CONVEX COCOMPACT SUBGROUPS OF $\mathrm{Out}(F_n)$

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Abstract. Call a finitely generated subgroup Γ of $\mathrm{Out}(F_n)$ convex cocompact if its orbit map of the free factor graph is a quasi-isometric embedding. We develop a characterization of convex cocompact subgroups of $\mathrm{Out}(F_n)$ via their action on Outer space similar to one of convex cocompact subgroups of mapping class groups.

1. Introduction

Motivated by the theory of Kleinian groups, Farb and Mosher define in [FM02] the notion of a *convex cocompact subgroup* Γ of the mapping class group $\mathrm{Mod}(S)$ of a surface S of genus $g \geq 2$ via geometric properties of the action of Γ on Teichmüller space $\mathcal{T}(S)$. Later, an equivalent characterization using the action of Γ on the curve graph of S was established in [H05] and [KeL08]. The aim of this article is to develop a similar theory for subgroups of the outer automorphism group of a free group. We begin by briefly recalling the results in the mapping class group case.

There is a compactification of $\mathcal{T}(S)$ by attaching the space $\mathcal{PML}$ of projective measured laminations. The *limit set* Λ_Γ of a subgroup Γ of $\mathrm{Mod}(S)$ is the smallest closed Γ-invariant subset of $\mathcal{PML}$.

Each Teichmüller geodesic corresponds to a pair of distinct points in $\mathcal{PML}$. Thus one can define the *weak hull* of the limit set Λ_Γ of Γ as the union of all Teichmüller geodesics with endpoints in Λ_Γ. This set is invariant under the action of Γ.

The mapping class group $\mathrm{Mod}(S)$ also acts on the *curve graph* $\mathcal{C}(S)$ of S as a group of isometries. The curve graph is a hyperbolic geodesic metric space, and its Gromov boundary $\partial\mathcal{C}(S)$ is the quotient of a $\mathrm{Mod}(S)$-invariant Borel subset of $\mathcal{PML}(S)$, with closed fibres.

Theorem 1 ([FM02, H05, KeL08]). *The following properties of a finitely generated subgroup Γ of $\mathrm{Mod}(S)$ are equivalent.*

(1) The orbit map on the curve graph $\mathcal{C}(S)$ of S is a quasi-isometric embedding.

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(2) *Γ is word hyperbolic, and there is a Γ-equivariant embedding $\partial\Gamma \to \mathcal{PML}(S)$ with image Λ_Γ such that the weak hull $\mathcal{H}_\Gamma$ of Λ_Γ is defined. The action of Γ on $\mathcal{H}_\Gamma$ is cocompact. If $F : \Gamma \to \mathcal{H}_\Gamma$ is any Γ-equivariant map then F is a quasi-isometry, and*
$$\overline{F} = F \cup \partial F : \Gamma \cup \partial\Gamma \to \mathcal{T}(S) \cup \mathcal{PML}(S)$$
is continuous.

(3) *A Γ-orbit on $\mathcal{T}(S)$ is quasi-convex: For any $x \in \mathcal{T}(S)$ and all $g, h \in \Gamma$, the Teichmüller geodesic connecting gx, hx is contained in a uniformly bounded neighborhood of Γx.*

The equivalence of (2) and (3) in the theorem below is due to Farb and Mosher [FM02]. The equivalence of (1) and (2) was established in [H05] and [KeL08]. A subgroup of $\mathrm{Mod}(S)$ which has the properties in Theorem 1 is called *convex cocompact.*

For the development of a theory of convex cocompact subgroups of the outer automorphism group $\mathrm{Out}(F_n)$ of a free group with $n \geq 3$ generators we replace Teichmüller space by Outer space equipped with the two-sided Lipschitz metric, and we use the free factor graph as an analog of the curve graph.

Definition 1. The *free factor graph* $\mathcal{FF}$ is the metric graph whose vertices are conjugacy classes of non-trivial free factors of F_n. Two vertices A, B are connected by an edge of length one if up to conjugation, either $A < B$ or $B < A$.

The free factor graph is hyperbolic in the sense of Gromov [BF14]. The outer automorphism group $\mathrm{Out}(F_n)$ of F_n acts on $\mathcal{FF}$ as a group of simplicial isometries. This action is coarsely transitive.

Definition 2. A finitely generated subgroup Γ of $\mathrm{Out}(F_n)$ is *convex cocompact* if one (and hence every) orbit map of its action on the free factor graph is a quasi-isometric embedding.

An element $\varphi \in \mathrm{Out}(F_n)$ is called *irreducible with irreducible power* (or *iwip* for short) if there is no $k \geq 1$ such that φ^k preserves a free factor. An element $\varphi \in \mathrm{Out}(F_n)$ of infinite order acts with unbounded orbits on $\mathcal{FF}$ if and only if it is iwip [KL09, BF14]. Thus any element of infinite order in a convex cocompact subgroup Γ of $\mathrm{Out}(F_n)$ is irreducible with irreducible powers. Since every subgroup of $\mathrm{Out}(F_n)$ has a normal torsion free subgroup of finite index, we may assume that Γ is torsion free and hence purely iwip, i.e. every nontrivial element is an iwip.

Outer space $cv_0(F_n)$ is the space of all minimal free actions of F_n on simplicial trees T with quotient T/F_n of volume one. It can be equipped with the *symmetrized Lipschitz metric* d. As this metric is not geodesic, we will work instead with coarse geodesics. By definition, a *c-coarse geodesic* is a path $\gamma : \mathbb{R} \to cv_0(F_n)$ such that for all $s, t \in \mathbb{R}$ we have
$$|s - t| - c \leq d(\gamma(s), \gamma(t)) \leq |s - t| + c.$$
Note that a coarse geodesic need not be continuous.

The projectivization $\mathrm{CV}(F_n)$ of Outer space admits a natural compactification by adding the boundary $\partial\mathrm{CV}(F_n)$. A point in $\partial\mathrm{CV}(F_n)$ is a projective minimal *very small* F_n-tree so that the action of F_n either is not simplicial, or it is not free. The action of $\mathrm{Out}(F_n)$ extends to an action on $\overline{\mathrm{CV}(F_n)} = \mathrm{CV}(F_n) \cup \partial\mathrm{CV}(F_n)$ by homeomorphisms. If $\Gamma < \mathrm{Out}(F_n)$ is any subgroup then we can define the *limit set* Λ_Γ of Γ as the smallest closed Γ-invariant subset of $\partial\mathrm{CV}(F_n)$.

The following result is the analog of the equivalence of (1) and (2) in Theorem 1. In its formulation, the c-weak hull of a set $\Lambda \subset \partial\mathrm{CV}(F_n)$ is the closure of the collection of all c-coarse geodesics for the symmetrized Lipschitz metric which converge to pairs of distinct points in Λ.

Theorem 2. *Let Γ be a finitely generated torsion free subgroup of $\mathrm{Out}(F_n)$. Then Γ is convex cocompact if and only if the following properties are satisfied.*

i) Γ *is word hyperbolic.*
ii) *There is a Γ-equivariant homeomorphism of the Gromov boundary $\partial\Gamma$ of Γ onto the limit set $\Lambda_\Gamma \subset \partial\mathrm{CV}(F_n)$.*
iii) *For sufficiently large $c > 0$, any two distinct points in Λ_Γ can be connected by a c-coarse geodesic, so the c-weak hull $\mathcal{H}_\Gamma$ of Λ_Γ is defined. The action of Γ on $\mathcal{H}_\Gamma$ is cocompact.*
iv) *If $F : \Gamma \to \mathcal{H}_\Gamma$ is any Γ-equivariant map then F is a quasi-isometry, and $\overline{F} : F \cup \partial F : \Gamma \cup \partial\Gamma \to \mathrm{CV}(F_n) \cup \partial\mathrm{CV}(F_n)$ is continuous.*

A subset A of $cv_0(F_n)$ is *coarsely strictly convex* if for any $c > 0$ there is a number $b(c) > 0$ such that any c–coarse geodesic with both endpoints in A is entirely contained in the $b(c)$-neighborhood of A.

Let d_L be the one-sided Lipschitz metric on $cv_0(F_n)$ and let as before d be the symmetrized Lipschitz metric. For a number $K > 1$, a *K-quasi-geodesic* for d_L is a path $\gamma : [a, b] \to cv_0(F_n)$ so that for all $s < t$ we have

$$|t - s|/K - K \le d_L(\gamma(s), \gamma(t)) \le K|t - s| + K.$$

Definition 3. A (coarse) geodesic $\gamma \subset (cv_0(F_n), d)$ is called *strongly Morse* if for any constant $K \ge 1$ there is a constant $M = M(K) > 0$ with the following property. The Hausdorff distance for the symmetrized Lipschitz distance between γ and any K-quasi-geodesic for d_L with endpoints on γ is at most M.

In the sequel we talk about families of M-Morse geodesics if all paths in the family satisfy the conditions in Definition 3 for the same constants.

The following establishes the analog of the equivalence (1) and (3) in Theorem 1.

Theorem 3. *A finitely generated subgroup Γ of $\mathrm{Out}(F_n)$ is convex cocompact if and only if the following holds true. Let $T \in cv_0(F_n)$. Then for all $g, h \in \Gamma$, the points gT, hT are connected by an M-Morse coarse geodesic which is contained in a uniformly bounded neighborhood of ΓT.*

Examples of convex cocompact groups are Schottky groups. In the case $n = 2g$ for some $g \geq 2$, convex cocompact subgroups of the mapping class group of a surface of genus g with one puncture, viewed as subgroups of $\mathrm{Out}(F_n)$, are convex cocompact as well. We discuss these examples in Section 6.

For mapping class groups of a closed surface S, there is another characterization of convex cocompact subgroups [FM02, H05]. Namely, Γ is convex cocompact if and only if the extension G of Γ given by the exact sequence

$$0 \to \pi_1(S) \to G \to \Gamma \to 0$$

is word hyperbolic. In contrast, the F_n-extension of a convex cocompact subgroup of $\mathrm{Out}(F_n)$ need not be word hyperbolic. As an example, the extension of a convex cocompact subgroup of the mapping class group of a surface with a puncture is not hyperbolic. The converse is also not true (as was pointed out to the authors by Ilya Kapovich).

However, Dowdall and Taylor [DT14] showed that the F_n-extension of a convex cocompact subgroup Γ all of whose elements are non-geometric is hyperbolic. There is substantial overlap of our work with recent results of Dowdall and Taylor [DT14] which were obtained independently and at the same time. Namely, in [DT14] they establish a local version of our Proposition 4.1. They also announced a local version of Proposition 3.2.

Organization: Section 2 collects the basic tools and background. In Section 3 we relate lines of minima as introduced in [H14a] to coarse geodesics in the thick part of Outer space whose shadows in the free factor graph are quasi-geodesics. Section 4 in turn shows that coarse geodesics in Outer space whose shadows are parametrized quasi-geodesics in the free factor graph arise from lines of minima. Section 5 completes the proof of the main results of this paper.

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2. Geometric tools

2.1. **The boundary of the free factor graph.** The free factor graph is hyperbolic. Its Gromov boundary can be described as follows [BR12, H12].

Unprojectivized Outer space $cv(F_n)$ of simplicial minimal free F_n-trees equipped with the equivariant Gromov Hausdorff topology can be completed by attaching a boundary $\partial cv(F_n)$. This boundary consists of all minimal *very small* actions of F_n on $\mathbb{R}$-trees which either are not simplicial or which are not free [CL95, BF92]. Here an F_n-tree is very small if arc stabilizers are at most maximal cyclic and tripod stabilizers are trivial. We denote by $\mathrm{CV}(F_n)$ the projectivization of $cv(F_n)$, with its boundary $\partial\mathrm{CV}(F_n)$. Also, from now on we always denote by $[T] \in \overline{\mathrm{CV}(F_n)} = \mathrm{CV}(F_n) \cup \partial\mathrm{CV}(F_n)$ the projectivization of a tree $T \in \overline{cv(F_n)} = cv(F_n) \cup \partial cv(F_n)$.

The space $\mathcal{ML}$ of *measured laminations* for F_n is a closed $\mathrm{Out}(F_n)$ invariant subspace of the space of all locally finite F_n-invariant Borel measures on $\partial F_n \times$

$\partial F_n - \Delta$, equipped with the weak*-topology. Dirac measures on pairs of fixed points of all elements in some primitive conjugacy class of F_n are dense in $\mathcal{ML}$ [Ma95]. The projectivization $\mathcal{PML}$ of $\mathcal{ML}$ is compact, and $\mathrm{Out}(F_n)$ acts on $\mathcal{ML}$ minimally by homeomorphisms [Ma95]. In the sequel we always denote by $[\mu] \in \mathcal{PML}$ the projectivization of a measured lamination $\mu \in \mathcal{ML}$.

By [KL09], there is a continuous length pairing

$$\langle , \rangle : \overline{cv(F_n)} \times \mathcal{ML} \to [0, \infty).$$

If $\mu \in \mathcal{ML}$ is arbitrary then $\langle T, \mu \rangle > 0$ for every tree $T \in cv_0(F_n)$.

Definition 2.1. A measured lamination $\mu \in \mathcal{ML}$ is *dual* to a tree $T \in \partial cv(F_n)$ if $\langle T, \mu \rangle = 0$.

Note that if μ is dual to T then any multiple of μ is dual to every tree obtained from T by scaling, so we can talk about a projective measured lamination which is dual to a projective tree. Every projective tree $[T] \in \partial \mathrm{CV}(F_n)$ admits a dual measured lamination μ [H12].

We say that a measured lamination μ is *supported in a free factor H* of F_n if it gives full mass to the F_n-orbit of $\partial H \times \partial H$. If $[T]$ has point stabilizers containing a free factor, then any measured lamination supported in the free factor is dual to T. If $[T] \in \partial \mathrm{CV}(F_n)$ is simplicial then the set of measured laminations dual to $[T]$ consists of convex combinations of measured laminations supported in a point stabilizer of $[T]$.

To describe the Gromov boundary of the free factor graph we need the following notions. A (projective) tree $[T]$ is called *indecomposable* if for any nondegenerate segments $I, J \subset T$ there are elements $u_1, \dots, u_n \in F_n$ with $I \subset u_1 J \cup \dots \cup u_n J$ and so that $u_i J \cup u_{i+1} J$ is a nondegenerate segment for all i.

Let $\sim$ be the smallest equivalence relation on $\partial \mathrm{CV}(F_n)$ with the following property. For every tree $[T] \in \partial \mathrm{CV}(F_n)$ and every $\mu \in \mathcal{ML}$ dual to $[T]$, any tree $[S] \in \partial \mathrm{CV}(F_n)$ dual to μ is equivalent to $[T]$.

Theorem 2.2 ([BR12, H12])**.** *The Gromov boundary $\partial \mathcal{FF}$ of $\mathcal{FF}$ can be identified with the set of equivalence classes under $\sim$ of indecomposable projective trees $[T]$ with the following additional property. Either the F_n-action on T is free, or there is a compact surface S with non-empty connected boundary, and there is a minimal filling measured lamination μ on S so that T is dual to μ.*

We call such a (projective) tree *arational* in the sequel.

By continuity of the length pairing, the set of all trees $[S]$ which are dual to some measured lamination μ is a closed subset of $\partial \mathrm{CV}(F_n)$. The topology on $\partial \mathcal{FF}$ is the quotient topology for the closed equivalence relation $\sim$ on the set of arational projective trees. It can be described as follows. A sequence of equivalence classes represented by elements S_i converges to the equivalence class represented by S if there is a sequence $(\nu_i) \subset \mathcal{ML}$ so that $\langle S_i, \nu_i \rangle = 0$ for all i and such that $\nu_i \to \nu$ with $\langle S, \nu \rangle = 0$ [H12].

Definition 2.3. A pair of measured laminations $(\mu,\nu)\in\mathcal{ML}\times\mathcal{ML}$ is called a *positive pair* if for any tree $S\in\overline{cv(F_n)}$ we have $\langle S,\mu+\nu\rangle>0$.

Positivity of a pair is invariant under scaling each individual component by a positive factor, so it is defined for pairs of projective measured laminations.

The following is Corollary 10.6 of [H12].

Lemma 2.4. *Let $[T]\neq[T']$ be arational trees which define distinct points in $\partial\mathcal{FF}$. Let $\mu,\mu'\in\mathcal{ML}$ be dual to $[T],[T']$; then (μ,μ') is a positive pair.*

2.2. **Lines of minima.** In this subsection we introduce the central tool used in this paper: *lines of minima* as defined in [H14a].

Let $\epsilon>0$ be a sufficiently small number and let

$$\mathrm{Thick}_\epsilon(F_n)$$

be the set of all trees $T\in cv_0(F_n)$ with volume one quotient so that the shortest length of any loop on T/F_n is at least ϵ.

For a tree $T\in cv_0(F_n)$ define

$$\Lambda(T)=\{\mu\in\mathcal{ML}\mid\langle T,\mu\rangle=1\}.$$

Then $\Lambda(T)$ is a compact subset of $\mathcal{ML}$, and the projection $\Lambda(T)\to\mathcal{PML}$ is a homeomorphism. Let moreover

$$\Sigma(T)=\{S\in cv(F_n)\mid\sup\{\langle S,\mu\rangle\mid\mu\in\Lambda(T)\}=1\}.$$

For a positive pair $(\mu,\nu)\in\mathcal{ML}\times\mathcal{ML}$ define

$$\mathrm{Bal}(\mu,\nu)=\{T\mid\langle T,\mu\rangle=\langle T,\nu\rangle\}\subset cv(F_n).$$

Let $(\mu,\nu)\in\mathcal{ML}^2$ be a positive pair. By Lemma 3.2 of [H14a], the function $S\to\langle S,\mu+\nu\rangle$ on $\mathrm{Thick}_\epsilon(F_n)$ is proper. This means that this function assumes a minimum, and the set

$$\mathrm{Min}_\epsilon(\mu+\nu)=\{T\in\mathrm{Thick}_\epsilon(F_n)\mid\langle T,\mu+\nu\rangle=\min\{\langle S,\mu+\nu\rangle\mid S\in\mathrm{Thick}_\epsilon(F_n)\}\}.$$

of all such minima is compact. Note that this set does not change if we replace $\mu+\nu$ by a positive multiple.

Call a primitive conjugacy class α *basic* for $T\in cv_0(F_n)$ if α can be represented by a loop of length at most two on the quotient graph T/F_n. Note that any $T\in cv_0(F_n)$ admits a basic primitive conjugacy class.

Definition 2.5. For $B>1$, a positive pair of points

$$([\mu],[\nu])\in\mathcal{PML}(F_n)\times\mathcal{PML}(F_n)-\Delta$$

is called *B-contracting* if for any pair $\mu,\nu\in\mathcal{ML}(F_n)$ of representatives of $[\mu],[\nu]$ there is some "distinguished" $T\in\mathrm{Min}_\epsilon(\mu+\nu)$ with the following properties.

1. $\langle T,\mu\rangle/\langle T,\nu\rangle\in[B^{-1},B]$.
2. If $\tilde\mu,\tilde\nu\in\Lambda(T)$ are representatives of $[\mu],[\nu]$ then $\langle S,\tilde\mu+\tilde\nu\rangle\geq 1/B$ for all $S\in\Sigma(T)$.

(3) Let $\mathcal{B}(T) \subset \Lambda(T)$ be the set of all normalized measured laminations which are up to scaling induced by a basic primitive conjugacy class for a tree $U \in \mathrm{Bal}(\mu, \nu)$. Then $\langle S, \xi \rangle \geq 1/B$ for every $\xi \in \mathcal{B}(T)$ and every tree

$$S \in \Sigma(T) \cap \left(\bigcup_{s \in (-\infty, -B) \cup (B, \infty)} \mathrm{Bal}(e^s \mu, e^{-s} \nu) \right).$$

Note that the requirement in part 3) of the definition is slightly stronger than in [H14a] as in [H14a] it was assume that the tree U is contained in $\mathrm{Thick}_\epsilon(F_n)$). We will establish below that this stronger property serves our needs.

Each B-contracting pair $(\mu, \nu) \in \mathcal{ML} \times \mathcal{ML}$ (i.e. the projectivized pair $([\mu], [\nu])$ is B-contracting in the sense of Definition 2.5) defines a *line of minima* γ by associating to each $t \in \mathbb{R}$ a point $\gamma(t) \in \mathrm{Min}_\epsilon(e^{t/2}\mu + e^{-t/2}\nu)$ which fulfills the above definition. Such a line of minima γ is not unique, but its Hausdorff distance (for the two-sided Lipschitz metric introduced below) to any other choice defined by any pair $(\hat{\mu}, \hat{\nu}) \in \mathcal{ML} \times \mathcal{ML}$ with $[\hat{\mu}] = [\mu]$ and $[\hat{\nu}] = [\nu]$ is uniformly bounded (see [H14a] for details).

Definition 4. Let (μ, ν) be a B-contracting pair and γ an associated line of minima. We define the *balancing projection* $\Pi_\gamma : cv_0(F_n) \to \gamma$ in the following way.

Given a tree $T \in cv_0(F_n)$, there is a unique number t so that $\langle T, e^{t/2}\mu \rangle = \langle T, e^{-t/2}\nu \rangle$. We then put $\Pi_\gamma(T) = \gamma(t)$ for that t.

The *one-sided Lipschitz metric* between two trees $S, T \in cv_0(F_n)$ is defined as

$$d_L(S, T) = \log \sup \left\{ \frac{\langle T, \nu \rangle}{\langle S, \nu \rangle} \mid \nu \in \mathcal{ML} \right\}.$$

The one-sided Lipschitz metric satisfies $d_L(S, T) = 0$ only if $S = T$, moreover it satisfies the triangle inequality, but it is not symmetric. Define the *two-sided Lipschitz metric*

$$d(S, T) = d_L(S, T) + d_L(T, S).$$

Proposition 5.2 of [H14a] shows the following.

Proposition 2.6. *For every $B > 0$ there is a number $\kappa = \kappa(B) > 0$ with the following property. Let $([\mu], [\nu])$ be a B-contracting pair, let γ be an axis for $([\mu], [\nu])$ and let $T \in cv_0(F_n)$.*

(1) If $S \in cv_0(F_n)$ is such that $d(\Pi_\gamma(T), \Pi_\gamma(S)) \geq \kappa$ then

$$d_L(T, S) \geq d_L(T, \Pi_\gamma(T)) + d_L(\Pi_\gamma(T), \Pi_\gamma(S)) + d_L(\Pi_\gamma(S), S) - \kappa.$$

(2) If $S \in cv_0(F_n)$ is such that $d(\Pi_\gamma(T), \Pi_\gamma(S)) \geq \kappa$ then

$$d(T, S) \geq d(T, \Pi_\gamma(T)) + d(\Pi_\gamma(T), \Pi_\gamma(S)) + d(\Pi_\gamma(S), S) - \kappa.$$

(3) If $S \in \gamma(\mathbb{R})$ is such that $d(T, S) \leq \inf_t d(T, \gamma(t)) + 1$ then $d(S, \Pi_\gamma(T)) \leq \kappa$.

(4) For all $s < t$,

$$|s - t| - \kappa \leq d(\gamma(s), \gamma(t)) \leq |s - t| + \kappa.$$

Proposition 5.2 as stated in [H14a] requires that the tree T is contained in $\text{Thick}_\epsilon(F_n)$. However, since we are using a stronger notion of B-contracting pair in this article, the proof given in [H14a] shows Proposition 2.6 without modification.

3. Lines of minima and their shadows

The goal of this section is to show that lines of minima are Morse coarse geodesics for the symmetrized Lipschitz distance whose shadows in the free factor graph are parametrized quasi-geodesics.

Fix once and for all a number $\epsilon > 0$ which is sufficiently small that all results in Subsection 2.2 hold true. Let

$$\Upsilon : cv_0(F_n) \to \mathcal{FF}$$

be a map which associates to a tree T the free factor generated by some basic primitive element for T (i.e. a primitive element $\alpha \in F_n$ so that it can be represented by a loop on T/F_n of length at most two).

Lemma 3.1. *For every $B > 0$ there is an $R > 0$ with the following property. Let $\gamma \subset \text{Thick}_\epsilon(F_n)$ be a line of minima defined by a B-contracting pair and let α be a primitive element of F_n.*

Suppose that $T, T' \in \text{Thick}_\epsilon(F_n)$ are two trees for which α is basic. Then

$$d(\Pi_\gamma(T), \Pi_\gamma(T')) \le R.$$

Proof. Recall that both T and T' are normalized so that the volume of the quotient graph $T/F_n, T'/F_n$ is 1. Let μ, ν be the measured laminations defining the line of minima γ, normalized so that $T \in \text{Bal}(\mu, \nu)$ and hence $\Pi_\gamma(T) = \gamma(0)$.

Suppose that $d(\Pi_\gamma(T), \Pi_\gamma(T')) \ge R > B + \kappa$ where $\kappa > 0$ is as in Proposition 2.6. By (4) of Proposition 2.6, this implies that $T' \in \text{Bal}(e^s\mu, e^{-s}\nu)$ for $|s| > B$.

Let $c > 0$ be so that $cT' \in \Sigma(\gamma(0))$. By property (3) in Definition 2.5 we have

$$\langle T', \alpha\rangle / \langle \gamma(0), \alpha\rangle \ge 1/Bc.$$

As $\gamma(0) \in \text{Thick}_\epsilon(F_n)$ we have $\langle \gamma(0), \alpha\rangle \ge \epsilon$ and therefore

$$2 \ge \langle T', \alpha\rangle \ge \epsilon/Bc.$$

In particular, $1/c \le 2B/\epsilon$.

On the other hand, from the definitions (see the detailed discussion in Section 4 of [H14a]), we get

$$d_L(\gamma(0), T') = 1/c.$$

Now by Proposition 2.6, a large distance of the projections $\Pi_\gamma(T) = \gamma(0)$ and $\Pi_\gamma(T')$ implies a large distance between $\gamma(0)$ and T', and hence a small c. This contradicts that $1/c \le 2B/\epsilon$ and finishes the proof. □

Proposition 3.2. *For every $B > 0$ there is a number $L = L(B) > 0$ with the following property. If $\gamma \subset \text{Thick}_\epsilon(F_n)$ is a line of minima defined by a B-contracting pair then the image of γ under Υ is an L-quasi-geodesic in $\mathcal{FF}$.*

Proof. Let $(\mu,\nu)\in\mathcal{ML}^2$ be a B-contracting pair with associated line of minima γ. Let

$$\mathcal{P}\subset F_n$$

be the collection of all primitive elements of F_n. Define a map

$$\Psi:\mathcal{P}\to\gamma$$

by associating to $\alpha\in\mathcal{P}$ a point $\Psi(\alpha)=\gamma(t)$ as follows. Choose a tree $T\in \mathrm{Thick}_\epsilon(F_n)$ such that α is basic for T. Define $\Psi(\alpha)=\Pi_\gamma(T)$ where $\Pi_\gamma: cv_0(F_n)\to \gamma$ is the balancing projection. By Lemma 3.1, this is a coarsely well-defined map.

Each element $\alpha\in\mathcal{P}$ generates a rank one free factor of F_n and hence $\mathcal{P}$ can be viewed as a subset of the vertex set of the free factor graph.

We claim that the map Ψ is a coarse R-Lipschitz retraction where $R=R(B)>0$ is as in Lemma 3.1. To this end let $\alpha,\beta\in\mathcal{P}$ be primitive elements which generate rank one free factors $\langle\alpha\rangle$ and $\langle\beta\rangle$ of distance two in the free factor graph. Up to conjugation, there is a proper free factor A of F_n so that $\langle\alpha\rangle<A,\langle\beta\rangle<A$. As a consequence, there is a primitive element ζ such that α,ζ and ζ,β can be completed to a free basis of F_n.

Choose a tree $T\in\mathrm{Thick}_\epsilon(F_n)$ so that both α,ζ are primitive basic for T, and choose a tree $S\in\mathrm{Thick}_\epsilon(F_n)$ so that both ζ,β are primitive basic for S. By Lemma 3.1, $\Psi(\alpha)$ and $\Psi(\zeta)$ as well as $\Psi(\zeta)$ and $\Psi(\beta)$ are R-close to each other. Hence, $\Psi(\alpha)$ and $\Psi(\beta)$ are $2R$-close.

Applying Lemma 3.1 again, we see that

$$d(\Psi(\xi),\gamma(t))\leq R$$

for any primitive basic element ξ for $\gamma(t)$. Thus for all t we have

$$d(\Psi\circ\Upsilon(\gamma(t)),\gamma(t))\leq R.$$

To summarize, the map Ψ is a $2R$-Lipschitz retraction for the distance function on $\mathcal{P}$ inherited from the distance $d_{\mathcal{FF}}$ on $\mathcal{FF}$ and the restriction to γ of the symmetrized Lipschitz metric on $cv_0(F_n)$. Here the number $R>0$ only depends on B.

By definition, the two-neighborhood of $\mathcal{P}$ is all of $\mathcal{FF}$. Thus Ψ can be extended to a coarsely well defined Lipschitz retraction from $\mathcal{FF}$ into γ.

Now the map $\Upsilon: cv_0(F_n)\to\mathcal{FF}$ is coarsely M-Lipschitz for some number $M>0$, i.e. we have $d_{\mathcal{FF}}(\Upsilon T,\Upsilon T')\leq Md(T,T')+M$ for all $T,T'\in cv_0(F_n)$ (see Section 2 of [H12] for a detailed discussion of this fact). As a consequence, $\Upsilon\circ\Psi$ is a coarsely MR-Lipschitz retraction of $\mathcal{FF}$ onto $\Upsilon(\gamma)$. In particular, it maps a point on $\Upsilon(\gamma)$ to a point of distance at most MR.

This shows that $\Upsilon(\gamma)$ is a parametrized $2MR$-quasi-geodesic in $\mathcal{FF}$. Namely, for $s<t$ let $g:[0,N]\to\mathcal{FF}$ be a simplicial geodesic joining $\Upsilon(\gamma(s))$ to $\Upsilon(\gamma(t))$. The retractions $\Psi(g(i))$ are points on γ which are of distance at most $2MR$ apart, and the endpoints $\Psi(g(0)),\Psi(g(N))$ are of distance at most $2MR$ from $\gamma(s)$ and

$\gamma(t)$. Thus, the length of the segment between $\gamma(s)$ and $\gamma(t)$ is at most $2MRN + 4MR$. □

Recall from Section 2.1 that the Gromov boundary $\partial\mathcal{FF}$ of $\mathcal{FF}$ is a set of equivalence class of projective trees in $\partial\mathrm{CV}(F_n)$. The equivalence relation $\sim$ is such that two trees $[S],[T]$ are equivalent if there is a measured lamination μ dual to both $[T],[S]$. As a consequence of Proposition 3.2, if (μ,ν) is a positive pair defining a line of minima, then both μ,ν define an equivalence class in $\partial\mathrm{CV}(F_n)$ for this relation $\sim$.

The next corollary is immediate from Lemma 3.2.

Corollary 3.3. *If (μ,ν) is a positive pair defining a line of minima then μ,ν define an equivalence class of a boundary point of $\mathcal{FF}$.*

The following definition is taken from Section 2 of [H14a].

Definition 3.4. A projective tree $[T] \in \partial\mathrm{CV}(F_n)$ is *doubly uniquely ergodic* if the following two conditions are satisfied.

(1) There exists a unique projective measured lamination $[\mu] \in \mathcal{PML}$ which is dual to $[T]$.
(2) If $[\mu]$ is dual to $[T]$ and if $[S]$ is dual to $[\mu]$ then $[S] = [T]$.

Denote by $\mathcal{UT} \subset \partial\mathrm{CV}(F_n)$ the $\mathrm{Out}(F_n)$-invariant set of doubly uniquely ergodic trees. Lemma 2.9 of [H14a] shows that a fixed point in $\partial\mathrm{CV}(F_n)$ of any iwip element of $\mathrm{Out}(F_n)$ is contained in $\mathcal{UT}$. Moreover, by Corollary 2.9 of [H14a], the action of $\mathrm{Out}(F_n)$ on the closure of $\mathcal{UT}$ is minimal.

The following is an immediate consequence of Proposition 3.2 and the main results of [NPR14].

Proposition 3.5. *An arational projective tree $[T] \in \partial\mathrm{CV}(F_n)$ which is an endpoint of a line of minima is doubly uniquely ergodic.*

The following is a reformulation of the "only if" implication in Theorem 2.

Corollary 3.6. *Let $\Gamma < \mathrm{Out}(F_n)$ be a word hyperbolic subgroup with the following properties.*

(1) There is a Γ-equivariant homeomorphism of the Gromov boundary $\partial\Gamma$ onto a compact subset Λ of $\mathcal{UT}$.
(2) There is some $B > 0$ so that for any two points $[S] \neq [T] \in \Lambda$, the pair of dual projective measured laminations $([\mu],[\nu])$ for $[S],[T]$ is a B-contracting pair.
(3) Γ acts cocompactly on the closure $\mathcal{H}_\Gamma$ of the union of all lines of minima defined by pairs of distinct points in Λ.

Then Γ is convex cocompact.

Proof. The set $\mathcal{H}_\Gamma$ is a closed Γ-invariant subset of $\text{Thick}_\epsilon(F_n)$. By assumption, the action of Γ on $\mathcal{H}_\Gamma$ is cocompact. As a consequence, up to replacing $\mathcal{H}_\Gamma$ by its closed one-neighborhood, we may assume that $\mathcal{H}_\Gamma$ is path connected and equipped with a Γ-invariant length metric. As Γ acts on $\mathcal{H}_\Gamma$ cocompactly, for $x \in \mathcal{H}_\Gamma$ the orbit map $g \in \Gamma \to gx$ is a quasi-isometry.

Let $F : \partial\Gamma \to \Lambda \subset \mathcal{UT}$ be the equivariant homeomorphism as in 1) of the corollary. Let γ be a geodesic in Γ with endpoints $\gamma(-\infty) \in \partial\Gamma, \gamma(\infty) \in \partial\Gamma$. There is a corresponding line of minima ζ in $\mathcal{H}_\Gamma$ connecting $F\gamma(-\infty)$ to $F(\gamma(\infty))$, and this line of minima is a c-coarse geodesic in $cv_0(F_n)$ for the symmetrized Lipschitz metric for a fixed number $c > 0$. In particular, it is a c-coarse geodesic in $\mathcal{H}_\Gamma$ equipped with the intrinsic path metric for some number $c' > 0$.

As an orbit map $\Gamma \to \mathcal{H}_\Gamma$ is a quasi-isometry, ζ determines an equivalence class of uniform quasi-geodesics in Γ, where two quasi-geodesics are equivalent if and only if their Hausdorff distance is uniformly bounded. By hyperbolicity, the geodesic γ is contained in this class. As a consequence, for some fixed $x \in \zeta$ the orbit γx is contained in a uniformly bounded neighborhood of ζ. Since the map $\Upsilon : cv_0(F_n) \to \mathcal{FF}$ is coarsely Lipschitz and coarsely $\text{Out}(F_n)$-equivariant and since it maps ζ to a parametrized uniform quasi-geodesic in $\mathcal{FF}$, this shows that an orbit map $g \in \Gamma \to gA \in \mathcal{FF}$ $(A \in \mathcal{FF})$ is a quasi-isometric embedding. □

Recall from the introduction the definition of a Morse coarse geodesic. The next observation shows that lines of minima are Morse.

Lemma 3.7. *For all $B > 0, K > 1$ there is a constant $M = M(B,K) > 0$ with the following property. Let $\gamma \subset \text{Thick}_\epsilon(F_n)$ be a B-contracting line of minima. Then every K-quasi-geodesic $\sigma \subset cv_0(F_n)$ for the one-sided Lipschitz metric or for the symmetrized Lipschitz metric with endpoints on γ is contained in $N_M(\gamma)$.*

Proof. The argument is standard; we follow the clear proof in Lemma 3.3 of [S14].

Namely, let $\Pi_\gamma : \text{Thick}_\epsilon(F_n) \to \gamma$ be the balancing projection. By Proposition 2.6 there is a number $\kappa > 1$ with the following property. If $d(\Pi_\gamma(x), \Pi_\gamma(y)) \geq \kappa$ then

$$d(x,y) \geq d(x, \Pi_\gamma(x)) + d(\Pi_\gamma(x), \Pi_\gamma(y)) + d(\Pi_\gamma(y), y) - \kappa.$$

Assume without loss of generality that σ is continuous. Set $A = 2\kappa K$. Let $[s_1, s_2] \subset [a,b]$ be a maximal connected subinterval such that $d(\sigma(s), \gamma) \geq A$ for all $s \in [s_1, s_2]$.

Let $s_1 = r_1 < \cdots < r_m < r_{m+1} = s_2$ be such that $d(\sigma(r_i), \sigma(r_{i+1})) = 2A$ for $i \leq m$ and $d(\sigma(r_m), \sigma(r_{m+1})) \leq 2A$. Since σ is a K-quasi-geodesic we have

$$d(\sigma(s_1), \sigma(s_2)) \geq (m-1)A/K - K.$$

Since the distance between $\sigma(r_i)$ and γ is at least A, by Proposition 2.6 we have

$$d(\Pi_\gamma(\sigma(r_i)), \Pi_\gamma(\sigma(r_{i+1})) \leq \kappa$$

and hence

$$d(\sigma(s_1), \sigma(s_2)) \leq 2A + (m-1)\kappa.$$

This shows $(m-1)A/K-K \leq 2A+(m-1)\kappa$ and hence $(m-1)(A/K-\kappa) \leq 2A+K$. As $A = 2\kappa K$ we conclude that $(m-1)\kappa \leq 4\kappa K + K$. Then m is uniformly bounded which is what we wanted to show. □

4. Limit sets of convex cocompact groups

In this section we show that a convex cocompact subgroup Γ of $\mathrm{Out}(F_n)$ satisfies properties i)- iv) in Theorem 2.

Let $\Gamma < \mathrm{Out}(F_n)$ be convex cocompact. Then Γ is finitely generated, and for one (and hence any) $A \in \mathcal{FF}$ the orbit map $g \in \Gamma \to gA \in \mathcal{FF}$ is a quasi-isometric embedding. As $\mathcal{FF}$ is hyperbolic, this implies that Γ is word hyperbolic. Moreover, the Gromov boundary $\partial\Gamma$ of Γ admits a Γ-equivariant embedding into $\partial\mathcal{FF}$. We denote by

$$Q_\Gamma \subset \partial\mathcal{FF}$$

its image. Since $\partial\Gamma$ is compact, the set Q_Γ is closed, Γ-invariant and minimal for the Γ-action.

Each point in Q_Γ is an equivalence class of arational trees and hence it determines a non-empty set of dual projective measured laminations. Lemma 2.4 shows that if μ, ν are dual measured laminations for arational trees defining distinct points in $\partial\mathcal{FF}$ then (μ, ν) is a positive pair.

Proposition 4.1. *Let $\Gamma < \mathrm{Out}(F_n)$ be convex cocompact. There is a number $B > 0$ with the following property. Let $(\mu, \nu) \in \mathcal{ML}^2$ be a pair of measured laminations which are dual to projective trees defining distinct points in Q_Γ. Then (μ, ν) is a B-contracting pair. For $R > 0$ the closed R-neighborhood of the union of all lines of minima obtained in this way is Γ-invariant and Γ-cocompact.*

Proof. Since $\partial\Gamma$ is Γ-equivariantly homeomorphic to the set Q_Γ, the group Γ acts cocompactly on the space of triples of pairwise distinct points in Q_Γ.

Let $\mathcal{FT} \subset \partial\mathrm{CV}(F_n)$ be the $\mathrm{Out}(F_n)$-invariant set of arational trees and let

$$\Pi : \mathcal{FT} \to \partial\mathcal{FF}$$

be the natural equivariant projection.

Write $\Theta = \Pi^{-1}(Q_\Gamma)$; we claim that Θ is a compact Γ-invariant subset of $\partial\mathrm{CV}(F_n)$. Namely, as $\partial\mathrm{CV}(F_n)$ is metrizable, it suffices to show that Θ is sequentially compact. To this end take a sequence $[T_i] \subset \Theta$ which limits to a tree $[T] \in \partial\mathrm{CV}(F_n)$. As $\mathcal{Q}_\Gamma$ is compact, up to subsequence, there is an element $\xi \in \mathcal{Q}_\Gamma$ so that $\Pi([T_i])$ converges to ξ. Now for each i choose a measured lamination ν_i dual to $[T_i]$. Since $\mathcal{PML}$ is compact, up to passing to a subsequence and normalization we may assume that $\nu_i \to \nu$. By continuity of the length pairing, $[T]$ is dual to ν. On the other hand, as $\Pi([T_i]) \to \xi$, we have $\langle S, \nu\rangle = 0$ if and only if $[S] \in \mathcal{FT}$ and $\Pi(S) = \xi$. Thus indeed, Θ is compact.

Denote by $\Delta \subset \partial\mathrm{CV}(F_n)^3$ the fat diagonal. The action of Γ on $Q_\Gamma^3 - \Delta$ is cocompact (as it is equivariantly homeomorphic to the action of Γ on $\partial\Gamma^3 - \Delta$) and therefore there is a compact fundamental domain $C \subset Q_\Gamma^3 - \Delta$.

Let $A = \Pi^{-1}(C) \subset \Theta \times \Theta \times \Theta - \Delta$ where by abuse of notation, Π also denotes the product projection $\Theta^3 \to Q_\Gamma^3$. By equivariance, A is a fundamental domain for the action of Γ on

$$M = \{([T_1], [T_2], [T_3]) \subset \Theta^3 - \Delta \mid \Pi([T_1], [T_2], [T_3]) \in Q_\Gamma^3 - \Delta\}.$$

Additionally, A is compact, arguing as above with the length pairing.

Since Θ is Γ-invariant, the group Γ acts diagonally on the set

$$D = \{(\mu_1, \mu_2, \mu_3) \in \mathcal{ML}^3 \mid \langle T_i, \mu_i \rangle = 0 \text{ for some } ([T_1], [T_2], [T_3]) \in M\}.$$

We define a subset

$$\Xi_0 = \{(\mu_1, \mu_2, \mu_3) \in \mathcal{ML}^3 \mid \langle T_i, \mu_i \rangle = 0 \text{ for some } ([T_1], [T_2], [T_3]) \in A\}.$$

The set Ξ_0 is closed, and we claim that it contains a fundamental domain for the action of Γ on D. Namely, let $(\mu_1, \mu_2, \mu_3) \in D$ be arbitrary, and let $([T_1], [T_2], [T_3])$ be a corresponding triple of trees. As A is a fundamental domain of the action of Γ, there is an element $\gamma \in \Gamma$ so that $(\gamma[T_1], \gamma[T_2], \gamma[T_3]) \in A$. Then

$$\gamma(\mu_1, \mu_2, \mu_3) = (\gamma\mu_1, \gamma\mu_2, \gamma\mu_3)$$

is dual to $(\gamma[T_1], \gamma[T_2], \gamma[T_3]) \in A$ since the length pairing is $\mathrm{Out}(F_n)$-invariant. Hence $\gamma(\mu_1, \mu_2, \mu_3) \in \Xi_0$, which shows the claim.

Construct a closed subset Ξ_1 of Ξ_0 as follows. A point $(\mu_1, \mu_2, \mu_3) \in \Xi_0$ is contained in Ξ_1 if there is some $([T_1], [T_2], [T_3]) \in A$ such that $\langle T_i, \mu_i \rangle = 0$ ($i = 1, 2, 3$) and that

$$\langle T_3, \mu_1 \rangle = \langle T_3, \mu_2 \rangle. \tag{1}$$

This makes sense since by assumption, $[T_3]$ is not equivalent to $[T_1]$ and $[T_2]$ and therefore $\langle T_3, \mu_i \rangle > 0$ for $i = 1, 2$. Note that this requirement determines the pair (μ_1, μ_2) up to a common scaling. Moreover, the identity (1) is invariant under scaling of the representative T_3 of $[T_3]$, i.e. it only depends on the projective class $[T_3]$.

Fix a point $T \in \mathrm{Thick}_\epsilon(F_n)$ and let $\Xi_2 \subset \Xi_1$ be the closed set of all triples with the additional property that $\langle T, \mu_1 + \mu_2 \rangle = 1$. Since $\{\mu \in \mathcal{ML} \mid \langle T, \mu \rangle = 1\}$ is a compact subset of $\mathcal{ML}$, the set

$$K = \{(\mu_1, \mu_2) \mid (\mu_1, \mu_2, \mu_3) \in \Xi_2 \text{ for some } \mu_3\}$$

is a compact subset of $\mathcal{ML} \times \mathcal{ML}$ consisting of positive pairs.

By Lemma 3.2 of [H14a], the family of functions $\{\langle\,, \mu + \nu\rangle \mid \mu + \nu \in K\}$ on $cv_0(F_n)$ is uniformly proper. Thus the closure

$$W = \overline{\bigcup_{(\mu_1, \mu_2) \in K} \mathrm{Min}_\epsilon(\mu_1 + \mu_2)}$$

is compact.

Note that the set W only depends on Ξ_1, i.e. on the identity (1), but not on the choice of the normalizing basepoint T. By equivariance, W is a compact fundamental domain for the action of Γ on

$$Z = \overline{\cup \mathrm{Min}_\epsilon(\mu + \nu)}$$

where the union is over all pairs of measured laminations which are dual to trees defining distinct points in Q_Γ.

Our goal is to show that each $(\mu,\nu)\in K$ is a B-contracting pair for some fixed number $B>0$. To this end we now use an argument from the proof of Proposition 3.8 of [H14a]. Namely, using the above notation, by continuity there is a number $B_1>0$ such that

$$\frac{\langle T,\mu\rangle}{\langle T,\nu\rangle}\in[B_1^{-1},B_1]$$

for all $T\in W$ and all $(\mu,\nu)\in K$. Together with equivariance, this implies the first requirement in the definition of a B-contracting pair.

For $S\in \mathrm{Thick}_\epsilon(F_n)$ let

$$\Lambda(S)=\{\nu\in\mathcal{ML}\mid \langle S,\nu\rangle=1\}.$$

If $S\in W$ and if $\tilde\mu,\tilde\nu\in\Lambda(S)$ are rescalings of $(\mu,\nu)\in K$ then using once more positivity, continuity and compactness, we have $\langle U,\tilde\mu+\tilde\nu\rangle\geq 1/B_2$ for all

$$U\in\Sigma(S)=\{V\mid \max\{\langle V,\nu\rangle\mid \nu\in\Lambda(S)\}=1\}$$

where $B_2>0$ does not depend on $S\in W$ and $(\mu,\nu)\in K$. This shows the second statement in the definition of a B-contracting pair.

For measured laminations $\mu,\nu\in\mathcal{ML}$ as before let

$$\mathrm{Bal}(\mu,\nu)=\{S\in cv_0(F_n)\mid \langle S,\mu\rangle=\langle S,\nu\rangle\}.$$

We claim that if $[T],[T']$ is a pair of projective arational trees defining two distinct boundary points of $\mathcal{FF}$ and if μ,ν are two measured laminations supported in the zero lamination of T,T' then the sets

$$U(p)=\{[S]\in\overline{[\mathrm{Thick}_\epsilon(F_n)]}\mid S\in \mathrm{Bal}(e^t\mu,e^{-t}\nu)\text{ for some }t>p\}$$

$(p>0)$ form a neighborhood basis in $\overline{[\mathrm{Thick}_\epsilon(F_n)]}$ for the set of all projective trees which are equivalent to $[T]$. By this we mean that for any open set $\mathcal{U}\subset\overline{[\mathrm{Thick}_\epsilon(F_n)]}$ which contains the set of all projective trees equivalent to $[T]$, we have $U(p)\subset\mathcal{U}$ for all sufficiently large p.

Namely, fix a tree $V\in\mathrm{Thick}_\epsilon(F_n)$. For $t\geq 0$ let

$$\beta(t)=e^t\mu+e^{-t}\nu/\langle V,e^t\mu+e^{-t}\nu\rangle.$$

Then $\{\beta(t)\mid t\geq 0\}$ is a compact subset of the set of all *currents* for F_n, i.e. F_n-invariant locally finite Borel measures on $\partial F_n\times\partial F_n-\Delta$. As $t\to\infty$, we have

$$\beta(t)\to\hat\mu=\mu/\langle V,\mu\rangle$$

in the space of currents equipped with the weak*-topology [H12]. As $\hat\mu$ is dual to an arational tree, we have $\langle S,\hat\mu\rangle=0$ if and only if $[S]$ is equivalent to $[T]$. The above claim now follows once more from continuity of the length pairing (as a pairing between F_n-trees and currents, see [KL09]).

Let $T\in\mathrm{Min}_\epsilon(\mu+\nu)$ and assume that the first and the second property in the definition of a B-contracting pair hold true for T. Let

$$\mathcal{B}(T)\subset\Lambda(T)$$

be the closure of the set of all normalized measured laminations which are up to scaling induced by a basic primitive conjugacy class for a tree $U \in \mathrm{Bal}(\mu, \nu)$. Then $\mathcal{B}(T)$ is a compact subset of $\Lambda(T)$ which does not contain the representatives $\hat{\mu}, \hat{\nu} \in \Lambda(T)$ of the measured laminations μ, ν.

Let $D(\mu), D(\nu) \subset \Sigma(T)$ be the set of all normalized arational trees which are dual to μ, ν. By continuity of the length pairing, the set of functions

$$\mathcal{F} = \{U \to \langle U, \zeta \rangle \mid \zeta \in \mathcal{B}(T)\}$$

is compact for the compact open topology on the space of continuous functions on $\Sigma(T)$. Thus by the above discussion, their values on the set $D = D(\mu) \cup D(\nu)$ are bounded from below by a positive number $c > 0$.

By continuity, there is some $p > 0$ so that these functions are bounded from below by $c/2$ on $\tilde{U}(p) = \{S \in \Sigma(T) \mid [S] \in U(p)\}$. Note that $\tilde{U}(p)$ is a neighborhood of D in $\Sigma(T)$. In the same way we can construct a neighborhood $\tilde{V}(q) \subset \Sigma(T)$ of the set of trees whose projective classes are equivalent to $[T']$ so that the functions from $\mathcal{F}$ are bounded from below on $\tilde{V}(q)$ by $c/2$. As a consequence, property (3) in Definition 2.5 holds true for T and for $B = \max\{p, q, 2/c\}$.

Now by compactness and continuity of the length pairing, the same property with the constant $2B$ holds true for pairs (μ', ν') in a small neighborhood of (μ, ν) and for trees S in a small neighborhood of T. As the set K is compact and hence the same holds true for

$$Q = \{((\mu, \nu), S) \in K \times W \mid S \in \mathrm{Min}_\epsilon(\mu + \nu)\}$$

it can be covered by finitely many open sets which are controlled in this way. The proposition now follows by invariance under the action of Γ. □

Corollary 4.2. *The limit set of a convex cocompact group is contained in the set $\mathcal{UT}$, and it is equivariantly homeomorphic to $\partial\Gamma$.*

Proof. By Proposition 3.5, the endpoint of a line a minima is contained in $\mathcal{UT}$. The above construction then shows that for a convex cocompact group Γ, the image of $\partial\Gamma$ under the canonical embedding $\partial\Gamma \to \partial\mathcal{FF}$ is contained in the image of the set $\mathcal{UT}$.

Now the restriction of the projection $\mathcal{FT} \to \partial\mathcal{FF}$ to the set $\mathcal{UT}$ is a homeomorphism onto its image. Thus the Γ-equivariant embedding $\partial\Gamma \to \partial\mathcal{FF}$ lifts to a Γ-equivariant embedding $\partial\Gamma \to \mathcal{UT}$. □

5. Morse

The goal of this section is to complete the proof of Theorem 3.

To this end let as before d_L be the one-sided Lipschitz metric on $cv_0(F_n)$. The metric d_L is not complete. Its completion consists of all simplicial F_n-trees $T \in \overline{cv_0(F_n)}$ with volume one quotient and no nontrivial edge stabilizers [A12].

As before, write

$$d(S,T) = d_L(S,T) + d_L(T,S).$$

Then d is a complete $\mathrm{Out}(F_n)$-invariant distance on $cv_0(F_n)$ [FM11].

By [AB12], for every $\delta > 0$ there is a number $c = c(\delta) > 0$ such that $d(S,T) \leq c d_L(S,T)$ for all $S, T \in \mathrm{Thick}_\delta(F_n)$. In particular, for both orientations and for any $C > 1$, any C-quasi-geodesic for d which is entirely contained in $\mathrm{Thick}_\delta(F_n)$ is a uniform quasi-geodesic for the one-sided Lipschitz metric as well.

Definition 5.1. A coarse geodesic $\gamma \subset \mathrm{Thick}_\epsilon(F_n)$ for d is *strongly Morse* if for any $K > 0$ there exists some M with the following property. The Hausdorff distance for the two-sided Lipschitz distance between γ and any K-quasi-geodesic for d_L with endpoints on γ is a most M.

As fast folding paths are geodesics for d_L and since any two points in $cv_0(F_n)$ can be connected by a fast folding path [FM11], this implies that a fast folding path with ordered endpoints on a strongly Morse coarse geodesic γ is contained in the M-neighborhood of γ where $M > 0$ only depends on the constants entering the definition of a strongly Morse coarse geodesic.

The next proposition is the main remaining step towards the proof of Theorem 3.

Proposition 5.2. *If $\gamma : \mathbb{R} \to \mathrm{Thick}_\epsilon(F_n)$ is strongly Morse then γ is contained in a uniformly bounded neighborhood of a B-contracting line of minima.*

Proof. Let $\gamma : [a,b] \to \mathrm{Thick}_\epsilon(F_n)$ be a strongly Morse coarse geodesic for the symmetrized Lipschitz distance d. Then γ is contained in the M'- neighborhood for the symmetrized Lipschitz metric of a fast folding path ζ with the same endpoints where $M > 0$ depends on the constants in Definition 5.1.

As the symmetrized Lipschitz distance on $cv_0(F_n)$ is complete [FM11], this implies that $\zeta \subset \mathrm{Thick}_\delta(F_n)$ for a number $\delta > 0$ only depending on ϵ and M'. Moreover, ζ is a geodesic for the one-sided Lipschitz metric which is an M-Morse quasi-geodesic in the sense of Definition 5.1 where M only depends on M'. We call such a fast folding path *M-stable.*

Shadows of folding paths in $\mathcal{FF}$ are uniform unparametrized quasi-geodesics [BF14]. By Lemma 2.6 of [H10] and its proof (more precisely, the last paragraph of the proof which is valid in the situation at hand without modification), it therefore suffices to show that for every $m > 0$ there is some number $k > 0$ so that the endpoints of any M-stable fast folding path in $\mathrm{Thick}_\delta(F_n)$ whose d_L-length is at least k are mapped by the map $\Upsilon : cv_0(F_n) \to \mathcal{FF}$ to points of distance at least m.

Assume to the contrary that this is not true. Then there is a number $m > 0$ and there is sequence β_i of M-stable fast folding paths in $\mathrm{Thick}_\delta(F_n)$ of length i whose endpoints are mapped by Υ to points in $\mathcal{FF}$ of distance at most m.

As the $\mathrm{Out}(F_n)$-action on $\mathrm{Thick}_\delta(F_n)$ is cocompact, by invariance under $\mathrm{Out}(F_n)$ and the Arzela-Ascoli theorem for folding paths [H12], up to passing to a subsequence we may assume that the paths β_i converge as $i \to \infty$ to a limiting folding path β. The path β is contained in $\mathrm{Thick}_\delta(F_n)$, it connects a basepoint to a tree $[T] \in \partial \mathrm{CV}(F_n)$, and it is guided by a *train track map* $f : \beta(0) \to T$ (see [H12] for details). Moreover, β is M-stable.

Our goal is to show that the tree T is arational. To do so, we will show that it does not have point stabilizers containing free factors, and that there is no tree $T' \in \partial \mathrm{CV}(F_n)$ which can be obtained from T by a one-Lipschitz alignment preserving map $\rho : T \to T'$ collapsing a nontrivial subtree of T to a point. Together, by [BR12, H12], this will imply arationality of T.

Recall that an *alignment preserving map* between two F_n-trees $T, T' \in \overline{cv(F_n)}$ is an equivariant map $\rho : T \to T'$ with the property that $x \in [y, z]$ implies $\rho(x) \in [\rho(y), \rho(z)]$. An F_n-equivariant map $\rho : T \to T'$ is alignment preserving if and only if the preimage of every point in T' is convex. The map ρ is continuous on segments.

Assume for contradiction that there is such a map $\rho : T \to T'$. Let $\overline{cv_0(F_n)}$ be the space of simplicial F_n-trees with volume one quotients which are not necessarily free. By Proposition 4.3 of [H12], there is a generalized folding path $\zeta \subset \overline{cv_0(F_n)}$ (i.e. folding may not be with constant speed, and there may be rest intervals as well) which connects a point $\zeta(0) \in \overline{cv_0(F_n)}$ to T' and with the following additional property. For each t, $\zeta(t)$ can be obtained from $\beta(t)$ by reducing the lengths of some edges of $\beta(t)$ and renormalizing the volume of the resulting quotient tree. Doing this length reduction and rescaling uniformly on the time interval $[0, 1]$ defines for each t a path $H_t : [0, 1] \to \overline{cv_0(F_n)}$ connecting $\beta(t)$ to $\zeta(t)$ which moreover depends continuously on t. As the length reduction of the edges is determined by the map $\rho : T \to T'$, we have $d_L(\zeta(s), \zeta(t)) \leq |s - t|$ for all s, t, moreover $d_L(\beta(t), \zeta(t)) \leq C$ where $C > 0$ is a universal constant (but we can not expect that $d_L(\zeta(t), \beta(t))$ is uniformly bounded).

We claim that $\zeta \subset \mathrm{Thick}_\rho(F_n)$ for some $\rho > 0$ only depending on M. Namely, otherwise along the infinite path ζ we can gradually rescale edges to construct a uniform quasi-geodesic in $cv_0(F_n)$ with endpoints on β which is not contained in a uniformly bounded neighborhood of β. If $\zeta \subset cv_0(F_n)$ then this can be done by replacing for large $\sigma < \tau$ and every $t \in [\sigma, \tau]$ the tree $\zeta(t)$ by $H_t(1 - (t - \sigma)/(\tau - \sigma))$. However, the existence of this path violates stability of β (since the Lipschitz distance of any point outside of Thick_δ to any point in Thick_ϵ is bounded below in terms of $\frac{\epsilon}{\delta}$). If ζ contains points in $\overline{cv_0(F_n)} - cv_0(F_n)$ then the same reasoning can be applied to the path $t \to H_t(\nu)$ for a number ν close to one.

As a consequence there is a number $A > 0$ and for each t there is an A-bilipschitz equivariant map $\beta(t) \to \zeta(t)$. As the existence of an A-bilipschitz ma between two metric spaces X, Y is invariant under rescaling the metric on the spaces X, Y by a common positive factor, by passing to a limit (compare [H12] for details on why this is possible), we obtain an F_n-equivariant A-bilipschitz map $T \to T'$. This violates the assumption that T' is obtained from T by collapsing some non-trivial subtree to a point.

Next we claim that $[T]$ does not have a point stabilizer containing a free factor. As before, we argue by contradiction and we assume otherwise. Choose a primitive element α of shortest translation length in $\beta(0)$ so that α stabilizes a point in T.

Let as before $f : \beta(0) \to T$ be a train track map guiding the folding path β. Then f is an F_n-equivariant edge isometry. Let $\xi \subset \beta(0)$ be an axis for α and let $x \in \xi$ be a vertex of $\beta(0)$ on ξ. As the translation length of α on $\beta(0)$ is positive and f is an edge isometry, we may assume that $f(x)$ is not stabilized by α.

Connect $f(x)$ by a minimal segment s to the fixed point set $\mathrm{Fix}(\alpha)$ of α in T. Let $y \in \mathrm{Fix}(\alpha)$ be the endpoint of s. As in the proof of Lemma 8.1 of [H12], we observe that the geodesic segment in T connecting $f(x)$ to $\alpha f(x)$ passes through y. The turn at x defining the two directions of the axis of α is illegal. This illegal turn is folded along the path β which decreases the translation length of α (since volume renormalization affects all edges of $\beta(0)$). Arguing as before, we conclude that we can connect two far enough points on β by a uniform quasi-geodesic which passes arbitrarily near the boundary of Outer space. As before, this violates the assumption on stability.

It now follows from the results in [BR12, H12] that the tree T is indeed arational. In particular, its shadow in $\mathcal{FF}$ has infinite diameter. As β is a limit of the path β_i and as the map $\Upsilon : cv_0(F_n) \to \mathcal{FF}$ is coarsely Lipschitz, for every $k > 0$ and all sufficiently large i there is a point $\beta_i(t_i)$ so that the distance between $\Upsilon(\beta_i(0))$ and $\Upsilon(\beta_i(t_i))$ is at least k. This is a contradiction to the assumption on the approximating paths and completes the proof of the proposition. □

We are now ready to complete the proof of Theorem 3. Namely, let $\Gamma < \mathrm{Out}(F_n)$ be convex cocompact. Suppose for simplicity that Γ is torsion free. Let $T \in cv_0(F_n)$ be arbitrary. By Proposition 4.1, there is a number $B > 0$, and for all $g, h \in \Gamma$ there is a B-contracting line of minima connecting two points in a uniformly bounded neighborhood of gT, hT. Moreover, this line of minima is contained in a uniformly bounded neighborhood of ΓT. By Lemma 3.7, such a line of minima is an M-Morse coarse geodesic where $M > 1$ only depends on B. Thus a convex cocompact subgroup of $\mathrm{Out}(F_n)$ satisfies the conclusion of Theorem 3.

On the other hand, if $\Gamma < \mathrm{Out}(F_n)$ is a finitely generated group which satisfies the conclusion of Theorem 3 then by Proposition 5.2 and Proposition 3.2, Γ is convex cocompact.

Remark 5.3.
(1) Similar to the case of Teichmüller space with the Weil-Peterssen metric (see [CS13] for a discussion), we believe that there are Morse coarse geodesics in the metric completion of Outer space. Note that by [A12], this metric completion is the space of simplicial F_n-trees with quotient of volume one and with all edge stabilizers trivial.
(2) We do not know whether every fast folding path which is entirely contained in $\mathrm{Thick}_\epsilon(F_n)$ for some ϵ is Morse.

6. Examples

6.1. **Schottky groups.** A *Schottky group* is a finitely generated free convex cocompact subgroup of $\mathrm{Out}(F_n)$. Such groups can be generated by a standard ping-pong construction [KL10, H14a]. Namely, an iwip-element acts with north-south dynamics on $\partial\mathrm{CV}(F_n)$. There is a unique attracting and a unique repelling fixed point. Each of these fixed points is a projective arational tree.

Call α, β *independent* if the fixed point sets for the action of α, β on $\partial\mathrm{CV}(F_n)$ are disjoint. If $\alpha, \beta \in \mathrm{Out}(F_n)$ are independent then there are $k > 0, \ell > 0$ such that α^k, β^ℓ generate a free convex cocompact subgroup of $\mathrm{Out}(F_n)$ ([KL10] and Section 6 of [H14a]). As in [FM02], this construction can be extended to groups generated by an arbitrarily large finite number of independent iwips.

6.2. **Convex cocompact subgroups of mapping class groups.** Let S be a compact surface of genus $g \geq 2$ with one puncture. Let $\mathrm{Mod}(S)$ be the mapping class group of S; then $\mathrm{Mod}(S)$ is the subgroup of $\mathrm{Out}(F_{2g})$ of all outer automorphisms which preserve the conjugacy class of the puncture of S.

Lemma 6.1. *If $\Gamma < \mathrm{Mod}(S)$ is convex cocompact in the sense of Farb-Mosher, then its image in $\mathrm{Out}(F_{2g})$ is convex cocompact in the sense of this article.*

To prove this lemma, we require several combinatorial complexes. For the surface, we require the arc graph $\mathcal{A}(S)$ and the arc-and-curve graph $\mathcal{AC}(S)$ (which is quasi-isometric to the curve graph). By i) of Theorem 1, a subgroup Γ of $\mathrm{Mod}(S)$ is convex cocompact if and only if the orbit map on the arc-and-curve graph is a quasi-isometric embedding.

On the free group side we use the free factor graph $\mathcal{FF}$ and the free splitting graph $\mathcal{FS}$. These four graphs naturally admit maps as follows

$$\begin{array}{ccc} \mathcal{A}(S) & \longrightarrow & \mathcal{FS} \\ \downarrow & & \downarrow \\ \mathcal{AC}(S) & \longrightarrow & \mathcal{FF} \end{array}$$

The map $\mathcal{A}(S) \to \mathcal{FS}$ associates to an arc $a \subset S$ the free factor $\pi_1(S - a)$. Similarly, the map $\mathcal{AC}(S) \to \mathcal{FF}$ associates to an arc or curve on S some primitive element in the complement.

In [HH14] the authors have shown that the map $\mathcal{A}(S) \to \mathcal{FS}$ is a quasi-isometric embedding. We conjecture that the same is true for the map $\mathcal{AC}(S) \to \mathcal{FF}$ – and that claim would immediately imply Lemma 6.1. Here, we instead sketch a proof of that lemma which bypasses this claim.

Proof. All four of these graphs are hyperbolic in the sense of Gromov. What is more, by work in [HOP13] outermost surgery paths (unicorn paths) in $\mathcal{A}(S)$ define unparametrized quasi-geodesics both in $\mathcal{A}(S)$ and in $\mathcal{AC}(S)$. The image of such a surgery path to $\mathcal{FS}$ is an unparametrized quasi-geodesic by work in [HH14]. Furthermore, unparametrized quasi-geodesics in $\mathcal{FS}$ map to unparametrized quasi-geodesics in $\mathcal{FF}$ by work in [KR12].

As a consequence, quasi-geodesics in $\mathcal{AC}(S)$ map to unparametrized quasi-geodesics in $\mathcal{FF}$.

We claim that there for any number $B > 0$ there is a number $K > 0$ with the following property: if any two points $g, g' \in \Gamma$ have distance at least K then $g\alpha, g'\alpha \in \mathcal{FF}$ have distance at least B.

We prove this by contradiction and suppose the conclusion would be false. Then there would be a sequence of geodesic segments γ_n of length n in Γ so that their images in $\mathcal{FF}$ would have bounded diameter (as we already know that the images are unparametrized quasi-geodesics with endpoints of distance at most B). Thus, we can take a limit in Γ to find a infinite geodesic γ_∞ whose image under the orbit map to $\mathcal{FF}$ is bounded. On the other hand, the image under the orbit map to $\mathcal{FS}$ and $\mathcal{AC}(S)$ is unbounded.

Thus, by the work in [H12], the limiting tree of the image of γ_∞ would have a point stabilizer which contains a free factor. On the other hand, the limiting tree is dual to an ending lamination on the surface S, and such trees do not have such stabilizers. This is a contradiction.

To finish the proof it now suffices to take B large enough to guarantee that the images of geodesic in Γ are parametrized quasi-geodesics. □

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