

The proof is similar to Lemma 3.4. For each combining matrix A we consider its action on the active columns, described by a matrix $\hat{A}$. It has a Perron-Frobenius eigenvector $[a_1,...,a_\tau]$ where $2 \leq \tau \leq 4$ and all the $a_i > 0$. It has corresponding eigenvalue μ. We consider the action of $\hat{A}$ on the invariant $(\tau-1)$-dimensional subspace not containing the Perron-Frobenius eigenvector $[a_1,...,a_\tau]$. Call the matrix describing this action $\tilde{A}$. There exists C, ν where $\nu < \mu$ and $\|\tilde{A}^k\|_{op} \leq C\nu^k$. Thus there exists n_A, $\gamma > 1$ so that $\frac{\mu^k}{C\nu^k} \geq \gamma^k$ for all $k \geq n_A$. Write $C_i(\tilde{A}_r) = \lambda[a_1,..,a_\tau] + w$ where λ is chosen to be the largest possible so that $C_i(\tilde{A}_r) - \lambda[a_1,...,a_\tau] \in \mathbb{R}^\tau_d$. There exists C, m_A so that for all $k > m_A$ with $\frac{\lambda}{|w|} > C\gamma^k$. Indeed, decompose $C_i(\tilde{A}_r) = \lambda'[a_1,...,a_\tau] + w'$ where w' is in the subspace $\hat{A}$ acts on. By above $\frac{\lambda'}{|w'|} > \gamma^k$ for all $k > n_A$. Now letting ξ be the minimal number so that $\xi[a_1,...,a_\tau] + w' \in \mathbb{R}^\tau_+$ for all large enough k we have the claim.[2]

Consider $A_r = A^k$ where A is a combining matrix and $k \geq n_A$. Consider the active columns $C_{i_1}(B),...,C_{i_\ell}(B)$ and the vector $\bar{a} \in \mathbb{R}^4_+$ where $\bar{a}_i$ is 0 if C_i is not active and is a_b where b is the corresponding column of $\hat{A}$ if is active. Because $\hat{A}$ captures the action of A on the active columns, $C_{i_b}(A_r)$ as $\lambda\tilde{a} + w_b$ where λ is chosen to be the largest possible with $C_{i_b} - \lambda\tilde{a} \in \mathbb{R}^4_+$ with $\frac{|w_b|}{\lambda} \leq C'\frac{1}{\gamma^k}$.

Now $Bw_b \in B_\Delta$ and so $\angle(w_b, \sum a_j C_{i_j}(B)) < \alpha$. It follows from Lemma 3.5 that $\sin(\angle(\sum a_j C_{i_j}(B), C_{i_b}(BA_r)) \leq D\gamma^{-k}\sin(\alpha)$. Hence the lemma follows for active columns. The case of passive column is analogous to the proof of Lemma 3.4 part(iii).

By choosing of K' large enough we absorb the various constants and are able to remove the requirement that $k \geq m_A$, n_A. $\square$

We next introduce some notation. If v, w are line segments in $\mathbb{R}^3$ let $|w|$ denote the length of w and Φ_v denote orthogonal projection onto the direction of v. We also need a fact from Euclidean geometry:

Lemma 6.11. *There exist numbers $0 < c_1, c_2 < 1$ with the following property. Let $\Delta' \subset \mathbb{R}^3$ be a (not necessarily regular) Euclidean simplex with diameter τ, and let w be a line segment in Δ' with length $|w| \geq c_1\tau$.*

Then there is a an edge e of Δ' with length at least $|w|$ so that $|\Phi_e(w)| \geq c_2|w|$.

Proof. First observe that the diameter of a simplex is given by its longest side. Second, observe that if w is is a line segment contained in Δ' then there exists a line segment $\tilde{w}$ contained in Δ', parallel to w, with $|\tilde{w}| \geq |w|$ and so that one endpoint of $\tilde{w}$ a vertex v of Δ'. We thus have $|\tilde{w}| \geq c_1\tau$, and by choosing $c_1 < 1$ large enough there is some edge e of Δ' emanating from v with $|e| \geq |\tilde{w}|$ and whose direction is within 1 degree of the direction of $\tilde{w}$. As a consequence, since w and $\tilde{w}$ are parallel we have that

$$|\Phi_e(w)| \geq \cos\left(\frac{\pi}{180}\right)|w|$$

Thus, $c_2 = \cos(\frac{\pi}{180})c_1$ has the desired property. $\square$

[2]In particular, $\xi \leq \frac{|w'|}{\min a_i}$

Lemma 6.12. *For every N there exists $F_1 < 1$ so that if B is a product of matrices from Corollary 6.1, $A = A_i \cdots A_{i+k}$ is a subproduct satisfying (P) or (R) with $k \leq c$ and $\|A\|_{\text{op}} \leq N$. Then*

$$\text{diam}(BA\Delta) \leq F_1 \text{diam}(B\Delta).$$

Proof. Consider the (normalized) simplices $B\Delta$ and $BA\Delta$ with vertices $p_1, ..., p_4$ and $q_1, ..., q_4$ respectively. These correspond to the directions of the columns $C_j(B)$ and $C_j(BA)$. Let τ be the length of the longest side of $B\Delta$ (which is also its diameter) and E be the set of sides of $B\Delta$ with length at least $c_1\tau$.

By Lemma 6.11 it suffices to show that there is a constant $0 < c_3 < 1$ so that

$$|\Phi_e(\overrightarrow{q_i q_j})| < c_3|e|$$

for all i, j and $e \in E$.

To prove this, let $e \in E$ be arbitrary. For concreteness, assume that $e = \overrightarrow{p_1 p_2}$. Let j be such that every entry in row j of A is positive, and $|C_j(B)| \geq |C_u(B)|$ (such a j exists as at least two rows of A are positive). Up to exchanging p_1 and p_2 we may also assume that $|\Phi_e(\overrightarrow{p_1 p_j})| \geq \frac{1}{2}|e|$.

Note that

$$C_i(BA) = \sum a_{k,i} C_k(B)$$

where $a_{k,i} \leq n$ for all k, i and $a_{j,i} \geq 1$ for all i. Thus, we have

$$q_i = \sum a_{k,i} \frac{|C_i(B)|}{|C_i(BA)|} p_k$$

Since $\|A\| < N$ and $|C_j(B)| \geq |C_u(B)|$, there exists some $\epsilon > 0$ (dependent only on N and the constant K from Theorem 6.3) so that $\frac{|C_j(B)|}{|C_j(BA)|} > \epsilon$.

Thus, we have that

$$\Phi_e(q_i) = \sum a_{k,i} \frac{|C_i(B)|}{|C_i(BA)|} \Phi_e(p_k)$$

In this expression, the coefficient of $\Phi_e(p_j)$ is at least ϵ, and all coefficients are bounded from above by N^2. Since $|\Phi_e(\overrightarrow{p_1 p_j})| \geq \frac{1}{2}|e|$, there is therefore some c_3 so that all $\Phi_e(q_i)$ are within $c_3|e|$ of p_2. This shows the claim. $\square$

Lemma 6.13. *There exists $N > 0$ and $F_2 < 1$ so that it B is a product of matrices from Corollary 6.1, $A = A_i \cdots A_{i+k}$ is a subproduct satisfying (P) or (R) with $k \leq c$ and $\|A\|_{\text{op}} > N$. Then*

$$\sin(\max \angle(C_i(BAA_{i+k+1} \cdot ... \cdot A_{r+k+3}), C_j(BAA_{i+k+1} \cdot ... \cdot A_{r+k+3})))$$
$$\leq F \sin(\max \angle(C_i(B), C_j(B))).$$

Proof. The number N will be given by Lemma 6.2, so that any A with $\|A\| > N$ contains a matrix corresponding to at least ℓ powers of a minimal depth 0 loop – where ℓ will be chosen below.

To begin, note that

$$\sin(\max \angle(C_i(XY), C_j(XY))) \leq \sin(\max \angle(C_i(X), C_j(X))).$$

for any matrices X, Y with non-negative entries (as $XY\Delta \subset X\Delta$). By the choice of N, there is an $s, i \leq s \leq i + k$ so that A_s is the matrix defined by a minimal depth 0 loop taken at least ℓ times. By the preceding remark it suffices to show that

$$\sin(\max\angle(C_i(BAA'),C_j(BAA')))\le F\sin(\max\angle(C_i(B),C_j(B))).$$

where $A=A_s$ is a matrix corresponding to the ℓ-th power of a combining matrix, and A' is a matrix corresponding to three building block steps. We emphasize that this is necessarily the product of three A_i, but may be a smaller product (if some of the A_i correspond to loops). In particular, we may assume that $\|A'\|_{\mathrm{op}}\le L$ for some universal L.

For simplicity of notation, we abbreviate $\beta=\sin(\max\angle(C_i(B),C_j(B)))$. By Lemma 6.10 and our choice of ℓ we then have that

$$\max_{i,j\text{active or passive}}\sin\angle(C_i(BA),C_j(BA))<K'\frac{\beta}{\xi^k}$$

where active and passive refers to the columns of A. Let b denote the index of the bad column of A.

Next, consider any column $C_k(BAA')$ and write it as a sum of columns of BA:

$$C_k(BAA')=a'_{kb}C_b(BA)+\sum_{i\neq b}a'_{ki}C_i(BA)$$

By Isolated Idle every column of A' has at least two nonzero entries, and thus $\sum_{i\neq b}a'_{ki}\ge 1$.

Furthermore, the sum $\sum_i a'_{ki}$ is at most $\|A'\|_{op}\le L$. Thus, we have

$$\frac{\|a'_{kb}C_b(BA)\|}{\|\sum_{i\neq b}a'_{ki}C_i(BA)\|}\le L\frac{\|C_b(BA)\|}{\|C_i(BA)\|}$$

for some index i corresponding to an active or passive column. Hence by Lemma 3.5

$$\sin\angle\left(C_k(BAA'),\sum_{i\neq b}a'_{ki}C_i(BA)\right)$$

$$\le L\frac{\|C_b(BA)\|}{\|C_i(BA)\|}\sin\angle\left(\sum_{i\neq b}a'_{ki}C_i(BA),a'_{bi}C_b(BA)\right)\le L\frac{\|C_b(BA)\|}{\|C_i(BA)\|}\beta$$

On the other hand, we have

$$\sin\angle\left(\sum_{i\neq b}a'_{ki}C_i(BA),\sum_{i\neq b}a'_{mi}C_i(BA)\right)$$

$$\le\sin\max_{r,s\text{active,passive}}\angle C_r(BA),C_s(BA)\le K'\frac{\beta}{\xi^k}$$

Since (absolute values of) angles satisfy the triangle inequality and $\sin(\theta+\phi)\le\sin(\theta)+\sin(\phi)$ for all $0\le\theta,\phi$ with $\theta,\phi<\frac{\pi}{2}$ we then have

$$\sin(\angle(C_k(BAA'),C_m(BAA')))$$
$$<L\frac{\|C_b(BA)\|}{\|C_i(BA)\|}\beta+K'\frac{\beta}{\xi^k}+L\frac{\|C_b(BA)\|}{\|C_j(BA)\|}\beta$$

$$\leq \left(L\frac{\|C_b(BA)\|}{\|C_i(BA)\|} + \frac{K'}{\xi^k} + L\frac{\|C_b(BA)\|}{\|C_j(BA)\|} \right)\beta$$

By Corollary 6.9, ℓ may be chosen large enough so that $L\frac{\|C_b(BA)\|}{\|C_j(BA)\|} < \frac{1}{4}$ for any active/passive index j, and by enlarging ℓ further, we may assume that $\frac{K'}{\xi^k} < \frac{1}{4}$ as well. Then

$$\sin(\angle(C_k(BAA'), C_m(BAA'))) < \frac{3}{4}\sin(\max\angle(C_i(B), C_j(B)))$$

which shows the desired outcome. □

The following proposition will be used to imply convergence, continuity and unique ergodicity at an infinite depth point.

Proposition 6.14. *Let A_i be as above. Then $(\prod A_i)\mathbb{R}^4_+$ converges to a single line.*

Proof. To prove the proposition we inductively group the product $(\prod A_i)$ into subproducts of length at most c that satisfy (P) or (R). Depending on the norm of the next block we apply Lemma 6.12 or 6.13 to conclude that either the diameter of the image of Δ, or $\max\sin\angle$ decreases by a definite factor. Since one of these happens an infinite number of times, the proposition follows. □

7. Connecting building block endpoints

Theorem 7.1. *Let $\mathcal{B}$ be a complete finite set of building blocks. and suppose that S_1, S_2 are respectively left and right endpoints for a building block. Then there exists a path of uniquely ergodic IETs connecting S_1 and S_2.*

Proof. Let $\pi = \pi(S_1) = \pi(S_2)$ be the permutation of the two points we want to connect. We will obtain the path c connecting S_1 to S_2 in Δ_π as the limit of approximating paths c_n. Before we define the actual paths c_n though, we first construct a combinatorial object that will serve as a guideline on how to build the paths.

We define a rooted oriented bivalent tree $\mathcal{T}$ in the following way. Each vertex will correspond to a building block $b \in \mathcal{B}$ and has two outgoing edges. The two vertices joined to b are the left and right compatible building blocks to b in $\mathcal{B}$. The root corresponds to the building block $(S_1, F, S_2) \in \mathcal{B}$, which exist by our hypothesis.

Suppose v is a vertex of $\mathcal{T}$. Then there is a unique path ρ which joins the root of $\mathcal{T}$ to v. This path ρ corresponds to a building block sequence, which in turn defines a Rauzy matrix $M(\rho)$. If v corresponds to the building block (T_1, F, T_2), then we say that the triple of IETs $(M(\rho)T_1, M(\rho)F, M(\rho)T_2) \in \Delta_\pi$ is defined by v.

Consider the set $v_0, \ldots, v_{2^n}$ of all vertices of distance n to the root of $\mathcal{T}$, and let $(T_1^{(i)}, F^{(i)}T_2^{(i)})$ be the triple of IETs defined by v_i. Note that (by construction) $T_2^{(i)} = T_1^{(i+1)}$. We define c_n as the concatenation of straight line segments in Δ_π connecting $T_1^{(i)}$ to $T_2^{(i)}$ for all i, parametrized so that each straight segment is traversed with constant speed on an interval of length 2^{-n}.

Explicitly, the first path c_0 is simply the straight line segment connecting S_1 to S_2. The path c_1 is the concatenation of the straight line connecting S_1 to F and F to S_2, where (S_1, F, S_2) is a building block in $\mathcal{B}$. Similarly, c_{i+i} is constructed by replacing each straight line segment in c_i by a concatenation of two line segments according to the building block which (after the suitable number of Rauzy steps) has the same endpoints.

Note that by definition c_i and c_{i+1} agree at all t which correspond to IETs defined by vertices of distance at most i to the root of $\mathcal{T}$.

We call any point $c_i(t)$ which corresponds to an IET defined by some vertex v of $\mathcal{T}$ a *problematic point* of c_i. The depth of $c_i(t)$ is defined to be the depth of the building block sequence corresponding to the unique path joining the root of $\mathcal{T}$ to v.

Explicitly, this means that the boundary points $c_1(0), c_1(1)$ are of depth 0. Suppose we have assigned a depth to all the problematic points of c_i. Then, each problematic point p of c_{i+1} is adjacent to two problematic points p_l, p_r of c_i. If d_l, d_r are the depths of these points, then the depth of p is $\min(d_l, d_r) + 1$.

By the definition of building blocks every problematic point corresponds to a uniquely ergodic IET.

For any $t \in [0, 1]$ which is not a multiple of 2^{-m} for some m we now claim that the sequence $c_n(t)$ converges in Δ_n. Namely, suppose that $t \in [k2^{-M}, (k+1)2^{-M}]$. Then for each $i > M$ the initial part of the Rauzy expansion of $c_i(t)$ agrees with the Rauzy expansion of the building block sequence corresponding to a suitable vertex v_M of distance M to the root of the tree $\mathcal{T}$. Increasing M to a $N > M$ it follows from the construction that v_M is contained in the (unique) oriented path joining the root of the tree $\mathcal{T}$ to v_N. Let ρ be the infinite oriented path in $\mathcal{T}$ which contains all v_i. Since t is not a dyadic fraction the path ρ cannot eventually only make left or only right turns (as that would, e.g. for left turns, mean that $k2^{-N} \leq t \leq k2^{-N} + 2^{-M}$ for all M).

Thus, the path ρ defines a building block sequence as in Section 6. Therefore, Proposition 6.14 implies that for any ϵ there is a $N > 0$ such that for all $i > N$ the values $c_i(t)$ differ by at most a distance of ϵ (since they share the same initial segment of building blocks). This shows that $c_i(t)$ converges.

In fact, the same argument shows that $\lim_{n\to\infty} c_n(t)$ is a uniquely ergodic IET by Theorem 2.6 and that the limiting function is continuous at such t.

It remains to show continuity at the problematic points. We show continuity using the limit formulation. By construction, points on c_n close to a depth L point follow the expansion P of the endpoint for longer and longer times. But in this situation, Lemma 5.2 shows that the normalized length vectors converge, and thus implies the desired continuity. □

Corollary 7.2. Let T_1, T_2 be two 4-IETs which each become the left or right endpoint of a building block after a finite number of Rauzy steps. Then T_1 and T_2 can be joined by a path of uniquely ergodic IETs.

Proof. Let K be large enough so that T_1, T_2 become endpoints after K Rauzy steps. Now consider the triangulation $\mathcal{P}_K$ of Δ_π defined by the fail planes of Rauzy induction within the first K steps, and choose endpoints of building blocks on all of the fail planes in $\mathcal{P}_K$ separating T_1 from T_2. Two such points which are not separated by a fail plane can be joined by a path as in the previous theorem, by applying Rauzy induction first. The Corollary follows by concatenating these finitely many paths. □

To finish the proof of Theorem 4.1, we need to be able to join IETs which are not themselves endpoints of building blocks. This will be done with a limiting argument via the following lemmas.

Lemma 7.3. *Let (X, d) be a metric space, let (p_i) be a sequence in X and $p_\infty \in X$ be so that*

(1) (p_i) converges to p_∞
(2) p_i, p_{i+1} are path connected by a path $P_i \subset X$ and
(3) $\lim_{i\to\infty} diam(P_i) = 0$.

Then p_1 is path connected to p_∞.

Proof. Let $\psi(t) = P_j(2^j t)$ where $t \in [1-2^{-j+1}, 1-2^{-j}]$ and $\psi(1) = p_\infty$ Because the P_i are continuous and $P_i(1) = P_{i+1}(0)$ ψ is continuous everywhere except possibly (1). Notice by conditions (1) and (3) $\lim_{t\to 1^-} \psi(t) = p_\infty$. So ψ extends to a path connecting p_1 to p_∞. □

Lemma 7.4. *Let $\mathcal{B}$ be a finite complete set of building blocks. Let T be an IET so that $R^k(T)$ is defined for all k. Then there exists a sequence of IETs (S_i) so that $\pi(R^k(S_j)) = \pi(R^k(T))$ for all $k < j$ and either $R^{i-1}(S_i) = R^{i-1}(S_{i+1})$ or they are left and right endpoints from a building block triple.*

Proof. Because $\mathcal{B}$ is complete, for each vertex of the Rauzy diagram there is a corresponding building block where the right hand side takes one (forward pointing) edge and the left hand side takes the other. Thus, we can follow the expansion of T and choose the desired building blocks. □

Theorem 7.5. *The set of uniquely ergodic unit length 4-IETs with permutation in the Rauzy class of (4321) is path connected.*

Proof. Let E by a uniquely ergodic 4-IET with permutation (4321) so that $R^k(E)$ is defined for all k. As paths may be concatenated, it suffices to show that for each uniquely ergodic S with permutation (4321) there is a path joining it to E.

The first case we consider is that S is a uniquely ergodic IETs that has $R^k(S)$ defined for all k. Let S_i be a sequence converging to S given by Lemma 7.4. By Corollary 7.2 for each i there is a path, l_i of uniquely ergodic IETs that connect S_i to S_{i+1}, and that stays in $M(S, i)\Delta$. These satisfy condition (2) of Lemma 7.3. Because S is uniquely ergodic and so $M(S, k)\Delta$ converges to a point, the l_i satisfy condition (3) of Lemma 7.3. For the same reason the endpoint of the l_i are converging to S satisfying condition (1) of Lemma 7.3. Repeat the same for E in place of S. Thus, Lemma 7.3 yields the desired path.

We next consider the case of a uniquely ergodic IET S that does not have $R^k(S)$ defined for all k. There exists an interval J so that $S|_J$ has the minimal number of intervals of any induced map of S. As an IET on fewer intervals, $S|_J$ has infinite Rauzy induction. By Section 2.1 there exists a path in the Rauzy diagram $\mathcal{R}(4)$ whose actions on the relevant columns reflect the action on $S|_J$ on the appropriate Rauzy class. Choose once and for all finite Rauzy paths P_i with positive associated matrix for each permutation in the Rauzy diagram.

Now we choose a sequence $\hat{T}_i$ of building block endpoints converging to T that share longer and longer segments of this path and then are followed by one of the

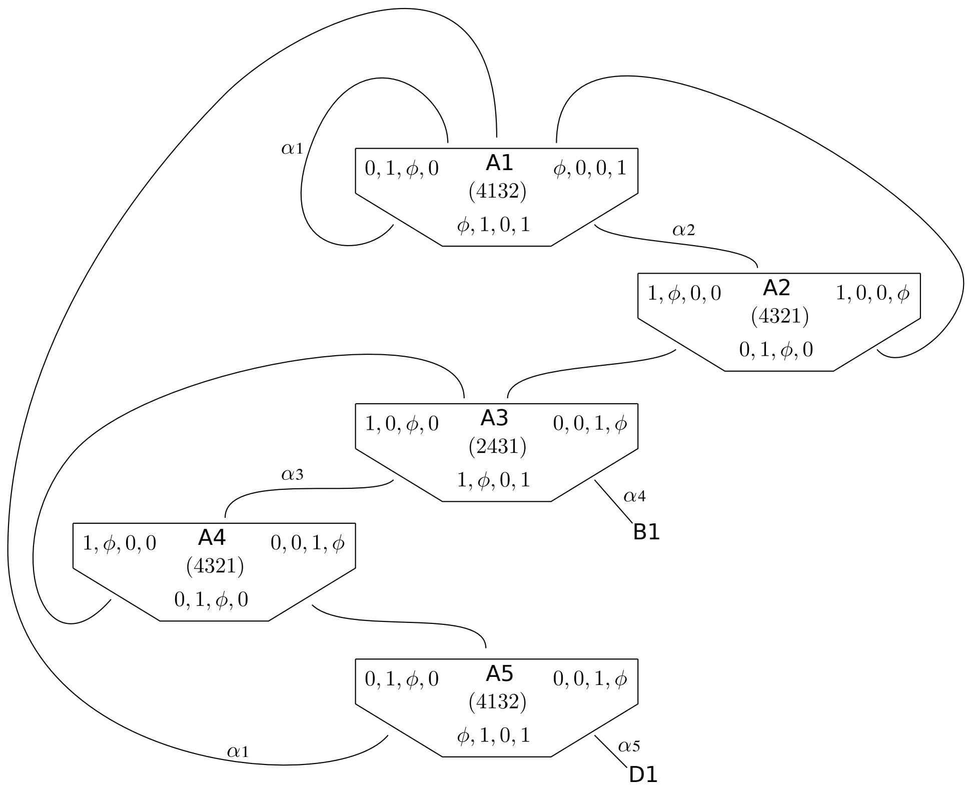

FIGURE 4. Diagram A

P_i. We can connect the $\hat{T}_i$ and satisfy condition (2) of Lemma 7.3. Following T in Rauzy induction leads to at least 3 columns pointing increasingly in the direction of the length vector of T and having increasing column norm by unique ergodicity. By Corollary 5.3 after the application of one of the P all four columns point in roughly this direction. Also by Corollary 5.3 the paths connecting $\hat{T}_i$ to $\hat{T}_{i+1}$ have diameter going to 0 as i goes to infinity.[3] By Corollary 5.3 the condition (1) of Lemma 7.3 is satisfied. Because any IET in $M(T_{n+k}, n+k)\Delta$ also satisfies the assumptions of Corollary 5.3 condition (3) of Lemma 7.3 is satisfied. □

8. Explicitly producing building blocks

In this section we prove Proposition 4.8 by explicit construction. The building blocks and their relations are depicted in Figures 4, 5 and 6. Unlabeled edges correspond to single Rauzy steps (left and right corresponds to type a and b Rauzy induction. Labeled edges of the graphs correspond to several common Rauzy steps (the Rauzy steps corresponding to each path can be found in Appendix A). Edges ending in capital letters lead to the corresponding blocks in a different diagram.

[3]This is because at some point they have (at least) 3 large columns that almost point in T's direction and all after that they follow building block matrices.

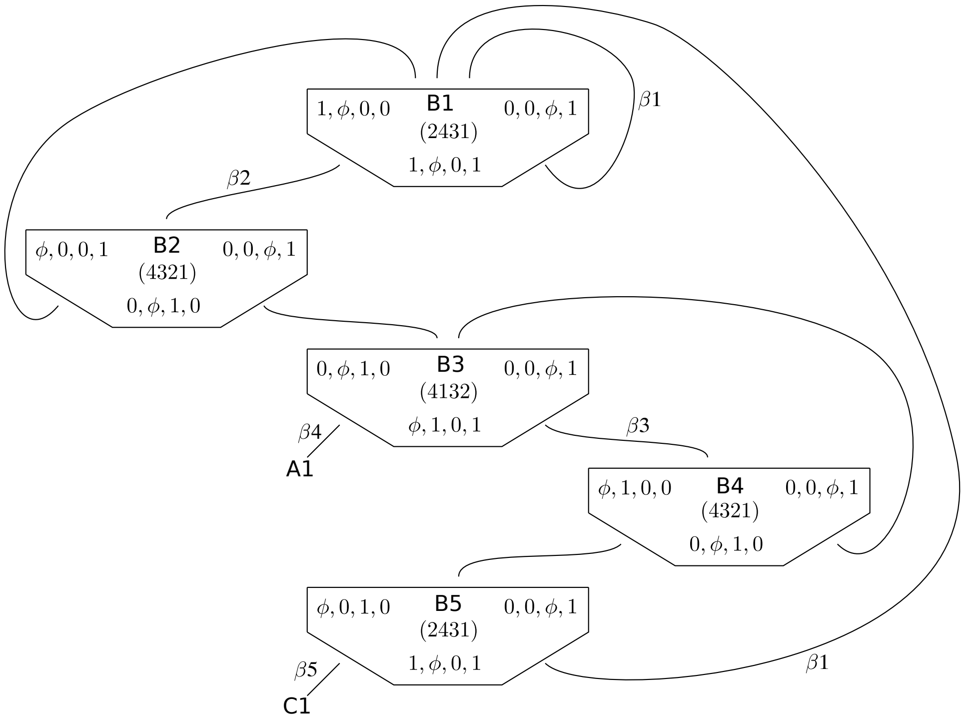

FIGURE 5. Diagram B

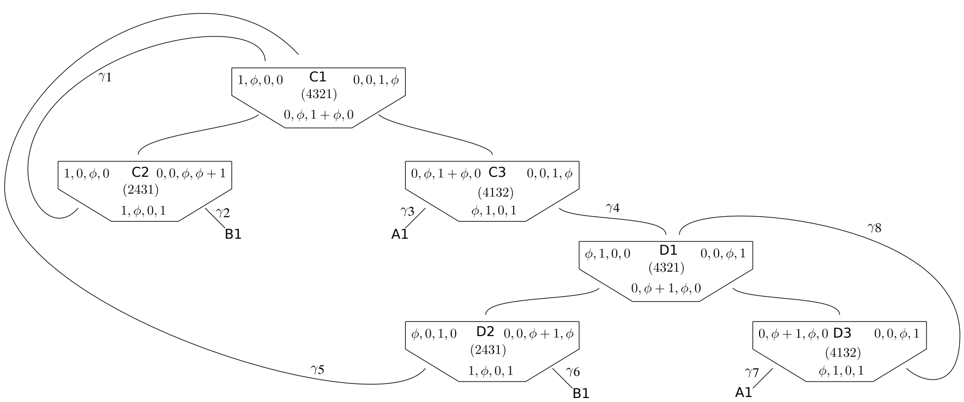

FIGURE 6. Diagram CD

By simply observing the Rauzy diagram and Figures 4 to 7 it is immediate that the set of building blocks satisfies **Transitivity** and **Completeness** claimed in Proposition 4.8.

To check **Combining Loops** and **Isolated Idle**, one checks in the diagrams that the following paths are the only periodic paths of depth 0, and that their matrices have the desired properties.

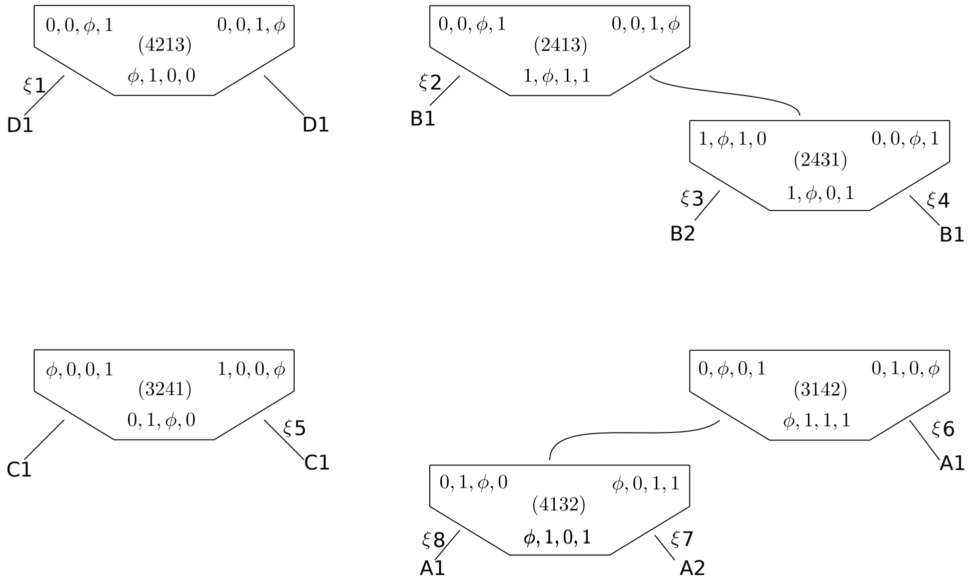

FIGURE 7. Diagram EFGH

(1) $A1 \to A1$ corresponds to $M_{\alpha 1}$.

$$M_{\alpha 1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Columns 2 and 3 are active, 4 is passive. Column 1 is idle. The matrices which can follow without repeating this loop are $\alpha_2 M_a^{(4321)} \alpha_3, \alpha_2 M_a^{(4321)} \alpha_4$, $\alpha_2 M_b^{(4321)} \alpha_1$ or $\alpha_2 M_b^{(4321)} \alpha_2$, all of which have at least two nonzero entries in each column. Thus, Isolated Idle holds for this loop.

(2) $A2 \to A1 \to A2$ corresponds to $M_b^{(4321)} M_{\alpha 2}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 2 \end{pmatrix}$$

Columns 1 and 4 are active, 2 and 3 are passive.

(3) $A3 \to A4 \to A3$ corresponds to $M_{\alpha 3} M_a^{(4321)}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Columns 1 and 3 are active, 2 and 4 are passive.

(4) $B1 \to B1$ corresponds to $M_{\beta 1}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Columns 3 and 4 are active, 1 is passive. Column 2 is idle. The matrices which can follow are $M_{\beta 2} M_a^{(4321)} \beta_1$, $M_{\beta 2} M_a^{(4321)} \beta_2$, $M_{\beta 2} M_b^{(4321)} M_{\beta 3}$ or $M_{\beta 2} M_b^{(4321)} M_{\beta 4}$, all of which have at least two nonzero entries in each column. Thus, Isolated Idle holds.

(5) $B1 \to B2 \to B1$ corresponds to $M_{\beta 2} M_a^{(4321)}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Columns 1 and 3 are active, columns 3 and 4 are passive.

(6) $B3 \to B4 \to B3$ corresponds to $M_{\beta 3} M_b^{(4321)}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Columns 3 and 4 are active, columns 1 and 2 are passive.

(7) $C1 \to C2 \to C1$ corresponds to $M_a^{(4321)} M_{\gamma 1}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Columns 1 and 2 are active, columns 3 and 4 are passive.

(8) $D1 \to D3 \to D1$ corresponds to $M_b^{(4321)} M_{\gamma 8}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Columns 3 and 4 are active, columns 1 and 2 are passive.

It remains to check **Almost Positivity**. We need to show that paths that increase depth enough yield almost positive matrices.

We begin by noting two reductions that describe why this is a finite task:

- It suffices to consider paths starting at $A1, A3, B1, B3, D1, C1$, as all paths we will construct will end at a such a point. Initial segments leading to one of these points have uniformly small length and do not affect the argument.
- All minimal depth 0 loops have matrices with positive entries on the diagonal. Since multiplying by such a matrix preserves the property of being positive, any path that yields a positive matrix will still have this property if any number of loops is inserted.

- If a path without loops yields an almost positive, yet not positive matrix, there is a finite number of modifications (add loops of length 1) that need to be considered. If all such modificiations yield positive matrices, we are done.

The desired property now follows by a lengthy case-by-case check. We only list the cases for paths starting at $A1$ in details. All other cases are checked in a completely analogous manner[4]

The following observation helps to reduce the number of cases:

Observation 8.1. Say that a (4×4)–matrix A is *weakly positive* if every entry is non-negative and there is at most one zero entry.

If A is a weakly positive matrix and B is Almost Positive or weakly positive, then AB has positive entries.

Thus, it suffices to show a slightly weaker version of Almost Positivity: *any path that increases depth enough yields a matrix that is Almost Positive, or is weakly positive.*

Paths that begin with the loop α_1 at A1: There are four possibilities that can follow the α_1-loop:

(1) The path $A1 \to A2 \to A3 \to B1$, corresponding to the matrix

$$\alpha_1\alpha_2 M_a^{(4321)}\alpha_4 = \begin{pmatrix} 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 4 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

which is weakly positive.

(2) The path $A1 \to A2 \to A3 \to A4 \to A5 \to D1$, corresponding to the matrix

$$\alpha_1\alpha_2 M_a^{(4321)}\alpha_3 M_b^{(4321)}\alpha_5 = \begin{pmatrix} 4 & 5 & 4 & 2 \\ 3 & 5 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 3 & 2 & 1 \end{pmatrix}$$

which is weakly positive.

(3) The loop $A1 \to A2 \to A3 \to A4 \to A5 \to A1$, which immediately leads to a positive matrix.

(4) The loop $A1 \to A2 \to A1$, leading to

$$\alpha_1\alpha_2 M_b^{(4321)} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

If this is followed by a path beginning with $A1 \to A3$, then one of the previous cases implies that we will reach a positive matrix. The only other possibility is to alternate with the $A1 \to A1$ loop. But, the square of the above matrix is positive, and thus this case also yields a positive matrix.

[4] which can also be automated and done with computer algebra systems – see the second author's webpage for a SAGE script doing this task.

As a consequence we record: *Any path that begins with the α_1–loop and increases depth will yield a positive matrix.* In particular, in the consequent analysis we never have to consider the case that the α_1–loop is inserted in the middle of some path – as then the resulting matrix will be positive upon increasing depth enough.

Paths that begin with the loop $A1 \to A2 \to A1$: (1) If the loop is followed by the α_1–loop, then the above show that the result will turn positive.

(2) If we follow by $A1 \to A3$ the resulting matrix is

$$\alpha_2 M_b^{(4321)} \alpha_2 M_a^{(4321)} = \begin{pmatrix} 2 & 5 & 4 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & 3 & 3 \end{pmatrix}$$

which is Almost Positive.

In consequence, *Any path that begins with the $(A1 \to A2 \to A1)$–loop and increases depth will yield an Almost Positive matrix.*

Paths that begin with $A1 \to A2 \to A3 \to B1$: Here, we start with the matrix

$$\alpha_2 M_a^{(4321)} \alpha_4 = \begin{pmatrix} 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

which is followed by $B1 \to B3$, leading to

$$\alpha_2 M_a^{(4321)} \alpha_4 \beta_2 M_b^{(4321)} = \begin{pmatrix} 6 & 5 & 6 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 3 & 3 & 4 & 1 \end{pmatrix}$$

From here, there are several possibilities:

(1) Returning to $A1$, leading to

$$\alpha_2 M_a^{(4321)} \alpha_4 \beta_2 M_b^{(4321)} \beta_4 = \begin{pmatrix} 6 & 5 & 11 & 7 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 3 & 3 & 7 & 4 \end{pmatrix}$$

which is weakly positive.

(2) Following $B3 \to B5 \to B1$, leading to a positive matrix.

(3) Following $B3 \to B5 \to C1$, leading to a weakly positive matrix.

Paths that begin with $A1 \to A2 \to A3 \to A5 \to D1$: This corresponds to the matrix

$$\alpha_2 M_a^{(4321)} \alpha_3 M_b^{(4321)} \alpha_5 = \begin{pmatrix} 4 & 5 & 4 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 2 & 1 \end{pmatrix}$$

There are several possibilities:

(1) following with $D1 \to D2 \to B1$ yields a weakly positive matrix.

(2) following with $D1 \to D2 \to C1$ yields a Almost Positive matrix.

(3) following with $D1 \to D3 \to D1 \to D2 \to C1$ (inserting the $D1$ loop into the previous case) leads to

$$\alpha_2 M_a^{(4321)} \alpha_3 M_b^{(4321)} \alpha_5 (M_b^{(4321)} \gamma_8) M_a^{(4321)} \gamma_5 = \begin{pmatrix} 6 & 13 & 16 & 12 \\ 1 & 3 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 3 & 7 & 8 & 6 \end{pmatrix}$$

Following with $C1 \to C3$ leads to a weakly positive matrix. Following with $C1 \to C2 \to B1$ leads to a positive matrix. Since both of these lead to positive matrices, we are done.

Paths that begin with $A1 \to A2 \to A3 \to A5 \to A1$: Note that the matrix describing $A5 \to A1$ in fact is the same as $A1 \to A1$, and thus this follows from the first case.

9. Extending to n-IETs

The goal of this section is to prove

Theorem 9.1. *Let π be a non-degenerate permutation for n-IETs where $n \geq 4$. The set of uniquely ergodic n-IETs with permutation π is path connected.*

The main tool in the proof of this is the following object.

Definition 9.2. *An IET S is called a* secret 4-IET at level k *if there exists $M(T, r)$ a matrix of Rauzy induction with $r \leq k$, v a non-negative vector with at most 4 non-zero entries so that $L(S) = M(T, r)v$.*

In other words, a secret 4-IET at level 0 is an n-IET which has at most 4 intervals of nonzero length. A general secret 4-IET is one that is of this form after applying some number of Rauzy steps.

Next we describe how π acts on subsets of $\{1, .., d\}$. Let $(p_1, \ldots, p_4)$ be a 4-tuple of numbers, labeled so that $p_1 < p_2 < p_3 < p_4$. Applying π to the p_i maps them to numbers $\pi(p_i)$ which may not be in order. Let $\pi|(p_1, p_2, p_3, p_4)$ be the permutation on the numbers $\{1, 2, 3, 4\}$ which describes how the order of p_i is rearranged by π. We call $\pi|(p_1, p_2, p_3, p_4)$ the *permutation π restricted to* $(p_1, \ldots, p_4)$.

To give a simple example, consider $\pi = (3142)$: then $\pi|(3, 4) = (12)$ and $\pi|(1, 3) = (21)$.

We say that a permutation π *acts as a rotation* on a pair (p_1, p_2) of symbols if $\pi|(p_1, p_2) = (21)$. A permutation π is *irreducible* if it does not preserve a set of the form $\{1, \ldots, k\}$ for $k < n$.

Definition 9.3. *Let π be a permutation on n symbols and $(p_1, \ldots, p_4)$, $(q_1, \ldots, q_4)$ be two 4-tuples of entries so that π restricts on both as a permutation in the Rauzy class of* (4321). *We call the tuples* accessible *if there exists index pairs (r_1, r_2), (s_1, s_2) so that π acts on $(p_{r_1}, p_{r_2}, q_{s_1}, q_{s_2})$ irreducibly, π acts as a rotation on (p_{r_1}, p_{r_2}) and (q_{s_1}, q_{s_2}) and we do not have $\pi|(p_{r_1}, p_{r_2}, q_{s_1}, q_{s_2}) = (4231)$.*

We require that $\pi|(p_{r_1}, p_{r_2}, q_{s_1}, q_{s_2}) \neq (4231)$ for technical reasons relating to the proof of the next lemma which does not work for this permutation. In particular, most of the analysis is for the case where $\pi|(p_{r_1}, p_{r_2}, q_{s_1}, q_{s_2})$ acts as a 3-IET with a pair of p_i, q_j behaving as one interval. Our argument does not work when this interval is the second one, which is the case of (4231).

Lemma 9.4. *Fix a non-degenerate permutation π on d symbols and suppose that T, S are uniquely ergodic IETs with permutation π. Further assume that the lengths vectors of T and S are 0 off of $(p_1, \dots, p_4)$ and $(q_1, \dots, q_4)$ respectively (i.e. that they are secret 4-IETs).*

If $(p_1, \dots, p_4)$ and $(q_1, \dots, q_4)$ are accessible then there exists a path of uniquely ergodic IETs connecting T to S.

Proof. If $\pi|(p_{r_1}, p_{r_2}, q_{s_1}, q_{s_2}) \in \mathcal{R}(4321)$ or is a rotation (for example if $\pi|(p_{r_1}, p_{r_2}, q_{s_1}, q_{s_2}) = (3412)$) then we can choose T_1 and S_1 secret 2-IETs at level zero on (p_{r_1}, p_{r_2}) and (q_{s_1}, q_{s_2}) which are irrational rotations by the same number. The set of all n-IETs whose length vectors are 0 off of the entries $(p_1, \dots, p_4)$ is a copy of 4-IET space. By Theorem 4.1, T can therefore be connected to T_1 by a continuous path. The analogous statement is follows for S, S_1. Applying Theorem 4.1 once more to connect T_1 to S_1 we obtain the desired path from T to S by concatenation.

If $\pi|(p_{r_1}, p_{r_2}, q_{s_1}, q_{s_2})$ acts as a rotation then (r_i, s_j) and (r_k, s_l) can be treated as one interval and we can change weight between them, without changing the self-map of the interval that the IET defines. This defines a path of uniquely ergodic IETs connecting T_1, S_1.

If $\pi|(p_{r_1}, p_{r_2}, q_{s_1}, q_{s_2})$ acts as an element in $\mathcal{R}(321)$ then one of the three intervals is split in two. Choose T_1, S_1 to be secret 2-IETs which are rotations by the same irrational number. These are path connected to T and S respectively by Theorem 4.1. Recall from the comments in the Introduction following Theorem 1.1 that 3-IETs can be thought of as the induced map of rotation by $\frac{1-L_1}{1+L_2}$ on $[0, \frac{1}{1+L_2})$ and they are uniquely ergodic if and only if $\frac{1-L_1}{1+L_2}$ is irrational. Now we connect T_1 and S_1 in the set of IETs with length zero except on $p_{r_1}, p_{r_2}, q_{s_1}, q_{s_2}$ keeping $\frac{1-L_1}{1+L_2} \notin \mathbb{Q}$ constant, which keeps the 3-IETs all uniquely ergodic. We can do this because T_1 and S_1 are rotations by the same irrational number.

To see that this is doable, consider $\alpha \in [0,1] \setminus \mathbb{Q}$ and the line segment

$$H_\alpha = \Delta_3 \cap \{(x_1, x_2, x_3) : \frac{x_2 + x_3}{x_1 + 2x_2 + x_3} = \alpha\}.$$

Observe that $(1-\alpha, 0, \alpha) \in H_\alpha$. If $\alpha < \frac{1}{2}$ then $(\frac{1-2\alpha}{1-\alpha}, \frac{\alpha}{1-\alpha}, 0)$ is in this set. If $\alpha > \frac{1}{2}$ then $(0, \frac{1-\alpha}{\alpha}, \frac{2\alpha-1}{\alpha}) \in H_\alpha$.[5] If we have $(p_1, q_1, p_2, q_2) \to (p_2, q_2, q_1, p_1)$ (where this notation means that $p_1 < q_1 < p_2 < q_2$ and π restricted to these intervals puts p_2 first, then q_2 then q_1 then p_1) choose $\alpha > \frac{1}{2}$. Start at $(1-\alpha, 0, \alpha, 0)$ and move in a straight line to $(1-\alpha, 0, 0, \alpha)$. From here move along the path $(a_t, b_t, 0, c_t)$ where (a_t, b_t, c_t) parametrizes the line between $(1-\alpha, 0, \alpha)$ and $(0, \frac{1-\alpha}{\alpha}, \frac{2\alpha-1}{\alpha})$ in H_α. The case of $(p_1, q_1, q_2, p_2) \to (q_2, p_2, q_1, p_1)$ is similar. The cases of $(p_1, q_1, p_2, q_2) \to (q_2, p_2, p_1, q_1)$ and (p_1, q_1, q_2, p_2) are also similar (if $L(T) = (a, b, c)$ then $L(T^{-1}) = (c, b, a)$). ☐

The proof of Theorem 9.1 relies mainly on the following combinatorial statement:

Proposition 9.5. *Let π be irreducible. Any two 4-tuples of entries so that π acts on both as a permutation in the Rauzy class of (4321) can be linked by an accessible*

[5] The fact that if $\alpha \notin \mathbb{Q}$ we only get one of $(1-\alpha, \alpha, 0)$ or $(0, 1-\alpha, \alpha)$ in H_α is why we rule out the case of (4231).

chain. That means there is a sequence of 4*-tuples, so that consecutive ones are accessible, which begins and ends with the given* 4*-tuples.*

Lemma 9.6. *If* $(p_1,\dots,p_4)$, $(q_1,\dots,q_4)$ *are not an accessible pair then up to swapping* $(p_1,\dots,p_4)$ *with* $(q_1,\dots,q_4)$ *we have* $p_i < q_j$, $\pi(p_i) < \pi(q_j)$ *for all* i, j.

Proof. We assume $p_1 < p_2 < p_3 < p_4$ and $q_1 < ... < q_4$ and $p_1 < q_1$. We show the lemma by contradiction by considering numerous cases.

We first handle the case that $p_1 < q_1 < p_4$.

Subcase a: If $\pi(p_i) < \pi(q_j)$ for all i, j then either

$$p_1 < q_1 < p_i < q_j \text{ or } p_1 < q_1 < q_j < p_i$$

where $\pi(p_i) < \pi(p_1)$ and $\pi(q_j) < \pi(q_1)$ (p_i and q_j both exist by the irreducibility of π). So we obtain the permutations (3142) or (4132) which are both in $\mathcal{R}(4321)$, and hence the tuples are accessible.

Subcase b: If $\pi(p_i) > \pi(q_j)$ for all i, j then we can either pick $p_\ell < q_1 < q_j < p_i$ where $\pi(q_j) < \pi(q_1) < \pi(p_\ell) < \pi(p_j)$ or $p_1 < p_i < q_j < q_\ell$ and $\pi(q_\ell) < \pi(q_j) < \pi(p_i) < \pi(p_1)$. These correspond to permutations in $\mathcal{R}(4321)$.

Subcase c: So we may now treat the case $p_1 < q_1 < p_i$ and $\pi(p_\ell) < \pi(q_j) < \pi(p_k)$. If j can be chosen to be 1 there exists q_a so that $\pi(q_a) < \pi(q_1)$. Also by the irreducibility of π on p_1, p_2, p_3, p_4 we may choose p_b so that $p_b < q_1$ and $\pi(p_b) > \pi(q_1)$ or $p_b > q_1$ and $\pi(p_b) < \pi(q_1)$. (Otherwise if $p_i < q_q < p_j$ then $\pi(p_i) < \pi(q_1) < \pi(p_j)$.) If both occur our permutation can by chosen to be (4321) or (3421). If there is only $p_b > q_1$ so that $\pi(p_b) < \pi(q_1)$ then by the irreducibility of π on $p_1, ..., p_4$ there exists $q_1 < p_c < p_b$ so that $\pi(p_b) < q_1 < \pi(p_c)$. So we obtain a permutation corresponding to (3214), (3241) by choosing q_1, q_a, p_c, p_b to be our symbols and treating the position of q_a relative to p_b, p_c we obtain the permutation $(4213), (4312)$ or (3421). If j can not be chosen to be 1 then we may choose our permutation to be (4312).

The next case is that $\pi(p_k) < \pi(q_j) < \pi(p_L)$ and is proved similarly.

The last case is that $p_4 < q_1$ and $\pi(p_i) > \pi(q_j)$ for all i, j. We pick $p_k < p_L$ and $q_i < q_j$ so that $\pi(p_L) < \pi(p_k)$ and $\pi(q_j) < \pi(q_i)$. $\pi|(p_L, p_k, q_i, q_j) = (4321)$ and we are done. □

We now begin the proof of Proposition 9.5. If the given 4-tuples are not accessible, then we may assume (by Lemma 9.6) that $q_j > p_i$ and $\pi(q_j) > \pi(p_i)$ for all i, j, and we do so from now on.

The proof proceeds by induction. To do that, we define the *distance* of two 4-tuples $(q_1, \dots, q_4), (p_1, \dots, p_4)$ to be

$$(\min_i\{q_i\} - \max_i\{p_i\}, \min_i\{\pi(q_i)\} - \max_i\{\pi(p_i)\})$$

ordered by lexicographic order. The distance is at least $(1, 1)$ by our assumption and Lemma 9.6.

If the distance is larger than that, then the following lemma allows to decrease the distance.

Lemma 9.7. *If $P = (p_1, \dots, p_4)$ and $Q = (q_1, \dots, q_4)$ are not an accessible pair then there exists $A = (a_1, \dots, a_4)$ so that A and Q are accessible and the distance from P to A is strictly less than the distance from P to Q.*

Proof. The proof is done in 2 cases.

Case 1: In this case we assume there exists $r < \min\{q_i\}$ so that $\pi(r) > \pi(q_i)$ for some i. Now if $\pi(r) < \pi(q_j)$ if and only if $\pi(q_1) < \pi(q_j)$ for $j \neq 1$, replace q_1 with r and $\pi|(r, q_2, q_3, q_4) = \pi|(q_1, q_2, q_3, q_4)$. Call this Case 1'. Otherwise, there are two sub-cases. If $\pi(r) < \pi(q_1)$ then by the assumption of Case 1 there exists q_i so that $\pi(q_i) < \pi(r)$. Since we are not in Case 1' there exists q_j so that $\pi(r) < \pi(q_j) < \pi(q_1)$. $\pi|(r, q_1, q_i, q_j) \in \{(3142), (4132)\}$, both of which are in the $\mathcal{R}(4321)$. If $\pi(r) > \pi(q_1)$ because we are not in Case 1', there exists q_i so that $\pi(q_1) < \pi(q_i) < \pi(r)$ and there exists q_j so that $q_1 < q_j < q_i$ so that $\pi|(r, q_1, q_i, q_j) \in \{(2413), (3241), (2431)\}$. (This is because the string 12 never appears as a consecutive pair in a permutation in the Rauzy class of (4321), since in this case the pair of intervals $I_1 \cup I_2$ could be treated as one interval.)

Case 2: If we are not in Case 1, then Lemma 9.8 implies we can have an accessible four-tuple which has $\min\{q_i\}$ and the absence of Case 1 implies $\min\{\pi(q_i)\} > \min\{q_i\}$. So some symbol greater than $\min\{q_i\}$ is sent before $\min\{\pi(q_i)\}$ (by counting). The proof is now finished similar to Case 1. □

Finally, the case of distance of $(1, 1)$ is handled by the next lemma.

Lemma 9.8. *For any i either there exists $r < i$ so that $\pi(r) > \pi(i)$ or there exists $r > i$ so that $\pi(r) < \pi(i)$.*

Proof. If $i = \pi(i)$ this simply follows by the irreducibility of π.

If $i \neq \pi(i)$ then this by a counting argument. Indeed if $i < \pi(i)$ then there exists $r > i$ so that $\pi(r) < \pi(i)$ because π is a bijection on a finite set and there is no injection from a set of large cardinality to a set of smaller cardinality. □

Let us see that this lemma implies Proposition 9.5 with distance $(1, r)$ for any r. Let $i = q_1$. So let us assume that there is $j < i$ so that $\pi(j) > \pi(i)$. Now similarly to Case 1 of Lemma 9.7 we can replace one of the q_k with j and obtain a new permutation on 4 symbols in the Rauzy class of 4-IETs. Now since $p_4 = q_1 - 1$ we have that that the new pair violates Lemma 9.6. The other possibility is similar.

This finishes the proof of Proposition 9.5.

Corollary 9.9. The set of secret 4-IETs of level k is path connected for any k.

Proof. For secret 4-IETs at level 0 this follows by Lemma 9.4 and Proposition 9.5. Otherwise, argue as in the proof of Corollary 7.2 with the triangulation $\mathcal{P}_N$. □

The final ingredient to the proof of Theorem 9.1 is the following lemma.

Lemma 9.10. *Let T be an IET so that $R^k(T)$ is defined for all k. Let S_1, S_2 be secret 4-IETs at depth r which are not secret at depth k. Let $S_1, S_2 \in M(T, r)\Delta$. Further assume that $M(R^k(T), r - k)$ is a positive matrix. Then S_1, S_2 are path connected by uniquely ergodic IETs in $M(T, k)\Delta$.*

The idea of this proof is that we build paths connecting $R^k S_1$ to $R^k S_2$ and the path we are interested in is the (projective) image of this path under $M(T,k) = M(S_i,k)$.

Proof. Because $M(R^k T, r-k)$ is positive, it follows that if $S \in M(T,r)\Delta$ is uniquely ergodic and $R^r(S)$ has two nonzero entries then it is not a secret 4-IET at level k. So it suffices to show that in the simplex the secret 4-IETs at level at most L are path connected for all L. This follows from Corollary 9.9. Indeed, let $S_i' = R^k(S_i)$ and P be the path given by Corollary 9.9 connecting S_1' and S_2'. $M(T,k)P$ is the path connecting S_1 and S_2 contained in $M(T,k)\Delta$. □

Proof of Theorem 9.1. If S, E have R^k defined for all k then the theorem follows analogously to the proof of Theorem 7.5 before via Lemma 7.3. In this case the p_i are secret 4-IETs at level i contained in $M(S,i)\Delta$ or $M(E,i)\Delta$. The paths P_i connecting p_i and p_{i+1} are contained in $M(S,m_i)$ or $M(E,q_i)$, where the m_i or q_i is given by Lemma 9.10. The m_i and q_i go to infinity with i. Since S and E are uniquely ergodic and have Rauzy induction defined for all k we have that $M(S,i)\Delta$ and $M(E,i)\Delta$ contract to S and E respectively. This verifies Lemma 7.3.

Otherwise, assume that S does not have all powers of Rauzy induction defined. We may without loss of generality assume that E does. We prove the theorem by assuming that we have inductively proved Theorem 9.1 for all $4 \leq m < n$. There exists an interval J so that the induced map $S|_J$ is a uniquely ergodic IETs on $k < d$ symbols. There is a $d \times k$ matrix M so that $L(S) = ML(S|_J)$ where in an abuse of notation $L(S|_J)$ is a vector with k entries. Because every irreducible permutation has a pair where it acts irreducibly, we have a secret 4-IET S' of the form $M\bar{v}$ for some v whose non-zero entries are a subset of $L(S|_J)$. By induction, the k-IET $S|_J$ can be joined by a path of uniquely ergodic IETs to S'. Applying M to this path connecting $S|_J$ to S' we obtain a path connecting a S to a secret 4-IET (with length vector given by $ML(S')$). By the first case, E is path connected by uniquely ergodic IETs to a secret 4-IET of some depth and this secret 4-IET is path connected by uniquely ergodic IETs to every uniquely ergodic secret 4-IET. We have just shown that one of these is path connected S by a path of uniquely ergodic IETs. □

Appendix A. Composite paths

In Figure 4

- **$\alpha1$:** (4132) → (3142) → (3142) → (4132)
- **$\alpha2$:** (4132) → (4213) → (4321) → (2431) → (3241) → (4321)
- **$\alpha3$:** (2431) → (3241) → (3241) → (4321)
- **$\alpha4$:** (2431) → (2413) → (2431)
- **$\alpha5$:** (4132) → (4213) → (4321)

In Figure 5

- **$\beta1$:** (2431) → (2413) → (2413) → (2431)
- **$\beta2$:** (2431) → (3241) → (4321) → (4132) → (4213) → (4321)
- **$\beta3$:** (4132) → (4213) → (4213) → (4321)
- **$\beta4$:** (4132) → (3142) → (4132)
- **$\beta5$:** (2431) → (3241) → (4321)

In Figure 6

$\gamma 1$**:** (2431) → (3241) → (3241) → (4321)
$\gamma 2$**:** (2431) → (2413) → (2431)
$\gamma 3$**:** (4132) → (3142) → (3142) → (4132)
$\gamma 4$**:** (4132) → (4213) → (4321)
$\gamma 5$**:** (2431) → (3241) → (4321)
$\gamma 6$**:** (2431) → (2413) → (2413) → (2431)
$\gamma 7$**:** (4132) → (3142) → (4132)
$\gamma 8$**:** (4132) → (4213) → (4213) → (4321)

In Figure 7

$\xi 1$**:** (4213) → (4213) → (4321)
$\xi 2$**:** (2413) → (2413) → (2431) → (2413) → (2413) → (2431)
$\xi 3$**:** (2431) → (3241) → (3241) → (4321) → (4132) → (4213) → (4321)
$\xi 4$**:** (2431) → (2413) → (2413) → (2431)
$\xi 5$**:** (3241) → (3241) → (4321)
$\xi 6$**:** (3142) → (3142) → (4132) → (3142) → (3142) → (4132)
$\xi 7$**:** (4132) → (4213) → (4213) → (4321) → (2431) → (3241) → (4321)
$\xi 8$**:** (4132) → (3142) → (3142) → (4132)

References

[AC] Athreya, J. and Chaika, J.: *The Hausdorff dimension of Non-Uniquely Ergodic directions in $\mathcal{H}(2)$ is almost everywhere* $1/2$. to appear in Geom. Topol.

[G09] Gabai, D: *Almost filling laminations and the connectivity of ending lamination space.* Geom. Topol. 13 (2009), no. 2, 1017–1041

[G11] Gabai, D: *On the topology of ending lamination space.* arXiv preprint arXiv:1105.3648 (2011).

[HP11] Hensel, S; Przytycki, P: *The ending lamination space of the five-punctured sphere is the Nöbeling curve.* J. Lond. Math. Soc. (2) 84 (2011), no. 1, 103–119.

[KS67] Katok, A. B; Stepin, A. M: *Approximations in ergodic theory.* Uspehi Mat. Nauk 22 1967 no. 5 (137), 81–106.

[K75] Keane, M: *Interval exchange transformations.* Math. Z. 141 (1975), 25–31.

[Ker85] Kerckhoff, S. P: *Simplicial systems for interval exchange maps and measured foliations.* Ergodic Theory Dynam. Systems 5 (1985), no. 2, 257–271.

[LS09] Leininger, C; Schleimer, S: *Connectivity of the space of ending laminations.* Duke Math. J. 150 (2009), no. 3, 533–575.

[LS11] Leininger, C; Schleimer, S: *Hyperbolic spaces in Teichmüller spaces.* arXiv preprint arXiv:1110.6526 (2011).

[M82] Masur, H: *Interval exchange transformations and measured foliations.* Ann. of Math. (2) 115 (1982), no. 1, 169–200.

[MS91] Masur, H; Smillie, J: *Hausdorff Dimension of Sets of Nonergodic Measured Foliations*, Ann. of Math. (2) 134, No. 3 (1991), pp. 455-543

[V06] Viana, M: *Ergodic theory of interval exchange maps.* Rev. Mat. Complut. 19 (2006), no. 1, 7–100.

[V78] Veech, W: *Interval exchange transformations.* J. Analyse Math. 33 (1978), 222–272.

[V82] Veech, W: *Gauss measures for transformations on the space of interval exchange maps.* Ann. of Math. (2) 115 (1982), no. 1, 201–242.

[Y10] Yoccoz, J-C: Interval exchange maps and translation surfaces. Homogeneous flows, moduli spaces and arithmetic, 169, Clay Math. Proc., 10, Amer. Math. Soc., Providence, RI, 2010.

[Z06] Zorich, A. Flat Surfaces. *Frontiers in Number Theory, Physics and Geometry I* (2006): 439–585.

University of Utah, Department of Mathematics, 209, 155 S 1400 E RM 233, Salt Lake City, UT, 84112-0090 USA

E-mail address: chaika@math.utah.edu

The University of Chicago, Department of Mathematics, 5734 South University Avenue, Chicago, Illinois 60637 USA

E-mail address: hensel@math.uchicago.edu