The letter R denotes a (commutative unital) noetherian ring.

Exercise 1. Let M_1 and M_2 be two finitely generated R-modules. Show that

$$\mathrm{Supp}(M_1 \otimes_R M_2) = \mathrm{Supp}(M_1) \cap \mathrm{Supp}(M_2).$$

Exercise 2. Consider the $\mathbb{Z}$-module $N = \bigoplus_{k\in\mathbb{N}} \mathbb{Z}/p^k$. Compute $\mathrm{Supp}_{\mathbb{Z}}(N)$ and $\mathrm{Ann}_{\mathbb{Z}}(N)$.

Exercise 3. Let M be a finitely generated R-module and $\mathfrak{p} \in \mathrm{Spec}(R)$. Show that

$$\mathfrak{p} \in \mathrm{Supp}(M) \Longleftrightarrow \mathrm{Hom}_R(M, R/\mathfrak{p}) \neq 0.$$

Exercise 4. Let k be a field, and $S = k[X_1, X_2, \cdots]/(X_1^2, X_2^2, \cdots)$. Show that S is not noetherian. Compute $\mathrm{Ass}_k(S)$ and $\mathrm{Ass}_S(S)$ (the definition of an associated prime immediately extends to non-noetherian rings).

Exercise 5. Let $x \in R$. For a prime $\mathfrak{p}$ of R, we denote by $x(\mathfrak{p}) \in \kappa(\mathfrak{p}) = R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})$ the image of x. To what (simple) condition on x is each of the following conditions equivalent?

- $x(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Ass}(R)$.
- $x(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in \mathrm{Ass}(R)$.

Exercise 6. (Primary decomposition) Let M be a finitely generated R-module. We are trying to find submodules $Q(\mathfrak{p}) \subset M$ for $\mathfrak{p} \in \mathrm{Ass}(M)$ satisfying

$$\mathrm{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\} \quad \text{and} \quad \bigcap_{\mathfrak{p}\in\mathrm{Ass}(M)} Q(\mathfrak{p}) = 0.$$

(i) Assuming that the $Q(\mathfrak{p})$'s exist, compute $\mathrm{Ass}(Q(\mathfrak{p}))$.

(ii) Show that the $Q(\mathfrak{p})$'s exist.

(iii) If $S \subset R$ is a multiplicatively closed subset, show that we have in $S^{-1}M$

$$\bigcap_{\substack{\mathfrak{p}\in\mathrm{Ass}(M)\\ \mathfrak{p}\cap S=\varnothing}} S^{-1}Q(\mathfrak{p}) = 0.$$

(iv) If $\mathfrak{p} \in \mathrm{Ass}(M)$ is minimal, show that $Q(\mathfrak{p}) = \ker(M \to M_{\mathfrak{p}})$.

Exercise 7. Let M, N be R-modules, with M finitely generated. Show that

$$\mathrm{Ass}(\mathrm{Hom}_R(M, N)) = \mathrm{Supp}(M) \cap \mathrm{Ass}(N).$$

(You may observe that $\mathrm{Hom}_R(M, N)$ is a submodule of $N^n = N \oplus \cdots \oplus N$ for some n.)

Exercise 8. Let $\varphi\colon R \to S$ be a ring morphism, and N an S-module. Show that

$$\mathrm{Ass}_R(N) = \{\varphi^{-1}\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Ass}_S(N)\}.$$

Exercise 9. (*) Let $R \to S$ be a flat ring morphism, and M an R-module. Show that

$$\mathrm{Ass}_S(M \otimes_R S) = \bigcup_{\mathfrak{p}\in\mathrm{Ass}_R(M)} \mathrm{Ass}_S(S/\mathfrak{p}S).$$

The letter R denotes a (commutative unital) noetherian ring.

Exercise 1. Let M be a nonzero finitely generated R-module. Prove directly (using Zorn's Lemma) that $\mathrm{Supp}(M)$ possesses a minimal element.

Exercise 2. Let M be a finitely generated R-module, and let $x \in R$. Show that the following are equivalent:

(i) Multiplication by x is a nilpotent endomorphism of M.

(ii) The element x belongs to every prime of $\mathrm{Ass}(M)$.

Exercise 3. Let M be a finitely generated R-module, and $M_i \subset M_{i+1}$ a chain of submodules such that $M_i/M_{i+1} \simeq R/\mathfrak{p}_i$ with $\mathfrak{p}_i$ a prime of R. Let $\mathfrak{p}$ be a minimal element of $\mathrm{Supp}(M)$. Show that the number of indices i such that $\mathfrak{p}_i = \mathfrak{p}$ does not depend on the choice of the chain, and express this number purely in terms of M.

Exercise 4. Let M be an R-module.

(i) Show that $\mathfrak{p} \in \mathrm{Supp}(M)$ if and only if there is a submodule $N \subset M$ such that $\mathfrak{p} \in \mathrm{Ass}(M/N)$. (Hint: take N of the form $\mathfrak{p}m$ for a well-chosen $m \in M$).

(ii) Assume that M is finitely generated, and let $\mathfrak{p} \in \mathrm{Supp}(M)$. Show that there is a chain of submodules $0 = M_0 \subsetneq \cdots \subsetneq M_n = M$ such that $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \mathrm{Spec}(R)$ and moreover $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots \mathfrak{p}_n\}$.

The letter R denotes a (commutative unital) noetherian ring.

Exercise 1. Let $\varphi\colon A \to B$ be a morphism of local noetherian rings making B a finite type A-module. Show that φ is a local morphism.

Exercise 2. Let $\rho\colon R \to S$ be a flat morphism and M a finitely generated R-module. Show that the map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ maps $\operatorname{Ass}_S(S \otimes_R M)$ into $\operatorname{Ass}_R(M)$.

Exercise 3. Assume that $\dim R \geq 2$. Show that $\operatorname{Spec} R$ is infinite.

Exercise 4. (i) Let $\mathfrak{p}$ be a prime of R. Show that the ideal $\mathfrak{p}R[t]$ of $R[t]$ is prime.

(ii) Show that $\dim R[t] \geq 1 + \dim R$

(iii) Show that $\dim R[t_1, \cdots, t_n] = n + \dim R$.

Exercise 5. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and consider the n-th symbolic power

$$\mathfrak{p}^{[n]} = \{u \in R \mid su \in \mathfrak{p}^n \text{ for some } s \in R - \mathfrak{p}\}.$$

(i) Show that $\operatorname{Ass}(R/\mathfrak{p}^n)$ may differ from $\{\mathfrak{p}\}$ by considering the case $R = k[x,y]/(xy)$ with k a field, and $\mathfrak{p} = xR$.

(ii) Show that $\operatorname{Ass}(R/\mathfrak{p}^{[n]}) = \{\mathfrak{p}\}$, and that $\mathfrak{p}^{[n]}$ is minimal among the ideals I containing $\mathfrak{p}^n$ and satisfying $\operatorname{Ass}(R/I) = \{\mathfrak{p}\}$.

Exercise 6. (i) Show that every prime of R has finite height.

(ii) Let M be a possibly non-finitely generated R-module. Assume that $M \neq 0$. Show that $\operatorname{Supp}(M)$ admits at least one minimal element.

Exercise 1. (i) Let M be an R-module such that id_M is in the image of the natural morphism

$$\mathrm{Hom}_R(M,R)\otimes_R M \to \mathrm{Hom}_R(M,M).$$

Show that M is projective.

(ii) Let M, N, Q three R-modules. Assume that Q is flat, M is finitely generated, and R is noetherian. Show that the natural morphism

$$\mathrm{Hom}_R(M,N)\otimes_R Q \to \mathrm{Hom}_R(M, N\otimes_R Q)$$

is bijective. (Hint: Introduce a finite presentation of M, that is, an exact sequence $F_1 \to F_0 \to M \to 0$, with F_0, F_1 free and finitely generated R-modules).

(iii) Assume that R is noetherian and let M is a finitely generated flat R-module. Show M is projective.

(iv) Give an example of a flat, non-projective, $\mathbb{Z}$-module.

Exercise 2. Let x be a nonzerodivisor in R. Express $\mathrm{Tor}_1(R/x, M)$ in an elementary way in terms of x and M.

Exercise 3. Let I, J be two ideals in a ring R. Express $\mathrm{Tor}_1^R(R/I, R/J)$ in an elementary way in terms of R, I, J.

Exercise 4. (i) Show that M is flat, resp. projective, if and only if $\mathrm{Tor}_1(N, M) = 0$, resp. $\mathrm{Ext}^1(M,N) = 0$, for every module N.

(ii) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence. Assume that M' and M'' are projective, resp. flat, and show that M is projective, resp. flat.

Exercise 5. Let M, N two R-modules. Assume that R is noetherian and that M is finitely generated. Show that $\mathrm{Tor}_n(M,N)$ and $\mathrm{Ext}^n(M,N)$ are finitely generated.

Exercise 6. Let $R \to S$ be a flat ring morphism, and M, N two R-modules.

(i) Show that

$$\mathrm{Tor}_n^R(M,N)\otimes_R S \simeq \mathrm{Tor}_n^S(M\otimes_R S, N\otimes_R S).$$

(ii) Assume that R is noetherian, and M finitely generated. Show that

$$\mathrm{Ext}_R^n(M,N)\otimes_R S \simeq \mathrm{Ext}_S^n(M\otimes_R S, N\otimes_R S).$$

Exercise 7 (Yoneda description of Ext^1)**.** We fix two modules A and B. Given an exact sequence α of type

$$0 \to B \to X \to A \to 0$$

we define $[\alpha] \in \mathrm{Ext}^1(A,B)$ to be the image of id_A under the morphism $\mathrm{Hom}_R(A,A) \to \mathrm{Ext}^1(A,B)$ (which is part of the long exact sequence of Ext-groups associated with the short exact sequence α).

(i) We say that α splits if there is a morphism $A \to X$ such that the composite $A \to X \to A$ is the identity. Show that α splits if and only if $[\alpha] = 0$.

We say that two exact sequences $0 \to B \to X \to A \to 0$ and $0 \to B \to X' \to A \to 0$ are *Yoneda equivalent* if there is an isomorphism $X \to X'$ fitting in the commutative diagram

$$\begin{array}{ccccc} B & \longrightarrow & X & \longrightarrow & A \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\ B & \longrightarrow & X' & \longrightarrow & A \end{array}$$

(ii) Show that a sequence splits if and if it is Yoneda equivalent to the sequence $0 \to B \to A \oplus B \to A \to 0$.

(iii) We let $E(A, B)$ be the set of exact sequences $0 \to B \to X \to A \to 0$ modulo Yoneda equivalence. Show that $\alpha \mapsto [\alpha]$ induces a map $E(A, B) \to \operatorname{Ext}^1(A, B)$.

We construct a map $\operatorname{Ext}^1(A, B) \to E(A, B)$ as follows. Take an exact sequence $0 \to K \to F \to A \to 0$ with F free. An element $u \in \operatorname{Ext}^1(A, B)$ is represented by a morphism $\varphi_u \colon K \to B$. Let X_u be the cokernel of the morphism $K \to F \oplus B$ given by $k \mapsto (j(k), -\varphi_u(k))$ where j is the injective morphism $K \to F$.

(iv) Show that we have an exact sequence $0 \to B \to X_u \to A \to 0$, and therefore an element of $E(A, B)$.

(v) Show that this gives a map $\operatorname{Ext}^1(A, B) \to E(A, B)$.

(vi) Show that $\operatorname{Ext}^1(A, B)$ and $E(A, B)$ are in bijection.

(vii) Let $\alpha, \beta \in E(A, B)$. Describe the element $\gamma \in E(A, B)$ such that $[\gamma] = [\alpha] + [\beta]$. Describe the functorialities of $E(A, B)$ in A and B.

Exercise 1. Let C, D be two chain complexes of R-modules. Their tensor product $C \otimes_R D$ is defined as follows. We let

$$(C \otimes_R D)_n = \bigoplus_{i \in \mathbb{Z}} C_i \otimes_R D_{n-i}$$

and for $x \in C_i$ and $y \in D_{n-i}$, the differential is given by

$$d_n^{C \otimes_R D}(x \otimes y) = d_i^C(x) \otimes y + (-1)^i x \otimes d_{n-i}^D(y).$$

(i) Show that $(C \otimes_R D, d^{C \otimes_R D})$ defines a chain complex.

(ii) Show that the complexes $C \otimes_R D$ and $D \otimes_R C$ are isomorphic.

Exercise 2. Let $f \colon B \to C$ be a morphism of chain complexes. We let

$$\mathit{cone}(f)_n = B_{n-1} \oplus C_n$$

and define a morphism $d_n \colon \mathit{cone}(f)_n \to \mathit{cone}(f)_{n-1}$ by

$$d_n(b, c) = (-d_{n-1}^B(b), d_n^C(c) - f_{n-1}(b)).$$

(i) Show that $(\mathit{cone}(f), d)$ defines a chain complex.

(ii) Show that we have an exact sequence of complexes

$$0 \to C \to \mathit{cone}(f) \to B[-1] \to 0,$$

where $B[-1]$ is the complex defined by $B[-1]_n = B_{n-1}$ and $d_n^{B[-1]} = -d_{n-1}^B$.

(iii) Deduce that we have a long exact sequence

$$\cdots \to H_{n+1}(\mathit{cone}(f)) \to H_n(B) \xrightarrow{\delta} H_n(C) \to H_n(\mathit{cone}(f)) \to \dots$$

(iv) Show that the morphism $\delta \colon H_n(B) \to H_n(C)$ may be chosen to coincide with the morphism induced by f.

(v) Deduce that f is a quasi-isomorphism if and only if $\mathit{cone}(f)$ is exact.

Let R be a ring, and $x_1, \cdots, x_n \in R$. We construct the associated *Koszul complex* as follows. Let $e_1, \cdots, e_n$ be the standard basis of the R-module R^n. Let $p \in \mathbb{Z}$. For $p \in \{1, \dots, n\}$, we let K_p be the free R-module with the basis consisting of the elements $e_{i_1} \wedge \cdots \wedge e_{i_p}$ where $1 \leq i_1 < \cdots < i_p \leq n$. We let $K_0 = R$, and $K_p = 0$ when $p \notin \{0, \dots, n\}$. We define a R-linear morphism $d \colon K_p \to K_{p-1}$ using the formula

$$d_p(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{r=1}^{p} (-1)^{r-1} x_{i_r} \cdot e_{i_1} \wedge \cdots \wedge e_{i_{r-1}} \wedge e_{i_{r+1}} \wedge \cdots \wedge e_{i_p}.$$

(the vector e_{i_r} is omitted.) When $p = 1$, the above formula must be understood as

$$d_1(e_i) = x_i \in R = K_0.$$

Exercise 1. Show that $d_{p-1} \circ d_p = 0$.

This gives a chain complex $K(x_1, \cdots, x_n) = (K, d)$. Let M be an R-module. We denote by $K(M; x_1, \cdots, x_n)$ the complex $K(x_1, \cdots, x_n) \otimes_R M$. Its p-th homology is denoted $H_p(M; x_1, \cdots, x_n)$.

Exercise 2. (i) Express $H_0(M; x_1, \cdots, x_n)$ and $H_n(M; x_1, \cdots, x_n)$ directly in terms of M and $x_1, \cdots, x_n$.

(ii) Describe the complex $K(M; x_1)$.

Exercise 3. (i) Show that the complexes $K(M; x_1, \cdots, x_n)$ and $K(x_1) \otimes_R \cdots \otimes_R K(x_n) \otimes_R M$ are isomorphic.

(ii) Let L be a chain complex of R-modules and $x \in R$. Show that we have an exact sequence of chain complexes of R-modules

$$0 \to L \to K(x) \otimes_R L \to L[-1] \to 0,$$

(where $L[-1]_n = L_{n-1}$ and $d_n^{L[-1]} = -d_{n-1}^L$) and deduce and exact sequence of R-modules

$$0 \to H_0(H_p(L); x) \to H_p(K(x) \otimes_R L)) \to H_1(H_{p-1}(L); x) \to 0.$$

Exercise 4. Let A be a local (noetherian) ring, M a finitely generated A-module, and $x_1, \cdots, x_n \in \mathfrak{m}$.

(i) Assume that $(x_1, \cdots, x_n)$ is an M-regular sequence. Show that $H_i(M; x_1, \cdots, x_n) = 0$ for $i > 0$.

(ii) Assume that $H_1(M; x_1, \cdots, x_n) = 0$. Show that $(x_1, \cdots, x_n)$ is an M-regular sequence.

Exercise 5. Let A be a local (noetherian) ring, and M a finitely generated A-module. Assume that $(x_1, \cdots, x_n)$ is an M-regular sequence.

(i) Let σ be a permutation of $\{1, \cdots, n\}$. Show that $(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ is an M-regular sequence.

(ii) Let $t_1, \dots, t_n$ be integers ≥ 1. Show that $(x_1^{t_1}, \dots, x_n^{t_n})$ is a regular M-sequence.

Exercise 6. (i) Let L be a complex of R-modules and $x \in R$. Show that $x \cdot H_p(K(x) \otimes_R L) = 0$.

(ii) Let $x_1, \cdots, x_n \in R$, and I be the ideal generated by these elements. Let M be an R-module. Show that $I \cdot H_p(M; x_1, \cdots, x_n) = 0$.

Exercise 7. (Depth sensitivity of the Koszul complex) Let A be a local ring, and $\{x_1, \cdots, x_n\}$ a generating set for its maximal ideal. Let M be a nonzero finitely generated A-module. Show that

$$\operatorname{depth} M = n - \max\{i | H_i(M; x_1, \cdots, x_n) \neq 0\}.$$

(use Exercise 6.)

Exercise 8. (A more functorial approach) Let U be a finitely generated free R-module. For an R-module, we denote by $V^\vee = \operatorname{Hom}_R(V, R)$ its dual. We consider the R-module

$$T(U) = \bigoplus_{p \geq 0} U^{\otimes p} = R \oplus U \oplus (U \otimes_R U) \oplus \cdots.$$

The R-module $\Lambda(U)$ is the quotient of $T(U)$ by the submodule generated by elements $x_1 \otimes \cdots \otimes x_p$ which are such that $x_i = x_j$ for some $i \neq j$. It is naturally graded; we denote by $\Lambda^p U$ the image of $U^{\otimes p}$ and by $u_1 \wedge \cdots \wedge u_p$ the image of $u_1 \otimes \cdots \otimes u_p$. An isomorphism $U \simeq R^n$ induces an isomorphism $\Lambda^p U \simeq K_p$.

(i) Show that the natural morphism $\rho_p \colon \Lambda^p(U^\vee) \to (\Lambda^p U)^\vee$ is an isomorphism.

(ii) Let $u \in U$, and $\varphi_u \colon \Lambda^p U \to \Lambda^{p+1} U$ be defined by $\varphi_u(v) = u \wedge v$. Show that if $e_1, \cdots, e_n$ is a basis of the R-module $U^\vee$, and $(x_1, \cdots, x_n)$ are the coordinates of u in the dual basis of U, then the differential d of the Koszul complex may be identified with

$$\Lambda^{p+1}(U^\vee) \xrightarrow{\rho_{p+1}} (\Lambda^{p+1} U)^\vee \xrightarrow{(\varphi_u)^\vee} (\Lambda^p U)^\vee \xrightarrow{\rho_p^{-1}} \Lambda^p(U^\vee).$$

(iii) Reprove without computation that $d \circ d = 0$.