EXERCISES 1 (INTERSECTION THEORY)

Let A be a noetherian commutative ring with unit, and M a finitely generated A-module.

Exercise 1. The length function is additive.

Exercise 2. The length of any maximal (i.e. saturated) chain of submodules of M is equal to the length of M.

A prime $\mathfrak{p}$ of A is *associated with* M if there is an element $m \in M$ such that $\mathfrak{p} = \mathrm{Ann}(m) = \{x \in A | xm = 0\}$. We write $\mathrm{Ass}(M)$ for the set of associated primes of M.

Exercise 3.
- (i) We have $\mathfrak{p} \in \mathrm{Ass}(M)$ if and only if M contains a submodule isomorphic to $A/\mathfrak{p}$.
- (ii) Let I be a maximal element of the set $\{\mathrm{Ann}(m) | m \in M - \{0\}\}$. Then I is a prime ideal.
- (iii) We have $M = 0$ if and only if $\mathrm{Ass}(M) = \varnothing$.
- (iv) Let $\mathfrak{p}$ be a prime of A. Then $\mathrm{Ass}(A/\mathfrak{p}) = \{\mathfrak{p}\}$.

Exercise 4. Consider an exact sequence of finitely generated A-modules

$$0 \to M' \to M \to M'' \to 0.$$

Then $\mathrm{Ass}(M') \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$.

Exercise 5. There is a chain of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that $M_i/M_{i-1} \simeq A/\mathfrak{p}_i$ with $\mathfrak{p}_i$ prime, for $i = 1, \cdots, n$. We have

$$\mathrm{Ass}(M) \subset \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}.$$

Exercise 6. Assume that A is local. Then the following are equivalent
- (i) $l_A(M) < \infty$.
- (ii) There is $n \in \mathbb{N}$ such that $(\mathfrak{m}_A)^n M = 0$.
- (iii) We have $\dim M \leq 0$.

Exercise 7. Consider an exact sequence of finitely generated A-modules

$$0 \to M' \to M \to M'' \to 0.$$

Then $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

Exercise 8. Show that the primes $\mathfrak{p}_i$ of Exercise 5 belong to $\mathrm{Supp}(M)$.

Exercise 9. Let $\mathfrak{p} \in \mathrm{Spec}\, A$. We view $\mathrm{Spec}\, A_{\mathfrak{p}}$ as a subset of $\mathrm{Spec}\, A$. Then

$$\mathrm{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = (\mathrm{Spec}\, A_{\mathfrak{p}}) \cap \mathrm{Ass}(M).$$

Exercise 10. We have $\mathrm{Ass}(M) \subset \mathrm{Supp}(M)$, and these sets have the same minimal elements.

Exercise 11. The set $\mathrm{Ass}(M)$ is finite, and so is the set of minimal primes in $\mathrm{Supp}(M)$.

EXERCISES 2 (INTERSECTION THEORY)

Exercise 1. When $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, we define

$$\mathrm{Ass}(\mathcal{F}) = \{x \in X | \mathfrak{m}_x \in \mathrm{Ass}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)\}.$$

(Here $\mathfrak{m}_x$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,x}$.)

A closed embedding $Z \to X$ is called *locally principal* if there is a covering by open affine subschemes $U_i = \operatorname{Spec} A_i$ and elements $s_i \in A_i$ such that $Z \cap U_i = \operatorname{Spec}(A_i/s_iA_i)$.

(i) If $X = \operatorname{Spec} A$, and $M = H^0(X, \mathcal{F})$, show that $\mathrm{Ass}(M) = \mathrm{Ass}(\mathcal{F})$.

(ii) Show that a closed embedding $D \to X$ is an effective Cartier divisor if and only if:
— $D \to X$ is locally principal,
— and $D \cap \mathrm{Ass}(\mathcal{O}_X) = \varnothing$.

(iii) Let $f \colon Y \to X$ be a morphism, and $Z \to X$ a locally principal closed embedding. Then show that $f^{-1}Z \to Y$ is a locally principal closed embedding.

(iv) Let $f \colon Y \to X$ be a morphism, and $D \to X$ an effective Cartier divisor. Show that $f^{-1}D \to Y$ is an effective Cartier divisor if and only if $f(\mathrm{Ass}(\mathcal{O}_Y)) \cap D = \varnothing$.

(v) Assume that f is flat. Show that $f(\mathrm{Ass}(\mathcal{O}_Y)) \subset \mathrm{Ass}(\mathcal{O}_X)$.

(vi) Explain how we can reprove the lemma concerning pull-backs of effective Cartier divisors.

Exercise 2. (i) Let M be a finitely generated A-module (A noetherian). Show that the following morphism is injective:

$$M \to \bigoplus_{\mathfrak{p} \in \mathrm{Ass}(M)} M_{\mathfrak{p}}.$$

Let X be a variety.

(ii) Show that every generic point of X is in $\mathrm{Ass}(\mathcal{O}_X)$.

(iii) Show that X is reduced if and only if :
— for every generic point $x \in X$, the ring $\mathcal{O}_{X,x}$ is reduced,
— and $\mathrm{Ass}(\mathcal{O}_X)$ is the set of generic points.

Exercise 3. Let us denote by P the closed point $0 \in \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, that is, the integral closed subscheme defined by the ideal (x,y). Find closed subschemes Z_1, Z_2 of $\mathbb{A}^2_k$ such that

$$[Z_1] = [Z_2] = 3[P] \in \mathcal{Z}(\mathbb{A}^2_k),$$

but $Z_1 \not\simeq Z_2$ as schemes (and thus as closed subschemes of $\mathbb{A}^2_k$).

(more exercises next page)

Exercise 4. (i) Let $f\colon Y \to X$ be a closed immersion. Show that f is an isomorphism if and only if there is an open subscheme U of X containing $\mathrm{Ass}(\mathcal{O}_X)$ such that $Y \cap U \to U$ is an isomorphism.

(ii) Find a closed immersion $Y \to X$ and an open dense subscheme U of X such that $Y \cap U \to U$ is an isomorphism (and thus $[Y] = [X] \in \mathcal{Z}(X)$), but $Y \not\simeq X$.

Exercise 5. Let $R = k[x, y, z]/(zx, zy)$ and $X = \operatorname{Spec} R$. Let D be the closed subscheme of X defined by $(z - x)$.

(i) Show that $D \to X$ is an effective Cartier divisor.
(ii) What is the multiplicity m_i of X at each irreducible component X_i of X?
(iii) Compare $[D]$ and $\sum_i m_i[D \cap X_i]$ in $\mathcal{Z}(X)$.
(iv) Is this compatible with Proposition 1.3.5?

Exercise 6. Prove the snake lemma: A commutative diagram of A-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

with exact rows induces a long exact sequence of A-modules

$$0 \to \ker \varphi' \to \ker \varphi \to \ker \varphi'' \to \operatorname{coker} \varphi' \to \operatorname{coker} \varphi \to \operatorname{coker} \varphi'' \to 0.$$

Exercise 7. Prove the going-down theorem: If $Y \to X$ is flat, then every irreducible component of Y dominates an irreducible component of X.

Exercise 8. Let $f\colon Y \to X$ be a flat morphism, with X irreducible and Y equidimensional. Show that f has relative dimension $\dim Y - \dim X$.

Exercise 9. Let $f\colon Y \to X$ be a finite morphism such that the $\mathcal{O}_X$-module $f_*\mathcal{O}_Y$ is locally free of rank $d > 0$. Show that f is flat of relative dimension 0, and that $f_* \circ f^*$ is multiplication with d on $\mathcal{Z}(X)$.

EXERCISES 3 (INTERSECTION THEORY)

Exercise 1. We will view $\mathbb{A}^1 = \operatorname{Spec} k[t]$ as the open complement of ∞ in $\mathbb{P}^1$. This defines an element $t \in k(\mathbb{P}^1)$ such that $\operatorname{div} t = [0] - [\infty] \in \mathcal{Z}(\mathbb{P}^1)$.

(i) Let Z be an integral variety and $f \colon Z \to \mathbb{P}^1$ a morphism whose image is not contained in $\{0, \infty\}$. Denote by f^*t the image of t under the induced morphism $k[t, t^{-1}] \to k(Z)$. Show that

$$\operatorname{div} f^*t = [f^{-1}0] - [f^{-1}\infty] \in \mathcal{Z}(Z).$$

(ii) Let X be an integral variety, and $\varphi \in k(X)^\times$. Show that there is an integral closed subscheme Z of $X \times_k \mathbb{P}^1$ such that $p \colon Z \to X$ is birational, the image of $f \colon Z \to \mathbb{P}^1$ is not contained in $\{0, \infty\}$, and

$$\operatorname{div} \varphi = p_* \circ \operatorname{div} f^*t \in \mathcal{Z}(X).$$

(iii) Let X be a variety. Let $\mathcal{Z}(X; \mathbb{P}^1)$ be the set of integral closed subschemes Z of $X \times_k \mathbb{P}^1$, such that the morphism $f \colon Z \to \mathbb{P}^1$ is dominant. For $\star \in \{0, \infty\}$, show that $f^{-1}\star$ may be identified to a closed subscheme of X, that will be denoted by $Z(\star)$.

(iv) Let X be a variety. Show that the subgroup of rationally trivial classes $\mathcal{R}(X) \subset \mathcal{Z}(X)$ is generated by the elements $[Z(0)] - [Z(\infty)]$, where Z runs over $\mathcal{Z}(X; \mathbb{P}^1)$.

Exercise 2. Let X be an integral variety, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.

(i) Show that we have a correspondence

$$\left\{ \begin{array}{l} \text{Integral closed subschemes } Z \subset \mathbb{P}(\mathcal{L} \oplus 1), \\ \text{with } Z \not\subset \mathbb{P}(1), Z \not\subset \mathbb{P}(\mathcal{L}), \\ \text{and } Z \to X \text{ birational.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{regular meromorphic} \\ \text{sections of } \mathcal{L}. \end{array} \right\}$$

(ii) Let s be a regular meromorphic section of $\mathcal{L}$, and $Z \subset \mathbb{P}(1 \oplus \mathcal{L})$ the corresponding closed subscheme, with morphism $p \colon Z \to X$. Show that

$$\operatorname{div}_{p^*\mathcal{L}}(p^*s) = [Z \cap \mathbb{P}(1)] - [Z \cap \mathbb{P}(\mathcal{L})] \in \mathcal{Z}(Z).$$

(iii) Show that $Z \cap \mathbb{P}(1)$ (resp. $Z \cap \mathbb{P}(\mathcal{L})$) may be viewed as a closed subscheme $Z(1)$ (resp. $Z(\mathcal{L})$) of X, and that we have

$$\operatorname{div}_{\mathcal{L}}(s) = p_*[Z \cap \mathbb{P}(1)] - p_*[Z \cap \mathbb{P}(\mathcal{L})] = [Z(1)] - [Z(\mathcal{L})] \in \mathcal{Z}(X).$$

Exercise 3. Prove directly (that is, without using Chapter 3 of the lecture) Weil's reciprocity law: For any $\varphi \in k(\mathbb{P}^1)^\times$, we have

$$\deg \circ \operatorname{div} \varphi = 0.$$

Exercise 4. Let $i \colon D \to X$ be an effective Cartier divisor, $f \colon X \to S$ a flat morphism with a relative dimension. Assume that $f \circ i \colon D \to S$ is flat and has a relative dimension. Show that

$$i^* \circ f^* = (f \circ i)^* \colon \operatorname{CH}(S) \to \operatorname{CH}(D)$$

EXERCISES 4 (INTERSECTION THEORY)

Exercise 1. Let A be a commutative ring. A *characteristic class* φ is the data of a group endomorphism $\varphi(E)$ of $\mathrm{CH}(X) \otimes A$ for every vector bundle $E \to X$, such that for every flat morphism $f\colon Y \to X$ having a relative dimension,

$$f^* \circ \varphi(E) = \varphi(f^*E) \circ f^*\colon \mathrm{CH}(X) \otimes A \to \mathrm{CH}(Y) \otimes A.$$

(i) Assume that a vector bundle E has a filtration by sub-bundles $E_{n+1} \subset E_n$ such that $L_n = E_n/E_{n+1}$ is a line bundle. Express the i-th Chern class $c_i(E)$ in terms of the classes $c_1(L_n)$.

(ii) Let $F \in A[x_1, \cdots, x_n]$ be a symmetric polynomial. Show that there is a unique characteristic class φ such that whenever E is a vector bundle with a filtration with successive quotients line bundles $L_1, \cdots, L_m$, then

$$\varphi(E) = F(c_1(L_1), \cdots, c_1(L_m)).$$

(iii) Let $P \in A[[t]]$ a power series. Show that there is unique characteristic class π_P such that

- If $0 \to E \to F \to G \to 0$ is an exact sequence of vector bundles, then $\pi_P(E) \circ \pi_P(G) = \pi_P(F)$.
- If $L \to X$ is a line bundle, then $\pi_P(L) = P(c_1(L))$.

(iv) Let $P \in A[[t]]$ a power series. Show that there is unique characteristic class γ_P such that

- If $0 \to E \to F \to G \to 0$ is an exact sequence of vector bundles, then $\gamma_P(E) + \gamma_P(G) = \gamma_P(F)$.
- If $L \to X$ is a line bundle, then $\gamma_P(L) = P(c_1(L))$.

(v) When $A = \mathbb{Q}$, and

$$P(t) = \sum_{n \geq 0} t^n/n!,$$

we define the *Chern character* $\mathrm{ch} = \gamma_P$. Show that

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \circ \mathrm{ch}(F)$$

for any vector bundles E, F.

Exercise 2. Let X be a smooth (or more generally locally factorial) variety. Show that the morphism $\mathrm{Pic}(X) \to \mathrm{CH}^1(X)$ mapping L to $c_1(L)[X]$ is a group isomorphism.

Exercise 3. When E is a vector bundle of rank r, its *determinant* $\det E$ is the line bundle $\Lambda^r E$. We say that E is *orientable* if $\det E$ is the trivial line bundle.

(i) Consider an exact sequence of vector bundles

$$0 \to E \to F \to L \to 0$$

where L is a line bundle. Show that $(\det E) \otimes L \simeq \det F$.

(ii) Show that $c_1(\det E) = c_1(E)$ for any vector bundle E, and deduce that $c_1(E) = 0$ when E is orientable.

(iii) Conversely, show that a vector bundle E over a smooth variety X is orientable as soon as $c_1(E)[X] = 0$.