ON THE EQUATION OF DEGREE NINE.

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The majority of the problems which I addressed in my lecture "Mathematical Problems"[1], and which belong to various areas of mathematics, have since been successfully addressed in manifold ways. In this note I want to return to some of these problems; in particular to those which require for their solution purely algebraic tools and methods, although the statements arise in other, non-algebraic disciplines.

A first class of such problems concerns finiteness questions of certain complete systems of functions – a question which we owe to the theory of algebraic invariants. The easiest problem of this sort seems to me to be the following.

Let a number m of rational integral functions $X_1, \ldots, X_m$ of a variable x be given; then clearly any rational integral combination of the $X_1, \ldots, X_m$ will after substitution of the expressions again be a rational integral function in x. Nevertheless, there may well be rational fractional combinations of the $X_1, \ldots, X_m$ which yield, after the substitution, also a rational integral function of x.

The problem now consists in showing that one can find a *finite* number of such fractional functions of $X_1, \ldots, X_m$ from which, together with the $X_1, \ldots, X_m$ themselves, every other such rational fractional function can be combined in a rational integral way[2].

This problem is an example which demonstrates that the situation, in which a question is very easy to state and yet presents considerable difficulty in answering, which is so common in arithmetic, can also occur in pure algebra.

A different class of algebraic problems of this sort can be found when one tries to algebraically realise topologically important curves, surfaces or other geometric objects, which might also be equipped with extra structures of the sort which current topological enquiries study[3].

A very simple example of this is the problem of projecting the projective plane bijectively and everywhere regularly into finite space. In three-dimensional space this is facilitated by the surface found by Boy in his dissertation – a surface which intersects itself in a space curve with a triple point. The question arises if in four-dimensional space there is a two-dimensional, singularity-free surface of such connection and *without* self-intersection.

This question has a positive answer in the simplest way by the observation that already a surface which is simply defined by quadratic functions allows the desired

[1]Given at the International Congress of Mathematicians at Paris: Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, sowie Archiv der Mathematik und Physik. 1900.

[2]Compare Problem 14 of the lecture cited above.

[3]This question should be seen as a generalisation of the 16th problem of my lecture

map to the projective plane; namely the surface defined by the following formulas:

$$\begin{aligned} x &= \eta\zeta, \\ y &= \zeta\xi, \\ z &= \xi\eta, \\ t &= \xi^2 - \eta^2, \\ \xi^2 + \eta^2 + \zeta^2 &= 1; \end{aligned}$$

here x, y, z, t denote the rectilinear coordinates of a four-dimensional space, and the parameters ξ, η, ζ the rectilinear coordinates in a three-dimensional space. The surface described by these formulas clearly is everywhere regular in four-dimensional space, and since a point ξ, η, ζ of the sphere in three-dimensional point defines the corresponding point x, y, z, t of the surface, as does its antipodal point $-\xi, -\eta, -\zeta$, the surface in the four-dimensional space does indeed have the connection of the projective plane. It therefore only remains to show that the surface considered does not have self-intersections, i.e. that to every point x, y, z, t of the surface corresponds only *one* pair of value triples ξ, η, ζ and $-\xi, -\eta, -\zeta$.

For this demonstration we observe that due to the formula

$$t^2 + 4z^2 = (\xi^2 + \eta^2)^2$$

t and z uniquely determine $\xi^2 + \eta^2$, and since t itself is $\xi^2 - \eta^2$, from t and z the values of ξ^2 and η^2 are uniquely determined, say $= p$ and q respectively.

Assuming first that $p > 0$, we obtain using the equations

$$z = \xi\eta \quad \text{and} \quad y = \zeta\xi$$

for ξ, η, ζ the triples

$$\xi = \sqrt{p}, \quad \eta = \frac{z}{\sqrt{p}}, \quad \zeta = \frac{y}{\sqrt{p}}$$

and

$$\xi = -\sqrt{p}, \quad \eta = -\frac{z}{\sqrt{p}}, \quad \zeta = -\frac{y}{\sqrt{p}}$$

and therefore the claim is proved. If on the other hand $p = 0$ then we consider the value q for η^2. Is $q > 0$, we obtain using the equation

$$x = \eta\zeta$$

for ξ, η, ζ the triples

$$\xi = 0, \quad \eta = \sqrt{q}, \quad \zeta = \frac{x}{\sqrt{q}}$$

and

$$\xi = 0, \quad \eta = -\sqrt{q}, \quad \zeta = -\frac{x}{\sqrt{q}}$$

which again corresponds to the claim. If finally we also have $q = 0$ then we obtain from the equation

$$\xi^2 + \eta^2 + \zeta^2 = 1$$

for ξ, η, ζ only the triples

$$\xi = 0, \quad \eta = 0, \quad \zeta = 1$$

and

$$\xi = 0, \quad \eta = 0, \quad \zeta = -1$$

and we observe that the claim is always true.

The surface constructed by Boy which we mentioned above has recently been presented in a very beautiful and intuitive way by F. Schilling; trying to represent this surface algebraically in the simplest possible way belongs to the realm of our problems.

A third class of problems of the described sort arises from nomography[4]. This discipline suggests to characterise if functions of an arbitrary number of arguments can be combined from functions of a certain fixed number of arguments.

From functions of a single argument we obtain by composition again only always functions of a single argument. If we want to obtain functions of several variables, then it is required to adjoin to the domain of functions of one argument at least one function of two variables. We choose to this end the sum $u+v$ of the variables u, v and immediately observe that the other three basic operations: subtraction, multiplication and division now also belong to the domain of the executable operations – after all, they can be combined from functions of *one* argument and sum in the following way;

$$\begin{aligned} u-v &= u+(-v), \\ u\cdot v &= \frac{1}{4}\{(u+v)^2-(u-v)^2\}, \\ \frac{u}{v} &= u\cdot\frac{1}{v}. \end{aligned}$$

The first question that arises is if apart from the sum there even are any other analytic functions which are essentially of two arguments, i.e. analytic functions which cannot be combined from functions of a single argument and the sum. The proof of the existence of such functions can be given in different ways. The most far-reaching results in this direction are due to A. Ostrowski[5]. His results imply that in particular the function of the two variables u, v

$$\zeta(u,v) = \sum_{n=1,2,3,\ldots} \frac{u^n}{n^v}$$

is one that cannot be combined from analytic functions of one argument and algebraic functions of an arbitrary number of arguments.

Another question concerns the existence of *algebraic* functions of this sort, i.e. the question if there is an algebraic function which cannot be obtained from functions of one argument and the sum.

In this direction the method of Tschirnhausen transformations gives some important insights. The method is as follows.

Let the equation of degree n

$$x^n + u_1x^{n-1} + u_2x^{n-2} + \cdots + u_n = 0$$

be given. To write the root x as a function of the n variables $u_1, u_2, \ldots, u_n$, we use the ansatz with indetermined coefficients

$$X = x^{n-1} + t_1x^{n-2} + \cdots + t_{n-1}$$

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[4] compare Problem 13 of my lecture

[5] Compare his article: "Über Dirichletsche Reihen und algebraische Differentialgleichungen", Math. Zeitschrift 8, S. 241

and derive the equation for X; which is

$$X^n + U_1 X^{n-1} + \cdots + U_n = 0$$

where in general

$$U_h = U_h(t_1, \ldots, t_{n-1})$$

is a function of degree h in $t_1, \ldots, t_{n-1}$. If one then determines the parameters $t_1, \ldots, t_{n-1}$ in such a way that

$$U_1 = 0, \quad U_2 = 0, \quad U_3 = 0$$

holds, then our original equation can be modified to have the form

$$X^n + U_4 X^{n-4} + U_5 X^{n-5} + \ldots + U_n = 0$$

using only rational processes and roots. Finally, putting

$$X = \sqrt[n]{U_n} Y,$$

we obtain for the new variable Y an equation of degree n, in which not only the coefficients of $Y^{n-1}, Y^{n-2}, Y^{n-3}$ have become zero, but also the first and last coefficient have become 1.

Therefore the equations of degrees five to nine take the following normal forms

$$\begin{aligned}
&x^5 + ux + 1 = 0,\\
&x^6 + ux^2 + vx + 1 = 0,\\
&x^7 + ux^3 + vx^2 + wx + 1 = 0,\\
&x^8 + ux^4 + vx^3 + wx^2 + px + 1 = 0,\\
&x^9 + ux^5 + vx^4 + wx^3 + px^2 + qx + 1 = 0.
\end{aligned}$$

Since the operation of taking a root is also a function of a single variable, we observe immediately that obtaining these normal forms requires only functions of a single argument and the sum. Concerning our question about algebraic functions essentially of two arguments, the equation of degree five can not yield such an example; since the normal form above of degree five contains only a single parameter u, obviously also the general equation of degree five can be solved using functions of a single argument and the sum.

For the normal form of degree six

$$x^6 + ux^2 + vx + 1 = 0$$

it seems that attempts to solve it using only functions of a single argument and the sum fail; it therefore stands to reason to conjecture that the root of this equation of degree six yields a function of the desired type.

Concerning the equation of degree seven

$$x^7 + ux^3 + vx^2 + wx + 1 = 0$$

I have already conjectured during the lecture mentioned in the beginning of this note that it is not even solvable using arbitrary continuous functions of two arguments; this conjecture also still requires proof.

Similarly, the root of the equation of degree eight can probably not be combined from functions of three arguments, rather it seems that the normal form of degree eight given above is a function essentially of the four arguments u, v, w, p.

In light of this, it seems even more remarkable to me that due to a combination of several factors, *the general equation of degree nine can also be resolved using only functions of four arguments*, because the five coefficients u, v, w, p, q of the normal form above allow a reduction to four variables and therefore are not essential variables in our sense.

To prove this, we apply the method of Tschirnhausen transformation and arrive at an equation for X of the form

$$X^9 + U_1 X^8 + U_2 X^7 + \ldots + U_9 = 0,$$

where $U_1, U_2, \ldots, U_9$ are integral rational functions of $t_1, \ldots, t_8$. We now express the parameter t_8 in terms of the other parameters $t_1, \ldots, t_7$ using the equation

$$U_1(t_1, \ldots, t_8) = 0$$

which is linear in $t_1, \ldots, t_8$. Substituting the resulting value of t_8 into U_2, U_3, U_4 yields expressions U_2', U_3', U_4', which are of degree two, three and four in $t_1, \ldots, t_7$, respectively.

By then writing U_2' as the sum of eight squares of linear functions $L_1, \ldots, L_8$ of the parameters $t_1, \ldots, t_7$ in the following way

$$U_2'(t_1, \ldots, t_7) = L_1^2 + L_2^2 + \ldots + L_8^2,$$

we observe that the equation

$$U_2'(t_1, \ldots, t_7) = 0$$

can be solved using the ansatz

$$\begin{aligned} L_1 + iL_2 &= 0, \\ L_3 + iL_4 &= 0, \\ L_5 + iL_6 &= 0, \\ L_7 + iL_8 &= 0. \end{aligned}$$

These are four linear equations of the parameters $t_1, \ldots, t_7$. Using these, we express t_4, t_5, t_6, t_7 in terms of t_1, t_2, t_3 and substitute the thus obtained values, which are linear in t_1, t_2, t_3, into U_3', U_4': obtaining expressions

$$U_3''(t_1, t_2, t_3) \quad \text{and} \quad U_4''(t_1, t_2, t_3)$$

which are cubic, respectively bi-quadratic, in t_1, t_2, t_3. Now it remains to determine t_1, t_2, t_3 in such a way that these two expressions vanish as well.

To this end, we observe that the equation

$$U_3''(t_1, t_2, t_3) = 0$$

determines a cubic two-dimensional surface in the three-dimensional $t_1 t_2 t_3$–space. On such a surface there are 27 straight lines. To find one of them, one needs to solve a question of degree 27, whose coefficients depend rationally on the coefficients of $U_3'' = 0$.

We want to study how to reduce the number of coefficients of the equation $U_3'' = 0$. As is well known, an integral rational function of degree three in three variables can always be written as a sum of five cubes as follows:

$$U_3''(t_1, t_2, t_3) = M_1^3 + M_2^3 + M_3^3 + M_4^3 + M_5^3,$$

where the M_1, M_2, M_3, M_4, M_5 are linear functions of t_1, t_2, t_3. This description is essentially unique: the cubes $M_1^3, M_2^3, M_3^3, M_4^3, M_5^3$ are roots of an equation of degree five, whose coefficients can be expressed rationally in the coefficients of the cubic function U_3''.

From this we observe that to describe the cubes $M_1^3, M_2^3, M_3^3, M_4^3, M_5^3$, and thus also the linear functions M_1, M_2, M_3, M_4, M_5 themselves, we require apart from the sum only functions of *one* argument. If we now introduce, instead of t_1, t_2, t_3 the linear fractional values

$$m_1 = \frac{M_1}{M_4}, \quad m_2 = \frac{M_2}{M_4}, \quad m_3 = \frac{M_3}{M_4},$$

as new variables, the equation $U_3'' = 0$ becomes an equation of the form

$$m_1^3 + m_2^3 + m_3^3 + 1 + (V_1 m_1 + V_2 m_2 + V_3 m_3 + V_4)^3 = 0,$$

which only contains the four parameters V_1, V_2, V_3, V_4 in its coefficients. This implies that, if the equations

$$\begin{aligned} t_1 &= \rho_1 s + \sigma_1, \\ t_2 &= \rho_1 s + \sigma_1, \\ t_3 &= \rho_3 s + \sigma_3 \end{aligned}$$

with the variable s describe one of the 27 straight lines of our surface $U_3'' = 0$, then the coefficients $\rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2, \sigma_3$ are also algebraic functions of the four parameters V_1, V_2, V_3, V_4.

Substituting finally also in U_4'' instead of t_1, t_2, t_3 the linear functions of s given above, then the equation

$$U_4''(t_1, t_2, t_3) = 0$$

will become the bi-quadratic equation

$$U_4'''(s) = 0$$

of s, and solving this again requires apart of the sum only functions of one argument.

The Tschirnhausen transformation we found now yields, if we make the corresponding substitutions in U_5, U_6, U_7, U_8, U_9, instead of the original equation of degree nine an equation of the form

$$X^9 + U_5^* X^4 + U_6^* X^3 + U_7^* X^2 + U_8^* X + U_9^* = 0$$

and, applying the substitution

$$X = \sqrt[9]{U_9^*} Y,$$

we arrive at an equation of the form

$$Y^9 + W_1 Y^4 + W_2 Y^3 + W_3 Y^2 + W_4 Y + 1 = 0,$$

in which only the four parameters W_1, W_2, W_3, W_4 appear. *The solution of the general equation of degree nine therefore requires only algebraic functions of four arguments*, and in fact functions of a single argument, the sum and two more special algebraic functions of four arguments are sufficient. It is improbable that for the general equation of degree nine the number of arguments can be reduced even further.

Obviously, for equations of higher degree there is a corresponding reduction in the number of arguments.