Abstract. We show that the Dehn function of the handlebody group is exponential in any genus $g \geq 3$. On the other hand, we show that the handlebody group of genus 2 is cubical, biautomatic, and therefore has a quadratic Dehn function.

1. Introduction

This article is concerned with the word geometry of the handlebody group $H_g$, i.e. the mapping class group of a handlebody of genus $g$. The core motivation to study this group is twofold. On the one hand, handlebodies are basic building blocks for three-manifolds – namely, for any closed 3–manifold $M$ there is a $g$ so that $M$ can be obtained by gluing two genus $g$ handlebodies $V, V'$ along their boundaries with a homeomorphism $\varphi$. Any topological property of $M$ is then determined by the gluing map $\varphi$. One of the difficulties in extracting this information is that $\varphi$ is by no means unique. In fact, modifying it on either side by a homeomorphism which extends to $V$ or $V'$ does not change $M$. In this sense, the handlebody group encodes part of the non-uniqueness of the description of a 3–manifold via a Heegaard splitting.

The other motivation stems from geometric group theory, and it is the more pertinent for the current work. Identify the boundary surface of $V_g$ with a surface $\Sigma_g$ of genus $g$. Then there is a restriction homomorphism of $H_g$ into the surface mapping class group $\text{Mcg}(\Sigma_g)$, and it is not hard to see that it is injective. On the other hand, considering the action of homeomorphisms of $V_g$ on the fundamental group $\pi_1(V_g) = F_g$ gives rise to a surjection of $H_g$ onto $\text{Out}(F_g)$. The handlebody group is thus immediately related to two of the most studied groups in geometric group theory.

But from a geometric perspective, neither of these relations is simple: in previous work [HH1] we showed that the inclusion of $H_g$ into $\text{Mcg}(\Sigma_g)$ is exponentially distorted for any genus $g \geq 2$. Furthermore, a result by McCullough shows that the kernel of the surjection $H_g \to \text{Out}(F_g)$ is infinitely generated.

In particular, there is no a priori reason to expect that $H_g$ shares geometric properties with either surface mapping class groups or outer automorphism groups of free groups.
The first main result of this paper shows that the geometry of $H_g$ for $g \geq 3$ seems to share geometric features with the (outer) automorphism group of a free group. In this result, we slightly extend our perspective and also consider handlebodies $V_{g,1}$ of genus $g$ with a marked point and their handlebody group $H_{g,1}$. We show.

**Theorem 1.1.** The Dehn function of $H_g$ and $H_{g,1}$ is exponential for any $g \geq 3$.

The Dehn function of a group is a combinatorial isoperimetric function, and it is a geometric measure for the difficulty of the word problem (see Section 2.2 for details and a formal definition). Theorem 1.1 should be contrasted with the situation in the surface mapping class group – by a theorem of Mosher [Mos], these groups are automatic and therefore have quadratic Dehn functions. On the other hand, Bridson and Vogtmann showed that $\text{Out}(F_g)$ has exponential Dehn function for $g \geq 3$ [BV1, BV2, HV].

Mapping class groups of small complexity are known to have properties not shared with properties of mapping class groups of higher complexity. For example, the mapping class group $\text{MCG}(\Sigma_2)$ of a surface of genus is a $\mathbb{Z}/2\mathbb{Z}$-extension of the mapping class group of a sphere with 6 punctures. This implies among others that the group virtually surjects onto $\mathbb{Z}$, a property which is not known for higher genus. On the other hand, the group $\text{Out}(F_2)$ is just the full linear group $GL(2,\mathbb{Z})$. Similarly, it is known that the genus 2 handlebody group surjects onto $\mathbb{Z}$ as well [IS].

Our second goal is to add to these results by showing that the handlebody group $H_2$ has properties not shared by or unknown for handlebody groups of higher genus.

**Theorem 1.2.** The group $H_2$ admits a proper cocompact action on a CAT(0) cube complex.

As an immediate corollary, using Corollary 8.1 of [Š] and Proposition 1 of [CMV]), we obtain among others that the genus bound in Theorem 1.1 is optimal.

**Corollary.** The group $H_2$ is biautomatic, in particular it has quadratic Dehn function, and it has the Haagerup property.

In order to prove Theorem 1.1, we recall from previous work [HH2] that the Dehn function of handlebody groups is at most exponential, and it therefore suffices to exhibit a family of cycles which requires exponential area to fill (compared to their lengths). These cycles will be lifted from cycles in automorphism groups of free groups used by Bridson and Vogtmann [BV1]. This construction occupies Section 3.

The proof of Theorem 1.2 is more involved, and relies on constructing and studying a suitable geometric model for the genus 2 handlebody group. In Section 4 we describe in detail the intersection pattern of disk-bounding curves in a genus 2 handlebody. The model of $H_2$ is built in two steps: in...
Section 5 we construct a tree on which $\mathcal{H}_2$ acts and in Section 6 we then use this tree to build our cubical model for $\mathcal{H}_2$ and prove Theorem 1.2. We also discuss some additional geometric consequences.

2. Preliminaries

2.1. Handlebody Groups. Let $V$ be a handlebody of genus $g \geq 2$. We identify the boundary of $V$ once and for all with a surface $\Sigma$ of genus $g$. Restrictions of homeomorphisms of $V$ to the boundary then induce a restriction map

$$r : \mathrm{MCG}(V) \to \mathrm{MCG}(\Sigma).$$

It is well-known that this map is injective, and we call its image the handlebody group $\mathcal{H}_g$.

When we consider a handlebody with a marked point $p$, we will always assume that the marked point is contained in the boundary. We then get a map

$$r : \mathrm{MCG}(V, p) \to \mathrm{MCG}(\Sigma, p)$$

whose image is the handlebody group $\mathcal{H}_{g,1}$.

We also need some special elements of the handlebody group. We denote by $T_\alpha \in \mathrm{MCG}(\Sigma)$ the positive (or left) Dehn twist about $\alpha$ (compare [FM, Chapter 3]). Dehn twists $T_\alpha$ are elements of the handlebody group exactly if the curve $\alpha$ is a meridian, i.e. a curve which is the boundary of an embedded disk $D \subset V$.

We have the following standard lemma which gives rise to another important class of elements.

**Lemma 2.1.** Suppose that $\alpha, \beta, \delta$ are three disjoint simple closed curves on $\Sigma$ which bound a pair of pants on $\Sigma$. Suppose that $\delta$ is a meridian. Then the product

$$T_\alpha T_\beta^{-1}$$

is an element of the handlebody group.

**Proof.** Let $P$ be the pair of pants with $\partial P = \alpha \cup \beta \cup \delta$, and let $D$ be the disk bounded by $\delta$. Then $P \cup D$ is an embedded annulus in $V$ whose two boundary curves are $\alpha$ and $\beta$. By applying a small isotopy we may then assume that there is a properly embedded annulus $A$ whose boundary curves are $\alpha, \beta$.

Consider the homeomorphism $F$ of $V$ which is a twist about $A$. To be more precise, consider a regular neighborhood $U$ of $A$ of the form

$$U = [0, 1] \times A = [0, 1] \times S^1 \times [0, 1].$$

The homeomorphism $F$ is defined to be the standard Dehn twist on each annular slice $[0, 1] \times S^1 \times \{t\} \subset U$. This map restricts to the identity on $\{0, 1\} \times A$, and thus extends to a homeomorphism of $V$. It restricts on the boundary of $V$ to the desired element, finishing the proof. \qed
The final type of elements we need are point-push maps. Recall (e.g. from [FM]) the Birman exact sequence

$$1 \to \pi_1(\Sigma, p) \to \text{Mcg}(\Sigma, p) \to \text{Mcg}(\Sigma) \to 1.$$ 

The image of $\pi_1(\Sigma, p) \to \text{Mcg}(\Sigma, p)$ is the point pushing subgroup. We need three facts about these mapping classes, all of which are well-known, and are fairly immediate from the definition.

**Lemma 2.2.**

i) The point-pushing subgroup is contained in $H_{g,1}$ (compare [HH1, Section 3]).

ii) If $\gamma \in \pi_1(\Sigma, p)$ is simple, then the point push about $\gamma$ is a product $T_\alpha T_\beta^{-1}$ of two Dehn twists, where $\alpha, \beta$ are the two simple closed curves obtained by pushing $\gamma$ off itself to the left and right, respectively (compare [FM, Fact 4.7]).

iii) The point push about $\gamma$ acts on $\pi_1(\Sigma, p)$ as conjugation by $\gamma$. Similarly, it acts on $\pi_1(V, p)$ as conjugation by the image of $\gamma$ in $\pi_1(V, p)$ (compare [FM, Discussion in Section 4.2.1]).

**2.2. Dehn functions.** Consider a finitely presented group $G$ with a fixed finite presentation $\langle S \mid R \rangle$. A word $w$ in $S$ (or, alternatively, an element of the free group $F(S)$ on the set $S$) is trivial in $G$ exactly if $w$ can be written (in $F(S)$) as a product

$$w = \prod_{i=1}^{n} x_i r_i x_i^{-1}$$

for elements $r_i \in R$ and $x_i \in F(S)$. We define the area of $w$ as the minimal $n$ for which such a description is possible. The Dehn function is the function

$$D(n) = \text{sup}\{\text{area}(w) \mid l(w) = n\}$$

where $l(w)$ denotes the length of the word $w$ (alternatively, the word norm in $F(S)$).

The Dehn function depends on the choice of the presentation, but its growth type does not (see e.g. [Alo]). We employ the convention that products in mapping class groups are compositions (i.e. the rightmost mapping classes are applied first).

**2.3. Annular subsurface projections.** In this subsection we briefly recall subsurface projections into annular regions, as defined in [MM], Section 2.4.

Let $A = S^1 \times [0,1]$ be a closed annulus. Recall that the arc graph $\mathcal{A}(A)$ of the annulus $A$ is the graph whose vertices correspond to embedded arcs which connect the two boundary circles $S^1 \times \{0\}, S^1 \times \{1\}$, up to homotopy fixing the endpoints. Two such vertices are joined by an edge if the corresponding arcs are disjoint except possibly at the endpoints (up to homotopy fixing the endpoints). It is shown in Section 2.4 of [MM] that the resulting graph is quasi-isometric to the integers.

Now consider a surface $\Sigma$ of genus at least two. Fix once and for all a hyperbolic metric on $\Sigma$. If $\alpha$ is any simple closed curve on $\Sigma$, let $\Sigma_\alpha \to \Sigma$
be the annular cover corresponding to $\alpha$, i.e. the cover homeomorphic to an (open) annulus to which $\alpha$ lifts with degree 1. By pulling back the hyperbolic metric from $\Sigma$ to $\Sigma_\alpha$, we obtain a hyperbolic metric on $\Sigma_\alpha$. This allows us to add two boundary circles at infinity which compactify $\Sigma_\alpha$ to a closed annulus $\hat{\Sigma}_\alpha$.

If $\beta$ is a simple closed curve on $\Sigma$, then any lift $\hat{\beta}$ of $\beta$ to $\Sigma_\alpha$ has well-defined endpoints at infinity (for example, since this is true for lifts to the universal cover). In addition, if $\beta$ has an essential intersection with $\alpha$, there is at least one lift $\hat{\beta}$ of $\beta$ to $\Sigma_\alpha$ which connects the two boundary circles of $\Sigma_\alpha$. Such a lift $\hat{\beta}$ has well-defined endpoints at infinity in $\hat{\Sigma}_\alpha$, and so it defines a vertex in $A(\hat{\Sigma}_\alpha)$. We define the projection $\pi_\alpha(\beta) \subset A(\hat{\Sigma}_\alpha)$ to be the set of all such lifts. Since $\beta$ is simple, this is a (finite) subset of diameter one.

Observe that if $\beta'$ is freely homotopic to $\beta$, then any lift of $\beta'$ is homotopic to a lift of $\beta$ with the same endpoints at infinity. Hence, the projection $\pi_\alpha(\beta)$ depends only on the free homotopy class of $\beta$.

If $\beta$ is disjoint from $\alpha$, the projection $\pi_\alpha(\beta)$ is undefined.

If $\beta_1, \beta_2$ are two simple closed curves which both intersect $\alpha$ essentially, then we define the subsurface distance

$$d_\alpha(\beta_1, \beta_2) = \text{diam}(\pi_\alpha(\beta_1) \cup \pi_\alpha(\beta_2)).$$

3. **Exponential Dehn functions in genus at least 3**

**Theorem 3.1.** Let $g \geq 3$. The Dehn function of $\mathcal{H}_g$ and $\mathcal{H}_{g,1}$ is at least exponential.

The core ingredient in the proof is the natural map

$$\mathcal{H}_{g,1} \to \text{Aut}(F_g)$$

induced by the action of homeomorphisms of the handlebody $V_g$ on the fundamental group $\pi_1(V_g) = F_g$. Our strategy will be to take a sequence of trivial words $w_n$ in $\text{Aut}(F_g)$ which have exponentially growing area and lift them into the handlebody group.

We will spend most of this section with discussing the case of $\mathcal{H}_{3,1}$ in detail; the other cases will be derived from this special case at the end of the section.

In [BV1] the following three automorphisms of $F_3$ are considered. Let $a, b, c$ be a free basis of $F_3$. Then define automorphisms

$$A : \begin{cases} 
    a &\mapsto a \\
    b &\mapsto b \\
    c &\mapsto ac
\end{cases}, \quad B : \begin{cases} 
    a &\mapsto a \\
    b &\mapsto b \\
    c &\mapsto cb
\end{cases}, \quad T : \begin{cases} 
    a &\mapsto a^2b \\
    b &\mapsto ab \\
    c &\mapsto c
\end{cases}$$

Observe that $B$ and $T^n AT^{-n}$ commute for all $n$, and therefore we have the following equation

$$T^n AT^{-n} BT^n A^{-1} T^{-n} B^{-1} = \text{id}$$
The crucial result we need is the following, which is proved in [BV1, Theorem A].

**Theorem 3.2** (Bridson-Vogtmann). Consider any presentation of $\text{Aut}(F_3)$ or $\text{Out}(F_3)$ whose generating set contains the automorphisms $A, B, T$. Then, if $f : \mathbb{N} \to \mathbb{N}$ is any sub-exponential function, the loops defined by the words

$$w_n = T^n A T^{-n} B T^n A^{-1} T^{-n} B^{-1}$$

cannot be filled with area less than $f(n)$ for large $n$.

In particular, since the words $w_n$ have length growing linearly in $n$, the theorem immediately implies that $\text{Aut}(F_3)$ and $\text{Out}(F_3)$ have exponential Dehn function.

To show Theorem 3.1, we will realize $A, B, T$ in a specific way as homeomorphisms of a genus 3 handlebody. This construction will be performed in several steps.

**Constructing the handlebody.** The first step is to give a specific construction of a genus 3 handlebody $V_3$ that will be particularly useful to us. We construct $V_3$ by attaching a single handle $H$ to an interval-bundle $V_2$ over a torus $S$ with one boundary component (which is a genus 2 handlebody).

To be more precise, denote by $S$ a surface of genus 1 with one boundary component. We pick a basepoint $p \in \partial S$. We define

$$V_2 = S \times [0, 1].$$

This is a handlebody of genus 2, and its boundary $\partial V_2$ has the form

$$\partial V_2 = (S \times \{0\}) \cup (\partial S \times [0, 1]) \cup (S \times \{1\}).$$

In other words, the boundary consists of two tori $S_i = S \times \{i\}, i = 0, 1$ and an annulus $A = \partial S \times [0, 1]$. We employ the convention that a subscript 0 or 1 attached to any object in $S$ denotes its image in $S \times \{0, 1\}$. For example, $p_0$ will denote the point $p \times \{0\}$.

Next, we want to attach a handle in $A$ to form the genus 3 handlebody. To this end, choose two disjoint embedded disks $D^-, D^+$ in the interior of $A$ which are disjoint from $p \times [0, 1]$. Gluing $D^-$ to $D^+$ (or, alternatively, attaching a 1-handle at these disks) yields our genus 3 handlebody $V_3$. We denote by $D$ the image of the disks $D^+, D^-$ in $V_3$.

Finally we will construct a core graph in $V_3$ in a way that is compatible with our construction. Begin by choosing two loops $a, b \subset S$ which intersect only in $p$, and which define a free basis of $\pi_1(S, p) = F_2$. Furthermore, choose points $q^-, q^+ \in \partial D^-, \partial D^+$ which are identified with each other in forming $V_3$. Then choose embedded arcs $c^+, c^- \subset A$ from $p_1$ to $q^+, q^-$ which only intersect in $p_1$. We denote by $c$ the loop in $V_3$ formed by traversing $c^+$ from $p_1$ to $q^+$, then $c^-$ from $q^-$ back to $p_1$.

Then the union

$$\Gamma = a_1 \cup b_1 \cup c$$
is an embedded three-petal rose in $\partial V_3$, so that the inclusion $\Gamma \to V_3$ induces an isomorphism on fundamental groups (recall that $a_1 = a \times \{1\}$ and similar for $b_1$). By slight abuse of notation, we will denote the images of the three petals in $\pi_1(V_3)$ by $a, b, c$, and note that they form a free basis.

**Realizing $T$ as a bundle map.** Recall that the mapping class group of a torus $S$ with one boundary component surjects to $\text{Aut}(F_2)$ [FM, Section 2.2.4 and Proposition 3.19], and therefore there is a homeomorphism $t$ of $S$ which restricts to the identity on $\partial S$, and so that the induced map $t : \pi_1(S, p) \to \pi_1(S, p)$ acts on the basis defined by the loops $a, b$ as follows:

$$t_*(a) = a^2b, \quad t_*(b) = ab.$$ 

The homeomorphism $t \times \text{Id}$ of $V_2$ preserves $S_0, S_1$ setwise and restricts to the identity on $A$. By the latter fact, the homeomorphism $t \times \text{Id}$ then defines a homeomorphism $\tau$ of $V_3$, which is the identity on $A$, and in particular fixed $c$ pointwise.

We summarize the important properties of $\tau$ in the following lemma.

**Lemma 3.3.** $\tau$ is a homeomorphism of $V_3$ fixing the marked point $p_1$ with the following properties:

i) The support of $\tau$ restricted to the boundary $\partial V_3$ is $S_0 \cup S_1$, and it preserves both subsurfaces $S_i$ set-wise.

ii) $\tau$ acts on $\pi_1(V_3, p_1)$ in the basis $a, b, c$ as the automorphism $T$:

$$\tau_*(a) = a^2b, \quad \tau_*(b) = ab, \quad \tau_*(c) = c.$$ 

**Realizing $A$ by a handleslide.** Intuitively, $\alpha$ will slide the end $D^+$ of the handle $H$ around the loop $a_0$ in the “bottom surface” $S_0$ of the interval bundle $V_2 \subset V_3$.

To be precise, let $z$ be an arc which joins $D^+$ to $p \times \{0\}$ inside $A$, and is disjoint from $c$. Consider a small regular neighbourhood of $\partial D^+ \cup z \cup a_0$
in $\partial V_3$. Its boundary consists of three simple closed curves, one of which is homotopic to $\partial D$, and the two others we denote by $a', a''$. One of them, say $a'$, is contained in $S_0$ and will intersect $a_0 \cup b_0$ in a single point (necessarily of $b_0$). The other curve $a''$ is disjoint from $S_1$, and intersects $A$ in a single arc. Note further that $a', a''$ and $\partial D$ bound a pair of pants in $\partial V_3$.

Let $\alpha$ be a homeomorphism of $\partial V_3$ which defines the product $T_{a''}T_{a'}^{-1}$ of Dehn twists in the mapping class group of $\partial V_3$ and is supported in a small regular neighbourhood of $a' \cup a''$. It extends to a homeomorphism of the handlebody $V_3$ by Lemma 2.1, and we will denote this extension by the same symbol.

Next, we compute the action of $\alpha$ on the fundamental group of $V_3$. We will do this using the core graph $\Gamma = a_1 \cup b_1 \cup c$ defined above. Since $a', a''$ are disjoint from $S_1$, we have that $\alpha(a_1) = a_1, \alpha(b_1) = b_1$ for $i = 1, 2$. Since $a' \subset S_0$, we see that $c$ is disjoint from $a'$. Finally, $c$ intersects $a''$ in a single point $q$. Thus, $\alpha(c)$ is a loop, which is formed by following $c$ until the intersection point $q$, traversing $a''$ once, and then continuing along $c$. Observe that by pushing $a''$ first through $D$, and then to the “top” half $S_1$ of the interval bundle, this loop $\alpha(c)$ is therefore homotopic in $V_3$ (relative to $p_1$) to the concatenation of $a_1$ and $c$. We thus have the following properties of $\alpha$:

**Lemma 3.4.** $\alpha$ is a homeomorphism of $V$ fixing $p_1$ with the following properties:

i) The restriction of $\alpha$ to $\partial V$ is supported in a small neighbourhood of $a', a''$, where $a' \subset S_0$, and $a''$ is disjoint from $S_1$ and intersects $A$ in a single arc.

ii) $\alpha$ acts on $\pi_1(V_3, p_1)$ as the automorphism $A$:

$$\alpha_*(a) = a, \quad \alpha_*(b) = b, \quad \alpha_*(c) = ac.$$  

**Realizing $B$ by a handleslide and a point push.** We will realise $B$ similar to $A$, by pushing $D^-$ along the loop $b$ of the “top” side of the interval bundle.

To do this, we first construct an auxiliary homeomorphism $\hat{\beta}$ of $V_3$ analogous to the previous step. Consider a regular neighbourhood of $\partial D \cup c^- \cup b_1$. Its boundary again consists of three curves; one of which is homotopic to $\partial D$, and we denote the others by $b', b''$. Let $b'$ be the one which is completely contained in $S_1$ (and thus freely homotopic to $b_1$).

As above, we can choose a homeomorphism $\hat{\beta}$ which is supported in a small neighbourhood of $b' \cup b''$ and defines the mapping class $T_{b''}T_{b'}^{-1}$. By Lemma 2.1 and the fact that $b', b''$ and $\partial D$ bound a pair of pants, it extends to $V_3$ and we denote the extension by the same symbol.

We now compute the effect of $\hat{\beta}$ on the core graph $\Gamma$. We begin with the petal $a_1$. It intersects both $b'$ and $b''$ in one point each. Hence, we have that
\( \hat{\beta}(a_1) \) is homotopic on \( \partial V_3 \), relative to the basepoint \( p_1 \), to the concatenation \( b_1 \ast a_1 \ast b_1^{-1} \). The loop \( b_1 \) is, by construction, disjoint from \( b', b'' \) and so we have \( \hat{\beta}(b_1) = b_1 \). Finally, \( c_+ \) intersects \( b'' \) in a single point, and is disjoint from \( b' \), while \( c_- \) is disjoint from both \( b', b'' \). Thus, \( \hat{\beta}(c) \) is homotopic on \( \partial V_3 \) to the concatenation \( b_1 \ast c \). In total, we see that \( \hat{\beta} \) acts on our chosen basis of \( \pi_1(V_3, p_1) \) as follows:

\[
\hat{\beta}_*(a) = bab^{-1}, \quad \hat{\beta}_*(b) = b, \quad \hat{\beta}_*(c) = bc.
\]

To define the homeomorphism \( \beta \), we post-compose \( \hat{\beta} \) with a point push \( P \) around \( b_1^{-1} \) (which is an element of the handlebody group by Lemma 2.2 i)). Since this point push has the effect on the level of fundamental group of conjugating by \( b_1^{-1} \) (Lemma 2.2 iii)), we see that therefore \( \beta \) will indeed realize the automorphism \( B \) as desired.

By Lemma 2.2 ii), the point push homeomorphism \( P \) can be chosen to be supported in the union of \( S_1 \) and a small neighborhood of \( p_1 \). In particular, we may assume that the support of the point push is disjoint from the arc \( a'' \cap A \) occurring in Lemma 3.4. We summarize the required properties of \( \beta \) in the following lemma.

**Lemma 3.5.** \( \beta \) is a homeomorphism of \( V \) fixing the marked point \( p \) with the following properties:

1. The restriction of \( \beta \) to \( \partial V \) is the product of four Dehn twists about curves \( d_i \subset S_0 \cup A \), all four of which are disjoint from the curves \( a', a'' \) occurring in Lemma 3.4.

2. \( \beta \) acts on \( \pi_1(V_3, p_1) \) as the automorphism \( B \):

\[
\beta_*(a) = a, \quad \beta_*(b) = b, \quad \beta_*(c) = cb.
\]

**Completing the proof.** Consider the homeomorphisms

\[
\tau^n \alpha \tau^{-n}
\]

of \( V_3 \). Their support is contained in a small regular neighbourhood of \( \tau^n (a'), \tau^n (a'') \). Recall that \( \tau \) preserves \( S_0 \) and hence \( \tau^n (a') \subset S_0 \). Thus \( \tau^n (a') \) is disjoint from all of the curves \( d_i \) from Lemma 3.5. Furthermore, \( \tau^n (a'') \cap A = a'' \cap A \) (since \( \tau \) restricts to the identity on \( A \)). Hence, since the \( d_i \) from Lemma 3.5 are disjoint from \( S_0 \) and intersect \( A \) in arcs which are disjoint from \( a'' \), we have that \( \tau^n (a'') \) is disjoint from all \( d_i \) for any \( n \).

As a consequence, the homeomorphisms \( \tau^n \alpha \tau^{-n} \) and \( \beta \) have (up to isotopy) disjoint supports, and therefore define commuting mapping classes in \( \mathcal{H}_{3,1} \). Therefore we conclude the following relation in \( \mathcal{H}_{3,1} \)

\[
1 = [\tau^n \alpha \tau^{-n}, \beta] = \tau^n \alpha \tau^{-n} \beta \tau^n \alpha^{-1} \tau^{-n} \beta^{-1},
\]

where we have denoted the mapping classes defined by the homeomorphisms \( \alpha, \beta, \tau \) by the same symbols. In other words, if we choose a generating set of \( \mathcal{H}_{3,1} \) which contains \( \alpha, \beta, \tau \) then

\[
\omega_n = \tau^n \alpha \tau^{-n} \beta \tau^n \alpha^{-1} \tau^{-n} \beta^{-1}
\]
are words in $H_{3,1}$ which define the trivial element. Furthermore, under the map $H_{3,1} \to \text{Aut}(F_3)$ they map exactly to the words $w_n$ occurring in Theorem 3.2. By the conclusion of that theorem, $w_n$ cannot be filled with sub-exponential area in $\text{Aut}(F_3)$. Since group homomorphisms coarsely decrease area, the same is therefore true for the words $\omega_n$. But, by construction, the length of the word $\omega_n$ grows linearly in $n$, showing that $H_{3,1}$ has at least exponential Dehn function. The same is true for $H_{3}$, since the words $w_n$ are also exponentially hard to fill in $\text{Out}(F_3)$ by Theorem 3.2.

To extend the proof of Theorem 3.1 to any genus $g \geq 3$, we argue as follows. Consider the handlebody $V$ of genus 3 constructed above. Take a connected sum of $V$ with a handlebody $V'$ of genus $g - 3$ at a disk $D_g$ in the annulus $A$, which is disjoint from all curves used to define $\alpha, \beta$, to obtain a handlebody $V_g$ of genus $g$. The homeomorphisms $\alpha, \beta, \tau$ can then be extended to homeomorphisms of $V_g$ which restrict to be the identity on $V'$. In this way the words $\omega_n$ define trivial words $\hat{\omega}_n$ in $H_{g,1}$.

There is a natural map

$$\iota : \text{Aut}(F_3) \to \text{Aut}(F_g)$$

which maps an automorphism $\varphi$ of $F(x_1, x_2, x_3)$ to its extension to the free group $\langle x_1, \ldots, x_g \rangle$ on $g$ generators which fixes all $x_i, i > 3$.

By construction, the words $\hat{\omega}_n$ map to the image of the words $w_n$ under $\iota$. Corollary 4 of [HM] (compare also [HH3]) shows that the image of $\iota$ is a Lipschitz retract of $\text{Aut}(F_g)$ and $\text{Out}(F_g)$, and therefore the words $\iota(w_n)$ are also exponentially hard to fill. By the same argument as above, the same is therefore true for the words $\hat{\omega}_n$.

4. Waves in genus 2

In this section we study intersection pattern between meridians in a genus two handlebody $V$. Recall that a cut system of a genus two handlebody is a pair $(\alpha_1, \alpha_2) \subset \partial V$ of disjoint meridians with connected complement. Equivalently, cut systems are the boundary curves of disjoint disks $D_1, D_2 \subset V$ so that $V - (D_1 \cup D_2)$ is a single 3–ball.

The next proposition (which is true in any genus) is well-known, see e.g. [Mas, Lemma 1.1] or [HH1, Lemma 5.2 and the discussion preceding it].

**Proposition 4.1.** Suppose that $(\alpha_1, \alpha_2)$ is a cut system and $\beta$ is an arbitrary (multi)meridian. Either $\alpha_1 \cup \alpha_2$ and $\beta$ are disjoint, or there is a subarc $b \subset \beta$, called a wave, with the following properties:

i) The arc $b$ intersects $\alpha_1 \cup \alpha_2$ only in its endpoints, and both endpoints lie on the same curve, say $\alpha_1$.

ii) The arc $b$ approaches $\alpha_1$ from the same side at both endpoints.

iii) Let $a, a'$ be the two components of $\alpha_1 \setminus b$. Then exactly one of

$$\langle a \cup b, \alpha_2 \rangle, \langle a' \cup b, \alpha_2 \rangle$$
is a cut system, which we call the surgery defined by the wave \( b \) in the direction of \( \beta \).

iv) The surgery defined by \( b \) has fewer intersections with \( \beta \) than \((\alpha_1, \alpha_2)\).

We say that a sequence \( (\alpha_i^{(n)})_n \) of cut systems is a surgery sequence in the direction of \( \beta \) if each \( (\alpha_i^{(n+1)}) \) is the surgery of \( (\alpha_i^{(n)}) \) defined by some wave \( b \) of \( \beta \). By Proposition 4.1, these exist for any initial cut system, and they end in a system which is disjoint from \( \beta \).

The following lemmas describe certain symmetry and uniqueness features of waves in genus 2. They are central ingredients in our study of projection maps in the next section. The results of these lemma are discussed and essentially proved in Section 4 of [Mas]. We include a proof for completeness and convenience of the reader.

To describe them, it is easier to take a slightly different point of view. Namely, let \((\alpha_1, \alpha_2)\) be a cut system of a genus 2 handlebody \( V \). Consider \( S = \partial V - (\alpha_1 \cup \alpha_2) \). This is a four-holed sphere, with boundary components \( \alpha_1^+, \alpha_1^-, \alpha_2^+, \alpha_2^- \) corresponding to the sides of the \( \alpha_i \). A wave \( b \) corresponds exactly to a subarc of \( \beta \) which joins one of the boundary components of \( S \) to itself.

At this point we want to emphasize that when considering arcs in \( S \), we always consider them up to homotopy which is allowed to move the endpoints (in \( \partial S \)).

**Lemma 4.2.** Let \( \beta \) be a meridian, and \( b \subset S \) be a wave of \( \beta \) joining a boundary component \( \alpha_i^+ \) to itself. Then \( b \) separates the two boundary components \( \alpha_{1-i}^+ \) and \( \alpha_{1-i}^- \).

**Proof.** Let \( b \) be such a wave, joining without loss of generality \( \alpha_1^+ \) to itself. Suppose that \( \alpha_2^- \) and \( \alpha_2^+ \) are contained in the same component of \( S - b \). Then \( b \) and a subarc \( a \subset \alpha_1 \) concatenate to a curve homotopic to \( \alpha_1^- \) on \( S \). Continue \( b \) beyond one of its endpoints across \( \alpha_1 \) to form a larger subarc of \( \beta \). It then exits from \( \alpha_1^- \) and, by minimal position, has its next intersection point with \( \alpha_1 \cup \alpha_2 \) in \( a \) again. This can be iterated, and leads to a contradiction as the curve \( \beta \) then cannot close up (compare Figure 2).

**Lemma 4.3.** Suppose \((\alpha_1, \alpha_2)\) is a cut system of \( V \), and \( \beta \) is any meridian. Suppose that \( \beta \) has a wave \( b^+ \) at \( \alpha_1^+ \).

i) There is a unique essential arc \( b^- \subset S \) with both endpoints on \( \alpha_i^- \) which is disjoint from \( b^+ \).

ii) The arc \( b^- \) from i) appears as a wave of \( \beta \).

iii) Any wave of \( \beta \) is homotopic to either \( b^+ \) or \( b^- \).

(The same is true with the roles of \( \alpha_i^-, \alpha_i^+ \) reversed)
Proof. Let $b^+$ be a wave as in the prerequisites, joining without loss of generality $\alpha_1^+$ to itself. Denote the complementary components of $b^+$ in $S$ by $C_1$ and $C_2$. The three boundary components $\alpha_1^-, \alpha_2^-, \alpha_2^+$ are contained in $C_1 \cup C_2$. By Lemma 4.2, $\alpha_2^-$ and $\alpha_2^+$ are not contained in the same $C_i$. We may therefore assume that $C_1$ contains $\alpha_2^-$, and $C_2$ contains $\alpha_2^+$ and $\alpha_1^-$. In particular, $C_1$ is a annulus with core curve homotopic to $\alpha_2^-$, and $C_2$ is a pair of pants.

Consider the boundary of a regular neighborhood of $\alpha_1^+ \cup b^+$ in $S$. This consists of two simple closed curves $\delta, \delta'$ which bound a pair of pants together with $\alpha_1^+$. Up to relabeling we have that $\delta \subset C_1, \delta' \subset C_2$. By the discussion above $\delta$ is then homotopic to $\alpha_2^-$, and $\delta'$ bounds a pair of pants with $\alpha_1^-, \alpha_2^+$ (compare Figure 3). In a pair of pants there is a unique isotopy class of arcs joining a given boundary component to itself, and hence assertion i) is true. In an annulus there is no (essential) arc joining a boundary component to itself, we thus also conclude that there cannot be a wave of $\beta$ based at $\alpha_2^-$. 

Next, observe that $(\alpha_1, \alpha_2, \delta')$ is a pair of pants decomposition consisting of three non-separating curves. Let $P_1, P_2$ be the two components of $\partial V - (\alpha_1 \cup \alpha_2 \cup \delta')$. Both $P_1$ and $P_2$ are pairs of pants whose boundary curves are $\alpha_1, \alpha_2, \delta'$. Suppose that $P_1$ contains $b^+$. Since $b^+$ joins the boundary component $\alpha_1$ of $P_1$ to itself, and $\beta$ is embedded, every component of $P_1 \cap \beta$ has at least one endpoint on $\alpha_1$. Since $b^+$ has both endpoints on $\alpha_1^+$ we
therefore conclude the following inequality on intersection numbers:

\[ i(\alpha_1, \beta) > i(\alpha_2, \beta) + i(\delta', \beta). \]

Now consider the situation in \( P_2 \). If there would not be an arc \( b^- \subset P_2 \) which joins \( \alpha_1 \) to itself, then any arc in \( P_2 \cap \beta \) which has one endpoint on \( \alpha_1 \) has the other on \( \alpha_2 \) or \( \delta' \). This would imply

\[ i(\alpha_1, \beta) \leq i(\alpha_2, \beta) + i(\delta', \beta) \]

which contradicts the inequality above. Hence, there is an arc \( b^- \subset \beta \) which joins \( \alpha_1 \) to itself in \( P_2 \). We have therefore found the desired second wave \( b^- \), showing ii).

In order to show iii), we only have to exclude a wave based at \( \alpha_2^+ \). However, this follows as above since \( \alpha_2^+ \) is contained in the annulus bounded by \( b^- \) and \( \alpha_2^- \).

We also observe the following corollary.

**Corollary 4.4.** Let \((\alpha_1, \alpha_2)\) be a cut system of a genus 2 handlebody. Let \( \beta \) be an arbitrary meridian and \( b^+, b^- \) be the two distinct waves guaranteed by Lemma 4.3. Then the surgeries defined by \( b^+ \) and \( b^- \) are equal.

In particular, there is a unique surgery sequence starting in \((\alpha_1, \alpha_2)\) in the direction of \( \beta \).

**Proof.** The first claim is obvious from the fact that (in the notation of the proof of Lemma 4.3) \( \delta' \) is homotopic to a boundary component of a regular neighborhood of \( \alpha_1^+ \cup b^+ \) and \( \alpha_1^- \cup b^- \), and is therefore equal to the surgery defined by both \( b^+ \) and \( b^- \). The second claim is immediate from the first. \( \square \)

## 5. Meridian Graphs

The purpose of this section is to construct a tree on which the handlebody group \( H_2 \) acts, and which is crucial for the construction of an action of \( H_2 \) on a CAT(0) cube complex in Section 6.

### 5.1. The wave graph (is a tree).

We begin with a construction of a graph which will model all possibilities to change a cut system \( Z \) to a disjoint cut system \( Z' \).

To this end, we fix in this subsection once and for all a cut system \( Z \). We are interested in describing all curves \( \delta \) in the complement of \( Z \) which are non-separating meridians on \( \partial V \). The following lemma allows us to encode them in a convenient way.

**Lemma 5.1.** Let \( Z = \{\alpha_1, \alpha_2\} \) be a cut system, and \( S = \partial V - Z \) its complementary subsurface. Fix a boundary component \( \partial_0 \) of \( S \). Then the following are true:

i) A curve \( \delta \subset S \) is non-separating on \( \partial V \) exactly if for both \( i = 1, 2 \), the two boundary components of \( S \) corresponding to the sides of \( \alpha_i \) are contained in different complementary components of \( \delta \).
Given any $\delta$ as in i), there is a unique embedded arc $w$ in $S$ with both endpoints on $\partial_0$ disjoint from $\delta$, and it separates the two boundary components corresponding to the curve of $Z$ that $\partial_0$ does not correspond to. We call such an arc an admissible wave.

Conversely, if $w$ is any admissible wave, there is a unique curve $\delta$ defining it via ii).

Two curves $\delta, \delta'$ as in i) intersect in two points exactly if the corresponding admissible wave $w, w'$ are disjoint.

**Figure 4.** On the left: Passing from an admissible wave to a meridian and back as in Lemma 5.1 ii) and iii). On the right: Disjoint admissible waves correspond to curves intersecting twice.

**Proof.** Assertion i) is clear. Assertion ii) follows because any curve $\delta$ as in ii) separates $S$ into two pairs of pants, and on a pair of pants there is a unique homotopy class of embedded arcs with endpoints on a specified boundary component. Assertion iii) follows since such an arc cuts $S$ into an annulus and a pair of pants (compare the proof of Lemma 4.3). It remains to show the final Assertion iv). First observe that if $w, w'$ are disjoint then the corresponding curves constructed in iii) indeed intersect in two points.

Finally, suppose that $\delta$ is a curve defined by an arc $w$ as in iii). The arc $w$ separates $S$ into $S_1$ and $S_2$. Without loss of generality we may assume that $S_1$ is an annulus containing $\alpha_2^-$ and $S_2$ is a pair of pants.

Suppose now that $\delta'$ intersects $\delta$ in two points. Then $\delta'$ will also intersect $w$ in two points, and hence there is a subarc $d' \subset \delta'$ in $S_2$ with both endpoints on $w$. Since $S_2$ is a pair of pants, there is a unique arc $w'$ in $S_2$ with endpoints on $\partial_0$ which is disjoint from $d'$, and therefore from $\delta'$. By the uniqueness of ii) this is the arc defining $\delta'$, and since it is contained in $S_1$ with endpoints on $\partial_0$ it is indeed disjoint from $w$.

Motivated by this lemma, we make the following definition.

**Definition 5.2.** The wave graph $W(Z)$ of $Z$ is the graph whose vertices correspond to admissible waves $w$ based at $\partial_0$, and whose edges correspond to disjointness.

For future reference, we record the following immediate corollary of Lemma 5.1:
Corollary 5.3. The wave graph $W(Z)$ is isomorphic to the graph whose vertices correspond to non-separating meridians $\delta$ which are disjoint from $Z$, and where vertices corresponding to $\delta, \delta'$ are connected by an edge if $i(\delta, \delta') = 2$.

Remark 5.4. As a consequence of Corollary 5.3, the wave graph $W(Z)$ can be identified with a subgraph of the Farey graph of the four-holed sphere $\Sigma_{0,4}$. To describe this subgraph, recall that every edge of the Farey graph is contained in two triangles. The three vertices of such a triangle correspond to the three different ways to separate the four punctures of $\Sigma_{0,4}$ into two sets. The condition for the meridian to be non-separating excludes one of these – so the wave graph is obtained from the Farey graph of $\Sigma_{0,4}$ by removing one vertex and two edges from each triangle. This point of view can be used to show that the wave graph is a tree (Theorem 5.9), but we will prove this theorem using different techniques which are useful later.

We also record the following

Lemma 5.5. The wave graph $W(Z)$ is connected.

Proof. This is a standard surgery argument, using induction on intersection number. Suppose $w, w'$ are any two admissible waves corresponding to vertices in $W(Z)$. If they are not disjoint, consider an initial segment $w_0 \subset w$ which intersects $w'$ only in its endpoint $\{q\} = w_0 \cap w'$. Let $w'_1, w'_2$ be the two components of $w' \setminus \{q\}$. Then both $x_i = w_0 \cup w'_i$ are arcs which are disjoint from $w'$ and have smaller intersection with $w$. It is easy to see that exactly one of them is admissible (compare also Figure 5), and hence $w'$ is connected to a vertex $x_i \in W(Z)$ which corresponds to an arc of strictly smaller intersection number with $w$. By induction, the lemma follows. □

In order to study $W(Z)$ further, we define the following projection maps.

First, given a vertex $w \in W(Z)$ corresponding to an admissible wave (which we denote by the same symbol), we define the set $A(w)$ to consist of embedded arcs $a$ in $S - w$ with one endpoint on $w$ and the second endpoint on $\partial_0$, which are not homotopic into $w \cup \partial_0$ (up to homotopy of such arcs).

We observe that any arc $a$ corresponding to a vertex in $A(w)$ cuts the pair of pants $S - w$ into two annuli. From this, we obtain the following consequence (see also Figure 5):

Lemma 5.6. i) No two arcs $a, a'$ corresponding to different vertices in $A(w)$ are disjoint.

ii) Given an arc $a$ corresponding to a vertex in $A(w)$ there is a unique admissible wave $w'$ which is disjoint from $w$ and from $a$.

iii) No two distinct admissible waves are disjoint from $w$.

Proof. If $a'$ is an arc with endpoint on $w$ and $\partial_0$, which is disjoint from $a$, then (since both components of $S - (w \cup a)$ are annuli) it is either homotopic into $\partial_0 \cup w$, or homotopic to $a$. This shows i). To see the second claim, note that there are two homotopy classes of arcs in $S$ with endpoints in $\partial_0$ disjoint
from \( w \). Exactly one of them is an admissible wave, showing ii). For the third claim, consider an admissible wave \( w' \) disjoint from \( w \). Since it is distinct from \( w \), it separates the two boundary components of \( S \) which are contained in \( S - w \). If \( w'' \) is now any other arc with endpoints on \( \partial_0 \) which is disjoint from \( w \) and \( w' \), then it is either homotopic to one of them, or not admissible, showing iii). \( \Box \)

Given a vertex \( w \in W(Z) \), we now define a map
\[
\pi_w : W(Z) \to A(w)
\]
in the following way:

i) If \( x \in W(Z) \) is not disjoint from \( w \), consider an initial segment \( x_0 \subset x \) one of whose endpoints is on \( \partial_0 \), the other is on \( w \), and so that its interior is disjoint from \( w \). We then put \( \pi_w(x) = x_0 \) (see Figure 5). This is well-defined by Lemma 5.6 i).

ii) If \( y \) is disjoint from \( w \), we let \( \pi_w(y) \) be the an arc in \( A(w) \), which is disjoint from \( y \). This is well-defined by Lemma 5.6 ii).

![Figure 5. Projecting a wave into A(w). An arbitrary wave x intersecting w has an initial segment x₀ joining ∂₀ to w, which is its projection. For a disjoint wave y there is a unique arc as above disjoint from it.](image)

We then get the following consequences.

**Proposition 5.7.** If \( x, y \in W(Z) \) are both not disjoint from \( w \), but \( x, y \) are disjoint from each other, then \( \pi_w(x) = \pi_w(y) \).

*Proof.* This is immediate from Lemma 5.6 i). \( \Box \)

**Proposition 5.8.** If \( x \in W(Z) \) is not disjoint from \( w \), and \( y \in W(Z) \) is disjoint from \( x \) and from \( w \), then \( \pi_w(x) = \pi_w(y) \).

*Proof.* This is immediate from Lemma 5.6 ii). \( \Box \)

We are now ready to prove the following central result.

**Theorem 5.9.** For any cut system \( Z \), the wave graph \( W(Z) \) is a tree.
Proof. We have already shown that $W(Z)$ is connected (Lemma 5.5), and so it suffices to show that there are no embedded cycles. As a first step, note that by Lemma 5.6 no two vertices in the link of $w$ are joined by an edge. Namely, if $w_1, w_2$ are in the link of $w$ (i.e. disjoint from $w$), Lemma 5.6 iii) states that they cannot be disjoint from each other. This in particular implies that there are no cycles of length $\leq 3$.

Hence, suppose that $w_0, w_1, \ldots, w_n$ is a cycle of length $n \geq 4$. By possibly passing to a sub-cycle, we may assume that

1. $w_1, w_n$ are the only vertices in the link of $w_0$ (i.e. the only arcs disjoint from $w_0$).
2. The arcs $w_1, w_n$ are distinct.

Consider now $\pi_{w_0}(w_1) = a \in A(w_0)$. Applying Proposition 5.8 (for $x = w_2, y = w_1$) we obtain that $\pi_{w_0}(w_2) = a$ as well. Inductively applying Proposition 5.7, we obtain that $\pi_{w_0}(w_i) = a$ for any $i = 1, \ldots, n - 1$. Finally, applying Proposition 5.8 again (for $x = w_{n-1}, y = w_n$) we see that $\pi_{w_0}(w_n) = a = \pi_{w_0}(w_1)$.

However, $w_n$ and $w_1$ are assumed to be distinct, and therefore cannot have the same projection by Lemma 5.6 ii).

5.2. The non-separating meridional pants graph (is a tree). We now come to the central object we will use to study $H_2$.

Definition 5.10. i) The non-separating meridional pants graph $P_{nm}^2$ has vertices corresponding to pants decompositions $X = \{\delta_1, \delta_2, \delta_3\}$ so that all $\delta_i$ are non-separating meridians. We put an edge between $X$ and $X'$ if they intersect minimally, i.e. in two points.

ii) For any cut system $Z$, let $P_{nm}^2(Z)$ be the full subgraph corresponding to all those pants decompositions $X = \{\delta_1, \delta_2, \delta_3\}$ which contain $Z$.

From Corollary 5.3 and Theorem 5.9 we immediately obtain

Corollary 5.11. For any cut system $Z$ the subgraph $P_{nm}^2(Z)$ is a tree. Any two such subtrees intersect in at most a single point.

We will use these subtrees in order to study $P_{nm}^2$. We begin with the following.

Lemma 5.12. The graph $P_{nm}^2$ is connected.

Proof. Let $X, Y$ be pants decompositions corresponding to vertices of $P_{nm}^2$. We will construct a path joining $X$ to $Y$ in $P_{nm}^2$. Choose two curves $\{\delta_1, \delta_2\} = Z_1$ from $X$ – these will form a cut system by the definition of $P_{nm}^2$. Now, consider the surgery sequence $(Z_i)$ starting in $Z_1$ in the direction of $Y$. Let $n$ be so that $Z_n$ is disjoint from $Y$. As $Y$ is a pants decomposition, this implies that actually $Z_n \subset Y$.

Also, by definition of surgery sequences, for any $i$ the cut systems $Z_{i-1}$ and $Z_{i+1}$ are contained in the complement of the cut system $Z_i$, and thus $Z_i \cup Z_{i-1}$ and $Z_i \cup Z_{i+1}$ correspond to vertices in the tree $P_{nm}^2(Z_i)$. Hence,
these vertices can be joined by a path. The desired path \((X_i)\) will now be obtained by concatenating all of these paths. To be more precise, we will have

1. There are numbers \(1 = m(0), m(1), \ldots, m(n)\) so that for all \(m(i - 1) < j \leq m(i)\) the pants decomposition \(X_j\) contains \(Z_i\).
2. For all \(m(i - 1) < j \leq m(i)\) the pants decomposition \(X_j\) are a geodesic in \(P^\text{nm}_2(Z_i)\).

From the description above it is clear that these sequences exist, showing Lemma 5.12.

We will now define projections of \(P^\text{nm}_2\) onto the subtrees \(P^\text{nm}_2(Z)\). To this end, let \(Z\) be a cut system. We define a projection \(\pi_Z: P^\text{nm}_2 \to P^\text{nm}_2(Z)\) in the following way.

i) If \(X\) is disjoint from \(Z\), we simply put \(\pi_Z(X) = X\).

ii) If \(X\) intersects \(Z\), then there is a wave \(w\) of \(X\) with respect to \(Z\), and we define \(\pi_Z(X) = Z \cup \{\delta\}\), where \(\delta\) is the surgery defined by the wave \(w\). Corollary 4.4 implies that this is well-defined.

Proposition 5.13. Suppose that \(X, Y \in P^\text{nm}_2\) are connected by an edge, and assume that both \(X, Y\) are not disjoint from \(Z\). Then \(\pi_Z(X) = \pi_Z(Y)\).

Proof. Since \(X\) and \(Y\) are not disjoint from \(Z\), there are waves \(w_X, w'_X, w_Y, w'_Y\) as in Lemma 4.3. We claim that unless \(\{w_X, w'_X\} = \{w_Y, w'_Y\}\), the total number of intersections between \(\{w_X, w'_X\}, \{w_Y, w'_Y\}\) is at least 4, contradicting that \(X, Y\) are joined by an edge.

However, this is seen in a similar way as the argument in Lemma 4.3 in different cases (compare Figure 6). First observe that as the waves are arcs in a four-holed sphere joining the same boundary to itself, two waves are either disjoint or intersect at least in two points.

Suppose first that the waves of \(Y\) are based at the same component of \(Z\) as the ones of \(X\), and assume that \(w_X, w_Y\) approach from the same side. If \(w_X\) and \(w_Y\) are disjoint, then by the uniqueness statement of Lemma 4.3 we have that \(\{w_X, w'_X\} = \{w_Y, w'_Y\}\), and thus the claim. If \(w_X, w_Y\) are not disjoint, then \(w_Y\) also intersects \(w'_X\) (in at least two points), and we are done.

The case where the waves of \(X\) and \(Y\) are based at different components is similar, noting that each of \(w_Y, w'_Y\) needs to intersect at least one of the \(w_X, w'_X\).

Proposition 5.14. Suppose that \(X, Y \in P^\text{nm}_2\) are joined by an edge, that \(X\) is not disjoint from \(Z\), but \(Y\) is disjoint from \(Z\). Then \(\pi_Z(X) = \pi_Z(Y)\).

Proof. Since \(X\) is not disjoint from \(Z\), it has a pair of waves \(w_X, w'_X\) as in Lemma 4.3. \(Y\) differs from \(X\) by exchanging a single curve of \(X\). Since \(Y\) is disjoint from \(Z\) but \(X\) is not, two curves \(x_1, x_2\) of \(X\) are disjoint from
In any configuration, different waves generate at least four intersection points.

$Z$, while a third one $x_3$ contributes the waves. The pair of curves $\{x_1, x_2\}$ which is disjoint from $Z$ has to be distinct from $Z$ as otherwise there could not be any waves. Hence, $X$ and $Z$ have precisely one curve in common, say $x_1$. The other curve $x_2$, being disjoint from one of the curves in $Z$ and the waves, is then necessarily the surgery along that wave (see Figure 7).

The move from $X$ to $Y$ replaces the curve $x_3$ contributing the waves, and therefore keeps $x_2$ – which will be the projection of both $X$ and $Y$. □

Together, these propositions can be rephrased as saying that the projection $\pi_Z$ from $P_{2}^{\text{nm}}$ to $P_{2}^{\text{nm}}(Z)$ can only change along a path while that path is actually contained within $P_{2}^{\text{nm}}(Z)$.

**Theorem 5.15.** The graph $P_{2}^{\text{nm}}$ is a tree.

*Proof.* The proof is very similar to the proof of Theorem 5.9. We have already seen connectivity of $P_{2}^{\text{nm}}$ in Lemma 5.12. Suppose that $P_1, \ldots, P_n$ is a nontrivial cycle in $P_{2}^{\text{nm}}$. We choose a cut system $Z \subset P_1$. Since $P_{2}^{\text{nm}}(Z)$ is a tree, the cycle cannot be completely contained in $P_{2}^{\text{nm}}(Z)$. Hence, by passing to a sub-cycle we may assume
There is a number \(k\), so that \(P_i\) is a vertex of \(\mathcal{P}_{2nm}(Z)\) exactly for \(1 \leq i \leq k\).

ii) The vertices \(P_1, P_k\) are distinct.

Applying Proposition 5.14 (with \(Y = P_k\) and \(X = P_{k+1}\)) we conclude that \(\pi_Z(P_{k+1}) = \pi_Z(P_k)\). Inductively applying Proposition 5.13 we conclude that \(\pi_Z(P_i) = \pi_Z(P_k)\) for \(i \leq n\). Applying Proposition 5.14 again (for \(X = P_n, Y = P_1\), we conclude that \(\pi_Z(P_1) = \pi_Z(P_k)\). But, since \(P_1, P_n \in \mathcal{P}_{2nm}(Z)\), we conclude \(P_1 = P_k\), violating assumption ii) above. This shows that \(\mathcal{P}_{2nm}\) admits no cycles and therefore is a tree.

5.3. Controlling Twists. In this subsection we study how subsurface projections to annuli around non-separating meridians \(\alpha\) behave along geodesics in \(\mathcal{P}_{2nm}\). We begin with the following lemma, which is likely known to experts.

**Lemma 5.16.** Suppose that \(Y \subset \partial V\) is a subsurface, and that \(\alpha \subset Y\) is an essential simple closed curve. Suppose that \(\beta_1, \beta_2\) are two curves which intersect \(\partial Y\), and suppose further that there is an arc \(b\) in \(Y\) which intersects \(\alpha\) and so that there are subarcs \(b_i \subset Y \cap \beta_i\) which are isotopic to \(b\). Then

\[d_\alpha(\beta_1, \beta_2) \leq 5\]

(here, the subsurface distance \(d_\alpha\) is seen as curves on \(S\), not \(Y\)).

**Proof.** Up to isotopy we may assume that the curves \(\beta_i\) both actually contain \(b\) (and are in minimal position with respect to themselves, \(\alpha\) and \(\partial Y\)).

Let \(S_\alpha \to \partial V\) be the annular cover corresponding to \(\alpha\), and let \(\hat{\alpha}\) be the unique closed lift of \(\alpha\). Suppose that \(b\) joins components \(\delta_1, \delta_2\) of \(\partial Y\).

Consider a lift \(\hat{b}\) of \(b\) which intersects \(\hat{\alpha}\). Its endpoints are contained in lifts \(\hat{\delta}_i\) of the curves \(\delta_i\). Observe that both the lifts \(\hat{\delta}_i\) (\(i = 1, 2\)) do not connect different boundary components of the annulus \(S_\alpha\) (as the curves \(\delta_i\) are disjoint from \(\alpha\)), and therefore \(\hat{\delta}_i\) bounds a disk \(D_i \subset S_\alpha\) whose closure in the closed annulus \(\overline{S_\alpha}\) intersects the boundary of \(\overline{S_\alpha}\) in a connected subarc.

Now consider lifts \(\hat{\beta}_i\) which contain the arc \(\hat{b}\). These are concatenations of an arc in \(D_1\), the arc \(\hat{b}\), and an arc in \(D_2\). As the arcs in \(D_i\) can intersect in at most one point (otherwise, minimal position of \(\beta_1, \beta_2\) would be violated!), this implies that there are two lifts of \(\beta_i\) which intersect in at most 2 points. This shows the lemma.

We can use this lemma to prove the following result how subsurface projections \(\pi_\alpha\) into annuli around meridians change along \(\mathcal{P}_{2nm}\)-geodesics if these geodesics never involve the curve \(\alpha\). This should be seen as the direct analog of the bounded geodesic projection theorem and its variants for hierarchies which are developed in [MM].

**Proposition 5.17.** Suppose that \(X_i\) is a geodesic in \(\mathcal{P}_{2nm}\), and that \(\alpha\) is a non-separating meridian. Suppose none of the pants decompositions \(X_i\)
contains $\alpha$. Then the subsurface projection $\pi_\alpha(X_i)$ is coarsely constant along $X_i$: there is a constant $K$ independent of $\alpha$ and the sequence, so that
$$d_\alpha(X_i, X_j) \leq K.$$  

Proof. First observe that as none of the $X_i$ contains $\alpha$, actually all $X_i$ intersect $\alpha$. Let $Z$ be a cut system completing $\alpha$ to a pants decomposition. Since $X_1$ intersects $\alpha$, we may assume that there is a wave $w$ of $X_1$ which intersects $\alpha$.

Now consider the path $X_i$. Observe that since $P_{nm}^2(Z)$ is a subtree of $P_{2}^{nm}$, the intersection of the path $X_i$ with $P_{nm}^2(Z)$ is a path, say $X_i, k \leq i \leq k'$. For $i = 1, \ldots, k - 1$, we have $X_i \notin P_{nm}^2(Z)$, and by Proposition 5.13 the pants decompositions $X_i$ will therefore all have the same wave $w$. By Lemma 5.16 this implies that the subsurface projection $\pi_\alpha$ is coarsely constant for $X_1, \ldots, X_{k-1}$ where $k$ is the first index so that $X_k \in P_{nm}^2(Z)$. The projections of $X_{k-1}$ and $X_k$ are uniformly close since $X_{k-1}, X_k$ are disjoint and both intersect $\alpha$. Similarly, the projections of $X_{k'}, X_{k'}$ are uniformly close, and applying Proposition 5.13 and Lemma 5.16, we see that the projection is coarsely constant for $i \geq k'$.

Hence, it suffices to show the statement of the proposition for paths which are completely contained in $P_{nm}^2(Z)$.

So, consider a path $X_j$ in $P_{nm}^2(Z)$ which is never disjoint from a curve $\alpha \subset \partial V - Z$. Let $w$ be the wave corresponding to $\alpha$. Then, the projection $\pi_w(X_i)$ is constant by Proposition 5.13, and therefore the projection $\pi_\alpha(X_i)$ is coarsely constant, arguing as in Lemma 5.16. \hfill \Box

Finally, we study the case where $\alpha$ does appear as one of the curves along a $P_{nm}^2$-geodesic.

**Corollary 5.18.** Let $\alpha$ be a non-separating meridian, and $X_i$ be a geodesic in $P_{nm}^2$, which does become disjoint from $\alpha$. Then there are $i_0 \leq i_1$ so that the following holds:

i) For $i < i_0$ the subsurface projection $\pi_\alpha(X_i)$ is coarsely constant.

ii) For $i_0 \leq i \leq i_1$, the curve $\alpha$ is contained in $X_i$.

iii) For $i > i_0$ the subsurface projection $\pi_\alpha(X_i)$ is coarsely constant.

Proof. In light of the previous Proposition 5.17 the only thing that remains to show is that an interval $i_0 \leq i_1$ exists with the property that $X_i$ contains $\alpha$ exactly for $i_0 \leq i \leq i_1$. This follows since the set $P_{nm}^2(\alpha)$ of non-separating meridional pants decompositions containing $\alpha$ is the union of $P_{nm}^2(Z)$ for $Z$ a cut system containing $\alpha$, which is a connected union of subtrees, hence itself a subtree. Therefore, a geodesic in $P_{nm}^2$ intersects $P_{nm}^2(\alpha)$ in a path. \hfill \Box

6. A geometric model for $H_2$

In this section we define a geometric model for the handlebody group (which is very similar to the one employed in [HH2]) and use the results from Section 5 in order to study the geometry of the genus 2 handlebody group. A first step is the following lemma.
Lemma 6.1. Suppose that \( X \in P_{2}^{\text{nm}} \) is a pants decomposition, and \( X = \{\delta_1, \delta_2, \delta_3\} \). Given \( i \in \{1, 2, 3\} \) there is a curve \( \beta_i \) with the following properties:

i) \( \beta_i \) is a non-separating meridian.

ii) \( \delta_i \) and \( \beta_j \) are disjoint for \( i \neq j \).

iii) \( \delta_i \) and \( \beta_i \) intersect in exactly two points.

Furthermore, the curve \( \beta_i \) is uniquely defined by these properties up to Dehn twist about the curve \( \delta_i \).

Proof. Assume without loss of generality that \( i = 3 \). Consider the surface \( S \) obtained by cutting \( \partial V \) at \( \delta_1, \delta_2 \) as in Section 4. The curve \( \delta_3 \) defines an admissible wave \( w \) as in Lemma 5.1, and by the same lemma any curve \( \beta_3 \) with the desired properties will be defined by an admissible wave \( w' \) which is disjoint from \( w \). Arguing as in Lemma 5.6, such an admissible wave \( w' \) exists and is unique up to Dehn twist in \( \delta_3 \). This shows both claims of the lemma. □

Definition 6.2. If \( X \in P_{2}^{\text{nm}} \), we call a curve \( \beta_i \) dual to \( \gamma_i \in X \) if it satisfies the conclusion of Lemma 6.1. A set \( \Delta = \{\beta_1, \beta_2, \beta_3\} \) containing a dual to each \( \gamma_i \in X \) is called a dual system to \( X \).

Since the handlebody group acts transitively on pants decompositions consisting of non-separating meridians, we see that the handlebody group also acts transitively on pairs \((X, \Delta)\) where \( X \in P_{2}^{\text{nm}} \) and \( \Delta \) is a dual system to \( X \).

Next, we will describe a procedure to canonically modify the dual system when changing the pants decomposition \( X \) to an adjacent one \( X' \) in \( P_{2}^{\text{nm}} \). Suppose that \( X \in P_{2}^{\text{nm}} \) is a pants decomposition, and that \( \Delta \) is a dual system. Suppose that \( \delta \in \Delta \) is the dual curve to a curve \( \gamma \in X \). Then the system \( X' = X \cup \{\delta\} \setminus \{\gamma\} \) obtained by swapping \( \gamma \) for \( \delta \) is also a pants decomposition consisting of non-separating meridians, and defines a vertex \( X' \) adjacent to \( X \) in \( P_{2}^{\text{nm}} \). We say that \( X' \) is obtained from \((X, \Delta)\) by switching \( \gamma \).

The curve \( \gamma \) is dual to \( \delta \) in \( X' \) in the sense of Lemma 6.1. However, the other curves \( \delta_1, \delta_2 \) of \( \Delta \setminus \{\delta\} \) are not – each of them will intersect \( \delta \) in four points. The following lemma will allow us to clean the situation up in a unique way.

Lemma 6.3. Suppose that \( X \) is a pants decomposition, and \( \Delta \) is a dual system. Let \( \gamma \in X \) be given, and suppose \( X' \) is obtained from \((X, \Delta)\) by switching \( \gamma \).

Let \( \gamma' \in X \) be distinct from \( \gamma \), and \( \delta' \in \Delta \) its dual. Then there is a dual curve \( c(\delta') \) to \( \gamma' \) for the system \( X' \), and the assignment \( \delta' \mapsto c(\delta') \) commutes with Dehn twists about \( \gamma' \).

Proof. Suppose that \( X = \{\gamma_1, \gamma_2, \gamma_3\} \), and that \( \Delta = \{\delta_1, \delta_2, \delta_3\} \), so that \( \delta_i \) is dual to \( \gamma_i \). Assume that we switch \( \gamma_2 \). Consider the surface \( S \) (as in
Figure 8. The cleanup move for dual curves in a switch.

Section 5) obtained as the complement of the cut system \( \{ \gamma_1, \gamma_3 \} \). Then both \( \gamma_2, \delta_2 \) are contained in \( S \) and intersect in two points.

The dual curve \( \delta_1 \) defines two waves \( w, w' \) with respect to \( X \). Consider \( w \), and note that it intersects \( \delta_2 \) in two points (compare Figure 8). There are two ways of surgering \( w \) in the direction of \( \delta_2 \), i.e. replacing a subarc of \( w \) by a subarc of \( \delta_2 \). Exactly one of them yields an essential wave, which we denote by \( v \). Note that \( v \) has the same endpoints as \( w \). We define \( v' \) similarly for the other wave \( w' \). We define \( c(\delta_1) \) to be the curve \( v \cup v' \). It intersects \( \gamma_1 \) in two points by construction, and is indeed nonseparating since it defines admissible waves (compare Lemma 5.1).

To see that the map \( c \) commutes with Dehn twists about \( \gamma_1 \), it suffices to note that such a Dehn twist can be supported in a small neighbourhood of \( \gamma_1 \), and therefore the assignment of \( v, v' \) to \( \delta_1 \) commutes with Dehn twists by construction. □

**Definition 6.4.** Suppose that \( X \in \mathcal{P}^{nm}_2 \) is a pants decomposition, that \( \Delta \) is a dual system, and \( \gamma \in X \) is given. We then say that \( (X', \Delta') \) is obtained from \( (X, \Delta) \) by switching \( \gamma \) if the following hold:

i) \( X' = X \cup \{ \delta \} \setminus \{ \gamma \} \).

ii) The dual curve to \( \delta \) is \( \gamma \).

iii) The dual curves to both other \( \delta' \in X' \) are obtained from the dual curves in \( \Delta \) by the map \( c \) from Lemma 6.3.

We record the following immediate corollary of the uniqueness statement in Lemma 6.3, which states that twisting about a curve different from \( \gamma \) commutes with switching \( \gamma \).

**Corollary 6.5.** Suppose that \( X \in \mathcal{P}^{nm}_2 \) is given, \( \gamma, \gamma' \) are two curves in \( X \). If \( (X', \Delta') \) is obtained from \( (X, \Delta) \) by switching \( \gamma \), then \( (X', T_{\gamma'} \Delta') \) is obtained from \( (X, T_{\gamma'} \Delta) \) by switching \( \gamma \).

We can now define our geometric model for the handlebody group of genus 2.
Definition 6.6. The graph $M_{2}^{nm}$ has vertices corresponding to pairs $(X, \Delta)$, where $X$ is a vertex of $P_{2}^{nm}$, and $\Delta$ is a dual system to $X$. There are two types of edges:

**Twist:** Suppose that $X = \{\gamma_1, \gamma_2, \gamma_3\}$ is a vertex of $P_{2}^{nm}$, and $\Delta$ is a dual system. Then we join, for any $i$

$$(X, \Delta) \text{ and } (X, T_{i}^{\pm} \Delta)$$

by edges $e_{i}^{\pm}$. We call these *twist edges* and say that $\gamma_i$ is involved in $e_{i}^{\pm}$.

If $\gamma$ is a curve that is involved in two oriented twist edges $e, e'$ we say that it is *involved with consistent orientation* if the corresponding Dehn twist has the same sign in both cases.

**Switch:** Suppose that $X \in P_{2}^{nm}$ is a vertex, and $\Delta$ is a dual system to $X$. Suppose further that $(X', \Delta')$ is obtained from $(X, \Delta)$ by switching some $\gamma \in X$.

We then connect $(X, \Delta)$ and $(X', \Delta')$ by an edge $e$. We say that $e$ is a *switch edge*, and that it corresponds to the edge between $X$ and $X'$ in $P_{2}^{nm}$.

Proposition 6.7. The handlebody group $H_{2}$ acts on $M_{2}^{nm}$ properly discontinuously and cocompactly.

Proof. The quotient of $M_{2}^{nm}$ by the handlebody group is finite, since $H_{2}$ acts transitively on the vertices of $P_{2}^{nm}$, and the group generated by Dehn twists about $X$ act transitively on dual systems of $X$. Since for a vertex $(X, \Delta)$ the union $X \cup \Delta$ cuts the surface into simply connected regions, the stabilizer of any vertex of $M_{2}^{nm}$ is finite. □

6.1. Cubical Structure. In this section we will turn $M_{2}^{nm}$ into the 1–skeleton of a CAT(0) cube complex.

In order to do so, we will glue in two types of cubes into $M_{2}^{nm}$. For the first, fix some $X \in P_{2}^{nm}$, and consider the subgraph $M_{2}^{nm}(X)$ spanned by those vertices whose corresponding pair has $X$ as its first entry. By definition, any edge in $M_{2}^{nm}(X)$ is a twist edge, and in fact $M_{2}^{nm}(X)$ is isomorphic as a graph to the standard Cayley graph of $\mathbb{Z}^{3}$. We call the subgraphs $M_{2}^{nm}(X)$ *twist flats*. We then glue standard Euclidean cubes to make $M_{2}^{nm}(X)$ the 1-skeleton of the standard integral cube complex structure of $\mathbb{R}^{3}$. We call these cubes *twist cubes*.

The second kind of cubes will involve switch edges, and to describe them we first need to understand all the switch edges corresponding to a given edge between $X, X'$ in $P_{2}^{nm}$. Let $\{\alpha_1, \alpha_2\} = X \cap X'$ be the two curves that the two pants decompositions have in common, and suppose $\gamma$ is switched to $\gamma'$. Then the possible switch edges will join vertices

$$((\alpha_1, \alpha_2, \gamma), (\delta_1, \delta_2, \gamma')) \text{ to } ((\alpha_1, \alpha_2, \gamma'), (c(\delta_1), c(\delta_2), \gamma))$$

Note that since $\delta_1, \delta_2$ are unique up to Dehn twists about $\alpha_1, \alpha_2$ (Lemma 6.1), and the map $c$ commutes with twists (Lemma 6.3), we conclude that the
switch edges corresponding to the edge between \(X, X'\) are exactly the edges between
\[
((\alpha_1, \alpha_2, \gamma), (T_{\alpha_1}^{n_1} \delta_1, T_{\alpha_2}^{n_2} \delta_2, \gamma')) \quad \text{and} \quad ((\alpha_1, \alpha_2, \gamma'), (T_{\alpha_1}^{n_1} c(\delta_1), T_{\alpha_2}^{n_2} c(\delta_2), \gamma))
\]
for any \(n_1, n_2\). Hence, in \(M_{2}^{nm}\), there is a copy of the 1–skeleton of a 3–cube with vertices
\[
(X, \Delta), (X, T_{\alpha_1} \Delta), (X, T_{\alpha_2} \Delta), (X, T_{\alpha_1} T_{\alpha_2} \Delta),
\]
\[
(X', \Delta'), (X', T_{\alpha_1} \Delta'), (X', T_{\alpha_2} \Delta'), (X', T_{\alpha_1} T_{\alpha_2} \Delta'),
\]
and we glue in a switch cube at this 1–skeleton. Similarly, we glue in three more switch cubes for the different possibilities of replacing \(T_{\alpha_1}\) and/or \(T_{\alpha_2}\) by their inverses.

For later reference, observe that by construction any square in our cube complex has either only twist edges, or exactly two nonadjacent switch edges in its boundary. This fairly immediately implies the following.

**Proposition 6.8.** The link of any vertex in the cube complex \(M_{2}^{nm}\) is a flag simplicial complex.

**Proof.** Since the link is a 2-dimensional simplicial complex, we only have to check that any boundary of a triangle in the 1–skeleton of the link bounds a triangle in the link. Vertices in the link correspond to edges in \(M_{2}^{nm}\) and are therefore of twist or switch type. Edges in the link are due to squares in the cubical structure, and by the remark above any edge has either both endpoints of twist type, or exactly one of switch type. Thus, there are only two types of triangles in the link: those were all three vertices are of twist type, and those where exactly one of the vertices is of switch type. But, in both of these cases, the three corresponding edges in \(M_{2}^{nm}\) are part of a common twist or switch cube, and therefore the desired triangle in the link exists. \(\Box\)

**Proposition 6.9.** The cube complex \(M_{2}^{nm}\) is connected and simply-connected.

**Proof.** The fact that \(M_{2}^{nm}\) is connected is an easy consequence of the fact that the tree \(P_{2}^{nm}\) is connected, and each twist flat \(M_{2}^{nm}(X)\) is connected as well.

Now, suppose that \(g(i) = (X_i, \Delta_i)\) is a simplicial loop in \(M_{2}^{nm}\). Then, the path \(X_i\) is a loop in \(P_{2}^{nm}\), and since the latter is a tree, it has backtracking. Thus we can write \(g\) as a concatenation
\[
g = g_1 * \sigma_1 * \tau * \sigma_2 * g_2
\]
where \(\sigma_1, \sigma_2\) are two switch edges corresponding to an edge from some vertex \(X\) to another vertex \(X'\), and from \(X'\) back to \(X\), respectively, and \(\tau\) is a path consisting only of twist edges. In fact, in order for \(\sigma_1 * \tau * \sigma_2\) to be a path, the total twisting about the curve \(\{\alpha\} = X' \setminus X\) has to be zero. Since the twist flat \(M_{2}^{nm}(X')\) is homeomorphic to \(R^3\) in our cubical structure, we may therefore homotope the path so that \(\tau\) does not twist about \(\alpha\) at all.
Next, consider the first twist edge \( t_1 \) in \( \tau \). Then, \( \sigma_1 \ast t_1 \) are two sides of a square in a switch cube, and thus \( g \) is homotopic to a path

\[
g = g_1 \ast t'_1 \ast \sigma'_1 \ast \tau' \ast \sigma_2 \ast g_2
\]

where now \( \tau' \) has strictly smaller length than \( \tau \), and \( \sigma'_1 \) and \( \sigma_2 \) still correspond to opposite orientations of the same edge in \( \mathcal{P}^{nm}_2 \). By induction, we can reduce the length of \( \tau' \) to zero, in which case \( g \) will have backtracking. An induction on the length of \( g \) then finishes the proof. \( \square \)

Hence, using Gromov’s criterion (e.g. [BH, Chapter II, Theorem 5.20]) we conclude:

**Corollary 6.10.** \( \mathcal{M}^{nm}_2 \) is a CAT(0) cube complex.

**Remark 6.11.** By our construction, the genus 2 handlebody group acts by semisimple isometries on a complete CAT(0)-cube complex of dimension 3. This should be contrasted to the main result of [Bri] which shows that any action of the mapping class group of a closed surface of genus \( g \geq 2 \) on a complete CAT(0)-space of dimension less than \( g \) fixes a point.

**Corollary 6.12.** The genus 2 handlebody group \( \mathcal{H}_2 \) is biautomatic, and has a quadratic isoperimetric inequality.

**Proof.** From [Š, Corollary 8.1] we conclude biautomaticity, since \( \mathcal{H}_2 \) acts properly discontinuously and cocompactly on the CAT(0) cube complex \( \mathcal{M}^{nm}_2 \) (Proposition 6.7 and Corollary 6.10). This implies that \( \mathcal{H}_2 \) has at most quadratic Dehn function \([BGSS, ECH^+]\). Observe that since \( \mathcal{H}_2 \) contains copies of \( \mathbb{Z}^2 \) (generated by Dehn twists about disjoint meridians) it is not hyperbolic, and therefore its Dehn function cannot be sub-quadratic \([Gro]\). \( \square \)

Using Proposition 1 of [CMV] we also conclude

**Corollary 6.13.** The genus 2 handlebody group has the Haagerup property.

### 6.2. Other geometric consequences.

The geometric model \( \mathcal{M}^{nm}_2 \) for the genus 2 handlebody group can also be used to conclude other facts about \( \mathcal{H}_2 \). For example, we have the following distance estimate in \( \mathcal{M}^{nm}_2 \), which should be compared to the Masur-Minsky distance formula for the surface mapping class group from [MM].

**Proposition 6.14.** There are constants \( c, C > 0 \) so that for all pairs of vertices \((X, \Delta_X), (Y, \Delta_Y) \in \mathcal{M}^{nm}_2 \) we have

\[
d_{\mathcal{M}^{nm}_2}((X, \Delta_X), (Y, \Delta_Y)) \precsim_c d_{\mathcal{P}^{nm}_2}(X, Y) + \sum_\alpha [d_\alpha(X \cup \Delta_X, Y \cup \Delta_Y)]_C
\]

where the sum is taken over all non-separating meridians \( \alpha \). Here, \( \precsim_c \) means that the equality holds up to a (uniform) multiplicative and additive constant \( c \), and \([\cdot]_C \) means that the term only appears of the argument is at least \( C \).
Proof. Consider a geodesic \( g : [0, l] \to \mathcal{M}_2^{nm} \) joining \((X, \Delta X)\) to \((Y, \Delta Y)\) in \(\mathcal{M}_2^{nm}\). We need to estimate the length \(l\) of \(g\). First, we claim that the projection of \(g\) to \(\mathcal{P}_2^{nm}\) is a path without backtracking in the tree \(\mathcal{P}_2^{nm}\) (but possibly with intervals on which it is constant).

Namely, suppose that this is not the case. Then, as the projection \(X_i\) of \(g\) backtracks, we can write \(g = g_1 * \sigma * \tau * \sigma' * g_2\) where \(\sigma, \sigma'\) are two switch edges corresponding to opposite orientations of the same edge in \(\mathcal{P}_2^{nm}\), and \(\tau\) is a path just consisting of twist edges. If \(\sigma\) switches a curve \(\gamma\), note that we may assume that \(\tau\) does not twist about \(\gamma\). Namely, the total twisting about \(\gamma\) has to be zero in order for \(\sigma'\) to be able to follow \(\tau\), and therefore any twists about \(\gamma\) can be canceled without changing the length or the projection to \(\mathcal{P}_2^{nm}\) of \(g\).

However, now the twists \(\tau\) can be moved to the end of \(g_1\) by Corollary 6.5 without changing the length of \(g\) or its endpoints. However, after this modification \(g\) has backtracking, which contradicts the fact that it is a geodesic. Similarly, arguing as above, we see that in a geodesic \(g\) all the twist edges involving a given curve \(\alpha\) need to have consistent orientation, as otherwise the geodesic could be shortened.

Using Corollary 6.5 again, we may also assume that all twist edges involving the same curve \(\alpha\) are adjacent in \(g\), and appear immediately after \(\alpha\) has become a curve in one of the \(X_i\).

Now, the number of switch edges in \(g\) is exactly \(d_{\mathcal{P}_2^{nm}}(X,Y)\). It therefore suffices to argue that the number of twist edges can be estimated by the right-hand side of the equality in the proposition.

Fix some non-separating meridian \(\alpha\). If \(\alpha\) never appears in \(X_i\), then by Proposition 5.17 the projection into the annulus around \(\alpha\) is coarsely constant. Hence, by choosing \(C\) large enough, these projections will not contribute to the sum in the statement of the proposition.

If \(\alpha\) does appear in \(X_i\), then by Corollary 5.18, it appears exactly for \(i_0 \leq i \leq i_1\), and the projection before \(i_0\) and after \(i_1\) is coarsely constant. If \(\alpha\) appears at \(i_0\), and \(g\) performs \(n\) twists about \(\alpha\) at this time, the projection \(\pi_\alpha\) changes by a distance of \(n\). For any subsequent switch edge corresponding to \(X_{i_0+1}, \ldots, X_{i_1}\), the projection can only change by a uniformly small amount in each cleanup move (given by Lemma 6.3) as the corresponding dual curves intersect in uniformly few points. In conclusion, we have that \(d_\alpha(X \cup \Delta X, Y \cup \Delta Y)\) differs from \(n\), where \(n\) is the length of the twist segment in \(g\) corresponding to \(\alpha\), by at most the length \(e_\alpha = i_1 - i_0\). However, observe that

\[
\sum_{\alpha \in X_i, i=1,\ldots,k} e_\alpha \leq 3k
\]

since any vertex in the geodesic \((X_i)\) can be in at most three of the intervals whose lengths are counted as the \(e_\alpha\). Hence, the sum of the error terms \(e_\alpha\) is bounded by \(d_{\mathcal{P}_2^{nm}}(X,Y)\), showing the proposition. \(\square\)
From this distance formula, we immediately see the following:

**Corollary 6.15.** The stabilizer of a nonseparating meridian $\delta$ in $V_2$ is undistorted in $H_2$.

**Corollary 6.16.** There is a quasi-isometric embedding of $H_2$ into a product of quasi-trees.

**Proof.** From Proposition 6.14, and the Masur-Minsky distance formula we see that the map

$$H_2 \to \mathcal{P}_2^{nm} \times \text{Mg}(\Sigma_2)$$

is a quasi-isometric embedding. By Theorem 5.15, the factor $\mathcal{P}_2^{nm}$ is already a tree. By the main result of [BBF], we can embed the second factor isometrically into a product of quasi-trees. $\square$

**Remark 6.17.** In fact, by arguing exactly like in [BBF], $H_2$ embeds into $\mathcal{P}_2^{nm} \times Y_1 \times \cdots \times Y_k$, where each $Y_k$ is a quasi-tree of metric spaces coming from applying the main construction of [BBF] to the set of all non-separating meridians, and projections into annuli around them. Since we do not need this precise result, we skip the proof.

References


Ursula Hamenstädt  
Rheinische Friedrich-Wilhelms Universität Bonn, Mathematisches Institut  
Endenicher Allee 60, 53115 Bonn  
Email: ursula@math.uni-bonn.de

Sebastian Hensel  
Mathematisches Institut der Universität München  
Theresienstraße 39, 80333 München  
Email: hensel@math.lmu.de