PRIME IDEALS AND TWISTING IN HOPF GALOIS EXTENSIONS

S. MONTGOMERY AND H.-J. SCHNEIDER

1. Introduction

This paper is a sequel to [MS]. We continue to study the relationship between the prime ideals of an algebra A and of a subalgebra R such that $R \subset A$ is a faithfully flat H-Galois extension for some finite-dimensional Hopf algebra H. In this paper we consider what happens when the Hopf algebra H and the extension $R \subset A$ are twisted by a Hopf 2-cocycle σ, so that $R_\sigma \subset A_\sigma$ becomes an H_σ-extension.

We are interested in versions of the classical Krull relations between prime ideals in finite extensions of commutative rings. In [MS] we introduced the Krull relations for an H-Galois extension $R \subset A$. See Section 2 of this paper for a definition of the basic Krull relations t-lying over, for some natural number t (t-LO), going up (GU), and incomparability (INC), and of the dual notions t-coLO, coGU and coINC. We define two new relations strong GU and strong coGU; these are stronger versions of going up and co going up, which we will need here. By definition, H has one of the Krull relations if the relation holds for all faithfully flat H-Galois extensions. We show in [MS] that H has one of the basic Krull relations if and only if H^* has the corresponding dual Krull relation. Also in [MS], we show that H has one of the Krull relations if it satisfies the relation for all H-Galois extensions of the form $A = R\#H$, where R is an H-module algebra.

To illustrate the Krull relations, consider a smash product extension $R \subset A = R\#H$ where R is prime, or more generally H-prime. If H has t-LO and INC, then P is a minimal prime of A precisely when $P \cap R = 0$, A has $n \leq \dim H$ minimal primes, say $P_1, \dots, P_n$, and if $N := \cap_i P_i$, then $N^t = 0$ and N is the largest nilpotent ideal of A [MS, 4.7].

Lorenz and Passman showed that the basic Krull relations hold for crossed products of group algebras $R \subset A = R\#_\sigma kG$, where G is a finite group (see Chapter 4 of [P] for an exposition of this theory); the analogous results for $H = (kG)^*$ were shown in [CM]. By the results of [MS] mentioned above, it follows that both kG and $(kG)^*$ satisfy all of the Krull relations. In [MS] we established all of the Krull relations for several additional classes

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of Hopf algebras, for example for solvable and cosolvable Hopf algebras. We also showed that finite-dimensional pointed Hopf algebras have GU and the three dual Krull relations; this used work of Cohen-Rainu-Westreich and Chin. However INC and t-LO remain open even if H is pointed and cocommutative, for example if H is the restricted enveloping algebra of a restricted Lie algebra in characteristic $p > 0$.

Our first main results in this paper, in Section 3, concern lifting the Krull relations from a Hopf subalgebra $K \subset H$ such that K contains the coradical H_0 of H, to H itself. Dually, we consider a quotient Hopf algebra H/I where I is a nilpotent Hopf ideal, and ask which of the Krull relations hold for H if they hold for H/I.

In particular we show for an $\mathbb{N}$-graded Hopf algebra H with K the homogeneous part of degree 0, that H has all the Krull relations if K does (Theorem 3.6). As a corollary we prove that coradically graded pointed Hopf algebras have all the Krull relations (Corollary 3.7). Consequently the Borel parts of Lusztig's Frobenius kernels $u_q(g)$, g a semisimple Lie algebra, have all the Krull relations, as do the finite-dimensional Hopf algebras $u(\mathcal{D})$ defined in [AS] when the linking elements $\lambda_{ij} = 0$.

Our second main topic, in Section 4, concerns when the Krull relations are preserved by twisting. We first review twistings of Hopf algebras and of H-comodule algebras. As a preliminary step, we prove (Theorem 4.3) that if $R \subset A$ is an H-Galois extension, then $R_\sigma \subset A_\sigma$ is an H_σ-Galois extension; moreover H_σ-Spec $R = H$-Spec$R$. We then prove that any one of coINC, t-coLO, or strong coGU are preserved under twisting (Theorem 4.7). In Corollary 4.8 we show that the quantum double $D(H)$ has the dual Krull relations if H has all Krull relations.

Finally in Section 5, we consider twistings of the comultiplication and apply our results about twisting, together with the classification theorems of Etingof and Gelaki [EG1] [EG2], to show that any (finite-dimensional) triangular Hopf algebra over an algebraically closed field of characteristic 0 has INC, t-LO and GU (Theorem 5.2). We also consider the double of a factorizable Hopf algebra, and as a consequence we prove that for a finite group G, $D(D(G))$ has all of the Krull relations.

We remark that it is possible that any finite-dimensional Hopf algebra H satisfies all of the Krull relations; no counterexamples are known. If the algebra R is Noetherian, then INC is true for any finite extension $R \subset A$ [Le]; however LO can fail even for finite extensions of Noetherian rings [HO].

2. The Krull relations revisited

In this section we first review the Krull relations from [MS], and introduce new versions of several of them which we shall need in this paper. Throughout H is a finite-dimensional Hopf algebra over a field k, and $R \subset A$ denotes a faithfully flat H-Galois extension. As in [MS, 1.1, 2.3], we say that an ideal I of R is **H-stable** if $IA = AI$, and let $(I : H)$ denote the largest H-stable ideal of R in I. I is an **H-prime ideal** of R if $I \neq R$, and whenever $JK \subset I$, for J, K H-stable ideals of R, either $J \subset I$ or $K \subset I$.

To avoid confusion, we will usually write P for a prime in $\mathrm{Spec}(A)$, Q for a prime in $\mathrm{Spec}(R)$, and I for an H-prime in H-$\mathrm{Spec}(R)$. We recall [MS, Lemma 2.2]:

Lemma 2.1. *(1) The map $f : \mathrm{Spec}(R) \to H\text{-}\mathrm{Spec}(R)$ given by $Q \mapsto (Q : H)$ is well-defined and surjective.*

(2) The map $g : \mathrm{Spec}(A) \to H\text{-}\mathrm{Spec}(R)$ given by $P \mapsto P \cap R$ is well-defined and surjective.

As in [MS], we say that $P \in \mathrm{Spec}(A)$ **lies over** $Q \in \mathrm{Spec}(R)$ if and only if $(Q : H) = P \cap R$. We will also say that $P \in \mathrm{Spec}(A)$ **lies over** $I \in H$-$\mathrm{Spec}(R)$ if and only if $I = P \cap R$. By Lemma 2.1, any $P \in \mathrm{Spec}(A)$ lies over some $Q \in \mathrm{Spec}(R)$; conversely for any $Q \in \mathrm{Spec}(R)$, there exists some $P \in \mathrm{Spec}(A)$ such that P lies over Q. Similarly any $P \in \mathrm{Spec}(A)$ lies over some $I \in H$-$\mathrm{Spec}(R)$; conversely for any $I \in H$-$\mathrm{Spec}(R)$, there exists some $P \in \mathrm{Spec}(A)$ such that P lies over I.

We note that the definition of P lying over Q reduces to the standard definition of lying over in non-commutative rings, that is that Q is minimal over $P \cap R$, under some additional assumptions; see [MS, 4.7].

We may use diagrams, as in [P], to represent many of the Krull relations. Thus for example the diagram in 2.2(3) means that given $Q_2 \subset Q_1$ in $\mathrm{Spec}(R)$ and $P_2 \in \mathrm{Spec}(A)$ which lies over Q_2, there exists some $P_1 \in \mathrm{Spec}(A)$ such that $P_2 \subset P_1$ and P_1 lies over Q_1. In the following definition, (1) - (3) and (1)$'$ - (3)$'$ appear in [MS]. It is shown in [MS, 4.3] that (1)$'$ - (3)$'$ are the duals of (1) - (3), in the sense that a condition (i) is true for H if and only if $(i)'$ is true for H^*. (4) and (4)$'$ are new; they will be useful since they are defined only in terms of R and not A.

Definition 2.2. The Krull relations

(1) The H-Galois extension $R \subset A$ has **t-lying over** (t-LO) if for any $Q \in \mathrm{Spec}(R)$, there exist $P_1, \dots, P_n \in \mathrm{Spec}(A)$, where $n \leq \dim H$, such that all P_i lie over Q, and such that $(\bigcap_{i=1}^n P_i)^t \subset (Q : H)A$.

$$\begin{tikzcd} & \{P_i\}^t \\ Q \arrow[ur, dotted, no head] & \end{tikzcd}$$

(2) $R \subset A$ has **incomparability** (INC) if for any $P_2 \subset P_1$ in $\mathrm{Spec}(A)$ with $P_2 \neq P_1$, then $P_2 \cap R \neq P_1 \cap R$.

(3) $R \subset A$ has **going up** (GU) if

$$\begin{array}{ccc}
 & & P_1 \\
 & \iddots & \vdots \\
Q_1 & & P_2 \\
\vert & / & \\
Q_2 & &
\end{array}$$

(4) $R \subset A$ has **strong going up** (strong GU) if

$$\begin{array}{ccc}
 & & P_1 \\
 & \iddots & \vdots \\
I_1 & & P_2 \\
\vert & / & \\
I_2 & &
\end{array}$$

(1)$'$ $R \subset A$ has t**-co-lying over** (t-coLO) if for any $P \in \operatorname{Spec}(A)$, there exist $Q_1, \dots, Q_m \in \operatorname{Spec}(R)$, where $m \leq \dim H$, such that P lies over all Q_j, and such that $(\bigcap_{j=1}^{m} Q_j)^t \subset P \cap R$.

$$\begin{array}{ccc}
 & & P \\
 & \iddots & \\
\{Q_j\}^t & &
\end{array}$$

(2)$'$ $R \subset A$ has **co-incomparability** (coINC) if for any $Q_2 \subset Q_1$ in $\operatorname{Spec}(R)$ with $Q_2 \neq Q_1$, then $(Q_2 : H) \neq (Q_1 : H)$.

(3)$'$ $R \subset A$ has **co-going up** (coGU) if

$$\begin{array}{ccc}
 & & P_1 \\
 & \iddots & \vert \\
Q_1 & & P_2 \\
\vdots & / & \\
Q_2 & &
\end{array}$$

(4)$'$ $R \subset A$ has **strong co-going up** (strong coGU) if

$$\begin{array}{ccc}
 & & Q_1 \\
 & \iddots & \vdots \\
I_1 & & Q_2 \\
\vert & / & \\
I_2 & &
\end{array}$$

Definition 2.3. We say the Hopf algebra H has one of the Krull relations above if for*all* faithfully flat H-Galois extensions $R \subset A$, the given Krull relation holds.

For later use we note

Remark 2.4. Let $\delta : A \to A \otimes H, a \mapsto a_{(0)} \otimes a_{(1)}$, be an H-comodule algebra with coinvariant elements $R = A^{\mathrm{co}H}$, and assume that $R \subset A$ is an H-Galois extension. Then the dual algebras $R^{\mathrm{op}} \subset A^{\mathrm{op}}$ form an H^{op}-Galois extension with comodule structure $\delta^{\mathrm{op}} : A^{\mathrm{op}} \to A^{\mathrm{op}} \otimes H^{\mathrm{op}}, a^{\mathrm{op}} \mapsto a^{\mathrm{op}}_{(0)} \otimes a^{\mathrm{op}}_{(1)}$. Thus if H satisfies any one of the Krull relations then so does H^{op}.

Proof. The Galois map $A^{\mathrm{op}} \otimes_{B^{\mathrm{op}}} A^{\mathrm{op}} \to A^{\mathrm{op}} \otimes H^{\mathrm{op}}, x^{\mathrm{op}} \otimes y^{\mathrm{op}} \mapsto (y_{(0}x)^{\mathrm{op}} \otimes y^{\mathrm{op}}_{(1)}$, is surjective hence bijective since $A \otimes A \to A \otimes H, y \otimes x \mapsto y_{(0)}x \otimes y_{(1)}$, is surjective. □

Lemma 2.5. *For any finite-dimensional Hopf algebra H, H has strong GU $\iff$ H^* has strong coGU. Moreover:*

(1) *If H has strong GU, then H has GU (that is,* 2.2(4) *implies* 2.2(3)*).*

(1)′ *If H has strong coGU, then H has coGU (that is,* 2.2(4)′ *implies* 2.2(3)′*).*

(2) *If H has GU, and either strong coGU or t-coLO , then H has strong GU (that is,* 2.2(3)*, and either* 2.2(4)′ *or* 2.2(1)′*, implies* 2.2(4)*).*

(2)′ *If H has coGU, and either strong GU or t-LO, then H has strong coGU (that is,* 2.2(3)′*, and either* 2.2(4) *or* 2.2(1) *implies* 2.2(4)′*).*

Proof. The fact that H has strong GU $\iff$ H^* has strong coGU follows similarly to the proof that H has GU $\iff$ H^* has coGU in [MS, Theorem 4.3(3)]. Thus (1) and (2) are the dual statements to (1)′ and (2)′, respectively, and so it suffices to show only (1) and (2).

(1) Assume that $Q_2 \subset Q_1$ in $\mathrm{Spec}(R)$ and that $P_2 \in \mathrm{Spec}(A)$ lies over Q_2. Let $I_i := (Q_i : H)$, for $i = 1, 2$; then $P_2 \cap R = I_2$. By strong GU, there exists $P_1 \in \mathrm{Spec}(A)$ such that $P_2 \subset P_1$ and $P_1 \cap R = I_1$. But $I_1 := (Q_1 : H)$. Thus H has GU.

(2) First assume H has GU and strong coGU. Assume that $I_2 \subset I_1$ in H-$\mathrm{Spec}(R)$ and that $P_2 \in \mathrm{Spec}(A)$ lies over I_2. By Lemma 2.1 there exists $Q_2 \in \mathrm{Spec}(R)$ with $(Q_2 : H) = I_2 = P_2 \cap R$. By strong coGU, there exists $Q_1 \in H$-$\mathrm{Spec}(R)$ such that $Q_2 \subset Q_1$ and $(Q_2 : H) = I_2$. Now use GU to find $P_1 \in \mathrm{Spec}(A)$ such that $P_2 \subset P_1$ and P_1 lies over Q_1. Then P_1 lies over I_1, and H has strong GU.

Now assume H has GU and t-coLO. Assume again that $I_2 \subset I_1$ in H-$\mathrm{Spec}(R)$ and that $P_2 \in \mathrm{Spec}(A)$ lies over I_2. By Lemma 2.1 there exists $Q \in \mathrm{Spec}(R)$ with $(Q : H) = I_1$. By t-coLO, there exist $Q_i \in H$-$\mathrm{Spec}(R)$, $i = 1, \ldots, m$, such that $(Q_i : H) = I_2$ for all i and that $(\cap Q_i)^t \subset I_2$. Since $I_2 \subset I_1 = (Q : H) \subset Q$ and Q is prime, some Q_i, call it Q_2, is contained in Q. Now use GU to find $P_1 \in \mathrm{Spec}(A)$ such that $P_2 \subset P_1$ and P_1 lies over Q. Then P_1 lies over I_1, and H has strong GU. □

Corollary 2.6. *If H is pointed, then H has strong GU and strong coGU.*

Proof. As noted earlier, any pointed Hopf algebra has GU, t-coLO, coGU, and coINC. Thus by Lemma 2.5(2), H has strong GU. We may now use Lemma 2.5(2)′ to see that H also has strong coGU. □

3. Galois extensions of the same base ring

In this section we consider two Hopf Galois extensions of the same base ring R, for two different Hopf algebras H and K. We will be particularly interested in the case when K is a Hopf subalgebra of H which contains the coradical H_0 of H.

Definition 3.1. Let H and K be Hopf algebras with $\dim K \leq \dim H$, and let R be a k-algebra. Assume that A and B are two ring extensions of R such that $R \subset A$ is faithfully flat H-Galois and that $R \subset B$ is faithfully flat K-Galois. We say that the triple (R, A, B) is (H, K)**-Krull admissible** if the following two conditions hold:

(1) For all ideals I of R, $((I : K) : H) = (I : H)$.

(2) There exists l such that for all ideals I of R, $(I : K)^l \subset (I : H)$.

Lemma 3.2. *Assume that (R, A, B) is (H, K)-Krull admissible. Then the following diagram is commutative*

$$\begin{array}{ccc} & \mathrm{Spec}(R) & \\ \swarrow & & \searrow \\ K\text{-}\mathrm{Spec}(R) & \xrightarrow{\cong} & H\text{-}\mathrm{Spec}(R) \end{array}$$

where $\mathrm{Spec}(R) \to K\text{-}\mathrm{Spec}(R)$ *is given by* $Q \mapsto (Q : K)$*,* $\mathrm{Spec}(R) \to H\text{-}\mathrm{Spec}(R)$ *is given by* $Q \mapsto (Q : H)$*, and the isomorphism* $\Phi : K\text{-}\mathrm{Spec}(R) \to H\text{-}\mathrm{Spec}(R)$ *is given by* $J \mapsto (J : H)$*. Moreover the map* Φ *respects inclusions in both directions.*

Proof. First, Φ is defined on all of K-Spec(R) since $Q \mapsto (Q : K)$ is surjective by Lemma 2.1. It is well-defined and the diagram commutes by Definition 3.1(1). To see that Φ is a bijection, first note that it is surjective since $P \mapsto (P : H)$ is surjective by Lemma 2.1. To see that it is injective and respects inclusions, choose $J_1, J_2 \in K$-Spec(R) such that $(J_1 : H) \subset (J_2 : H)$. Then by 3.1(2),

$$J_1^l = (J_1 : K)^l \subset (J_1 : H) \subset (J_2 : H) \subset J_2.$$

Since J_2 is K-prime, it follows that $J_1 \subset J_2$. □

Proposition 3.3. *Assume that (R, A, B) is (H, K)-Krull admissible.*

(1) If $R \subset B$ has coINC, then $R \subset A$ has coINC.

(2) If $R \subset B$ has s-coLO, then $R \subset A$ has ls-coLO.

(3) If $R \subset B$ has strong coGU, then $R \subset A$ has strong coGU.

Proof. (1) Let $Q_1 \subset Q_2$ in Spec(R) with $(Q_1 : H) = (Q_2 : H)$. Then by Lemma 3.2, $(Q_1 : K) = (Q_2 : K)$. Thus $Q_1 = Q_2$ since $R \subset B$ has coINC.

(2) Let $P \in$ Spec(A). We want $Q_1, \ldots Q_m \in$ Spec(R), for some $m \leq \dim H$, such that $(Q_j : H) = P \cap R$ for all j and $(\cap_{j=1}^m Q_J)^{ls} \subset P \cap R$. Now by Lemma 2.1, there exist $Q \in$ Spec(R) and $\widetilde{P} \in$ Spec(B) such that $(Q : H) = P \cap R$ and $(Q : K) = \widetilde{P} \cap R$. Since $R \subset B$ has s-coLO, there exist $Q_1, \ldots, Q_m \in$ Spec(R), for $m \leq \dim H$, such that $(Q_j : K) = (Q : K) =$

$\widetilde{P} \cap R$ for all j and $(\cap_{j=1}^{m} Q_J)^s \subset (Q : K)$. By 3.1(1), $(Q_j : H) = (Q : H) = P \cap R$, and by 3.1(2), $(Q : K)^l \subset (Q : H)$. Thus

$$(\cap_{j=1}^{m} Q_j)^{ls} \subset (Q : K)^l \subset (Q : H) \subset P \cap R.$$

(3) We need to complete the diagram

$$\begin{tikzcd}[arrows=dash] & Q_1 \arrow[d, dotted] \\ I_1 \arrow[ur, dotted] \arrow[d] & Q_2 \arrow[dl] \\ I_2 & \end{tikzcd}$$

where $I_1, I_2 \in H$-Spec(R) and $Q_2 \in$ Spec(R) with $(Q_2 : H) = I_2$. By Lemma 3.2, there exist $J_2 \subset J_1 \in K$-Spec(R) such that $(J_i : H) = I_i$, for $i = 1, 2$. Now by Lemma 3.2, $(Q_2 : H) = I_2$ implies that $(Q_2 : K) = J_2$. Since $R \subset B$ has strong coGU, there exists $Q_1 \in$ Spec(R) such that the diagram

$$\begin{tikzcd}[arrows=dash] & Q_1 \arrow[d, dotted] \\ J_1 \arrow[ur, dotted] \arrow[d] & Q_2 \arrow[dl] \\ J_2 & \end{tikzcd}$$

is complete. Hence $(Q_1 : K) = J_1$, and so by 3.1(1), $(Q_1 : H) = (J_1 : H) = I_1$ and we are done. □

Using the main result in Section 3 of [MS], we will apply the preceding Proposition to our case of interest, namely to a Hopf subalgebra $K \subset H$ containing the coradical H_0 of H. We let $J(H)$ denote the Jacobson radical of H, and let l be the index of nilpotency of $J(H^*)$ (that is, l is the smallest $n \geq 1$ such that $J(H^*)^n = 0$). Note that the length of the coradical filtration of H is $l - 1$.

Theorem 3.4. *Let K be a Hopf subalgebra of H such that $H_0 \subset K$. Let $R \subset A$ be a faithfully flat H-Galois extension, with comodule structure map $\delta : A \to A \otimes H$, and let $B := \delta^{-1}(A \otimes K)$. Then (R, A, B) is (H, K)-Krull admissible, and*

(1) K has coINC implies that H has coINC;

(2) K has s-coLO implies that H has sl-coLO, where l is the index of nilpotency of $J(H^*)$;

(3) K has strong coGU implies that H has strong coGU.

Proof. First note that $R \subset B$ is K-Galois by [MS, 3.11]. By [MS, Lemma 6.3], part (1) of 3.1 holds. Moreover by [MS, Theorem 3.7], $(I : H_0)^l \subset (I : H)$, where l is the nilpotency index of $J(H^*)$. By [MS, Lemma 3.3(1)] with $C = H_0 = K_0$, $(I : K)$ is H_0-stable and thus $(I : K) \subset (I : H_0)$. It follows that $(I:K)^l \subset (I:H)$ and so 3.1(2) holds. Thus (R,A,B) is (H,K)-Krull admissible. (1)–(3) now follow from Proposition 3.3. $\square$

Corollary 3.5. *Let I be a nilpotent Hopf ideal of H and let $\overline{H} = H/I$ be the quotient Hopf algebra. Then*

(1) $\overline{H}$ has INC implies that H has INC;

(2) $\overline{H}$ has s-LO implies that H has sl-LO, where now l is the index of nilpotency of $J(H)$;

(3) $\overline{H}$ has strong GU implies that H has strong GU.

Proof. Since $I \subset J(H)$, H/I maps surjectively to $H/J(H)$, and so

$$H^* \supset \overline{H}^* \supset (H^*)_0 = (H/J(H))^*.$$

Letting $K = \overline{H}^*$, we see that the corollary is just the dual of Theorem 3.4. The result follows by [MS, 4.3] and Lemma 2.5. $\square$

We give an application of the results of this section to $\mathbb{N}$-graded finite dimensional Hopf algebras. That is, $H = \oplus_{n\geq 0} H(n)$, where the grading is both as an algebra and as a coalgebra, and the antipode is a graded map; see [Sw2, p. 237]. By [Sw2, 11.1.1], $H(0) \supseteq H_0$, the coradical.

Theorem 3.6. *Let H be a finite dimensional $\mathbb{N}$-graded Hopf algebra and let $K = H(0)$. If K has any one of the Krull relations* 2.2(1), (2), (4) *or* 2.2(1)′, (2)′, (4)′*, then so does H. Moreover if K has all the Krull relations, then so does H.*

Proof. The projection $\pi : H \longrightarrow K$ is a surjective Hopf algebra map with nilpotent kernel $\oplus_{n\geq 1} H(n)$, and $H_0 \subset K$. The first part of the theorem now follows from Theorem 3.4 and Corollary 3.5. The second part follows from the first part together with Lemma 2.5. $\square$

As an example of such a Hopf algebra, we could begin with any Hopf algebra H such that H_0 is a Hopf subalgebra, and consider its coradical filtration $\{H_n\}$. Let $gr(H)$ be the associated graded algebra; that is, $gr(H) = \oplus_{n\geq 0} H(n)$, where $H(n) = H_n/H_{n-1}$. Then $gr(H)$ satisfies the hypotheses in the theorem. When H is graded and $H(n) = H_n/H_{n-1}$, H is said to be *coradically graded*.

Corollary 3.7. *Assume that H is pointed and coradically graded. Then H has all the Krull relations.*

Proof. Since H is pointed, $H(0) = H_0 = kG$ for some finite group G. It is known that kG has all the Krull relations; see the discussion in [MS, 4.9]. Thus the previous theorem applies. $\square$

If H is any pointed Hopf algebra, it is known that H satisfies GU and the dual Krull relations, but INC and t-LO remain open even if H is cocommutative. Passing to the associated graded Hopf algebra might be helpful in this problem, although going back up from $gr(H)$ to H seems very difficult.

We give some examples to which these results apply.

Example 3.8. Let g be a semisimple Lie algebra, let $u_q(g)$ be the finite-dimensional quantum group of Lusztig, q a root of unity, and let $H = u_q(g)^{\geq 0}$ be a Borel subalgebra. That is, $H = k\langle E_i, K_i \mid 1 \leq i \leq l\rangle$. Then in fact H is coradically graded with $H_0 = k\langle K_i \mid 1 \leq i \leq l\rangle$, a group algebra. By Corollary 3.7, H has all of the Krull relations.

Example 3.9. The finite-dimensional Hopf algebras $u(\mathcal{D})$ defined in [AS, 5.17] in terms of a linking datum $\mathcal{D}$ of finite Cartan type are coradically graded if all the linking elements λ_{ij} are 0. Thus they satisfy all of the Krull relations by Corollary 3.7

4. The Krull relations under twistings

We first review the idea of a Hopf algebra twisted by a cocycle. This was introduced in [Do], although twisting a Hopf algebra by a dual cocycle was done earlier in [Dr].

First, recall from [Sw1] that for a Hopf algebra H, a (left) *2-cocycle* on H is a convolution-invertible map $\sigma : H \otimes H \to k$ satisfying the equality

$$\sigma(h_{(1)}, l_{(1)})\sigma(h_{(2)}l_{(2)}, m) = \sigma(l_{(1)}, m_{(1)})\sigma(h, l_{(2)}m_{(2)}) \tag{4.1}$$

for all $h, l, m \in H$. We assume also that σ is normal, that is,

$$\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)$$

for all $h \in H$.

We may now form a new Hopf algebra H_σ by leaving the coalgebra structure of H unchanged but twisting the algebra structure by σ. That is, H_σ has new multiplication

$$h \cdot_\sigma l := \sigma(h_{(1)}, l_{(1)})h_{(2)}l_{(2)}\sigma^{-1}(h_{(3)}, l_{(3)}).$$

for all $h, l \in H$. One can also define a new antipode.

Also, given a right H-comodule algebra A, we may form the algebra A_σ, with twisted multiplication

$$a \cdot_\sigma b = \sigma^{-1}(a_{(1)}, b_{(1)})a_{(0)}b_{(0)}$$

for all $a, b \in A$. Then A_σ is a right H_σ-comodule algebra, using the same comodule structure map as for A. We note that we need σ^{-1} here because of the mixture of a left cocycle with a right comodule.

A reference for the above facts is [KS, 10.2.3]; see also [M, Sec. 7.5] for a discussion of A_σ. We first prove a result about twisting Galois extensions.

Theorem 4.3. *Let $R \subset A$ be an H-extension, let σ be a cocycle on H, and consider the twisted algebra A_σ. Then $R_\sigma = R$, and an ideal I of R is H-stable if and only if it is H_σ-stable. Moreover*

(1) $R \subset A$ is H-Galois if and only if $R \subset A_\sigma$ is H_σ-Galois;

(2) $R \subset A$ is H-cleft if and only if $R \subset A_\sigma$ is H_σ-cleft; moreover if $A = R\#_\tau H$, then $A_\sigma \cong R\#_{\tau^\sigma} H_\sigma$, where $\tau^\sigma = \tau * \sigma^{-1}$.

(3) H-$\mathrm{Spec}(R) = H_\sigma$-$\mathrm{Spec}(R)$

Proof. First, since $R = A^{coH} = A_\sigma^{coH_\sigma}$, it is easy to see that $r \cdot_\sigma a = ra$ and $a \cdot_\sigma r = ar$for any $r \in R, a \in A$. It follows that $R_\sigma = R$. Moreover if I is any ideal of R and $AI = IA$, then clearly $A \cdot_\sigma I = I \cdot_\sigma A$. Thus the fact about stability follows.

(1) Consider the two canonical Galois maps for A and A_σ; that is, $\beta : A \otimes_R A \longrightarrow A \otimes H$ via $a \otimes b \mapsto ab_{(0)} \otimes b_{(1)}$ and $\beta^\sigma : A_\sigma \otimes_R A_\sigma \longrightarrow A_\sigma \otimes H_\sigma$ via $a \otimes b \mapsto a \cdot_\sigma b_{(0)} \otimes b_{(1)} = a_{(0)}b_{(0)} \otimes b_{(2)}\sigma^{-1}(a_{(1)}, b_{(1)})$.

Define $\Phi, \Psi : A \otimes H \longrightarrow A \otimes H$ by

$$\Phi(a \otimes h) = a_{(0)} \otimes \sigma^{-1}(a_{(1)}, Sh_{(3)})\sigma(h_{(1)}, Sh_{(2)})h_{(4)}$$

and

$$\Psi(a \otimes h) = a_{(0)} \otimes \sigma(a_{(1)}, Sh_{(1)})\sigma^{-1}(Sh_{(2)}, h_{(3)})h_{(4)}.$$

We claim that $\Psi = \Phi^{-1}$ and that $\beta = \Phi \circ \beta^\sigma$. Thus β is bijective if and only if β^{-1} is bijective. To show this we require the cocycle condition (4.1), and in addition the identities

$$\sigma(h_{(1)}, Sh_{(2)})\sigma^{-1}(Sh_{(3)}, h_{(4)}) = \varepsilon(h) \tag{4.4}$$

$$\sigma^{-1}(Sh_{(1)}, h_{(2)})\sigma(h_{(3)}, Sh_{(4)}) = \varepsilon(h). \tag{4.5}$$

Identity (4.4) appears in [BM] and [Do]; see also [KS]. (4.5) can be obtained from (4.4) as follows: apply (4.4) to the left cocycle σ^{-1} on H^{cop} and use that $S_{H^{cop}} = \overline{S}_H$. Then

$$\sigma^{-1}(h_{(4)}, \overline{S}h_{(3)})\sigma(\overline{S}h_{(2)}, h_{(1)}) = \varepsilon(h).$$

Now replace h by Sh and we have (4.5).

We can now show that $\Psi = \Phi^{-1}$. First,

$$\begin{aligned}
(\Psi \circ \Phi)(a \otimes h) &= \Psi(a_{(0)} \otimes \sigma^{-1}(a_{(1)}, Sh_{(3)})\sigma(h_{(1)}, Sh_{(2)})h_{(4)}) \\
&= a_{(0)(0)} \otimes \sigma(a_{(0)(1)}, Sh_{(4)(1)})\sigma^{-1}(Sh_{(4)(2)}, h_{(4)(3)}) \cdot \\
&\qquad \cdot \sigma^{-1}(a_{(1)}, Sh_{(3)})\sigma(h_{(1)}, Sh_{(2)})h_{(4)(4)} \\
&= a_{(0)} \otimes \sigma(a_{(1)}, Sh_{(4)})\sigma^{-1}(Sh_{(5)}, h_{(6)})\sigma^{-1}(a_2, Sh_{(3)}) \cdot \\
&\qquad \cdot \sigma(h_{(1)}, Sh_{(2)})h_{(7)} \\
&= a_{(0)} \otimes \sigma^{-1}(Sh_{(3)}, h_{(4)})\sigma(h_{(1)}, Sh_{(2)})h_{(5)} \\
&= a \otimes h,
\end{aligned}$$

using (4.4) in the last step. Similarly, using (4.5), we see that

$$\begin{aligned}(\Phi\circ\Psi)(a\otimes h) &= \Phi(a_{(0)}\otimes\sigma(a_{(1)},Sh_{(1)})\sigma^{-1}(Sh_{(2)},h_{(3)})h_{(4)})\\ &= a_{(0)(0)}\otimes\sigma(a_{(1)},Sh_{(1)})\sigma^{-1}(Sh_{(2)},h_{(3)})\sigma^{-1}(a_{(0)(1)},Sh_{(4)(3)})\cdot\\ &\qquad\cdot\sigma(h_{(4)(1)},Sh_{(4)(2)})h_{(4)(4)}\\ &= a_{(0)}\otimes\sigma(a_{(2)},Sh_{(1)})\sigma^{-1}(Sh_{(2)},h_{(3)})\sigma^{-1}(a_{(1)},Sh_{(6)})\cdot\\ &\qquad\cdot\sigma(h_{(4)},Sh_{(5)})h_{(7)}\\ &= a_{(0)}\otimes\sigma(a_2,Sh_{(1)})\sigma^{-1}(a_{(1)},Sh_{(2)})h_{(3)}\\ &= a\otimes h.\end{aligned}$$

Thus $\Psi=\Phi^{-1}$. Finally we check that $\beta=\Phi\circ\beta^\sigma$, using (4.1):

$$\begin{aligned}\Phi(\beta^\sigma(a\otimes h)) &= \Phi(a_{(0)}b_{(0)}\otimes b_{(2)}\sigma^{-1}(a_{(1)},b_{(1)})\\ &= a_{(0)}b_{(0)}\otimes\sigma^{-1}(a_{(0)(1)}b_{(0)(1)},Sb_{(2)(3)})\sigma(b_{(2)(1)},Sb_{(2)(2)})\cdot\\ &\qquad\cdot\sigma^{-1}(a_{(1)},b_{(1)})b_{(2)(4)}\\ &= a_{(0)}b_{(0)}\otimes\sigma^{-1}(a_{(1)}b_{(1)},Sb_{(5)})\sigma(b_{(3)},Sb_{(4)})\sigma^{-1}(a_2,b_{(2)})b_{(6)}\\ &= a_{(0)}b_{(0)}\otimes\sigma^{-1}(a_{(1)},b_{(1)}Sb_{(6)})\sigma^{-1}(b_{(2)},Sb_{(5)})\sigma(b_{(3)},Sb_{(4)})b_{(6)}\\ &= a_{(0)}b_{(0)}\otimes\sigma^{-1}(a_{(1)},b_{(1)}Sb_{(2)})b_{(3)}\\ &= ab_{(0)}\otimes b_{(1)}\\ &= \beta(a\otimes b).\end{aligned}$$

This proves (1).

(2) Assume that A is H-cleft, via the H-comodule map $\gamma: H\longrightarrow A$ with convolution inverse γ^{-1}. We claim that A_σ is H_σ-cleft, via the same map $\gamma^\sigma=\gamma$ on vector spaces, but with convolution inverse $(\gamma^\sigma)^{-1}(h)=\gamma^{-1}(h_{(3)})\sigma(h_{(1)},Sh_{(2)})$. First, the fact that γ is an H_σ-comodule map follows since $H=H_\sigma$ as coalgebras and $A=A_\sigma$ as comodules. Also, since γ is a comodule map, $\delta(\gamma(h))=\gamma(h_{(1)})\otimes h_{(2)}$ and $\delta(\gamma^{-1}(h))=\gamma^{-1}(h_{(2)})\otimes Sh_{(1)}$, where δ is the comodule structure map of A. It follows that

$$\delta((\gamma^\sigma)^{-1}(h))=\gamma^{-1}(h_{(4)})\otimes Sh_{(3)}\sigma(h_{(1)},Sh_{(2)}).$$

Now in $\mathrm{Hom}(H,A_\sigma)$,

$$\begin{aligned}\gamma(h_{(1)})\cdot_\sigma(\gamma^\sigma)^{-1}(h_{(2)}) &= \gamma(h_{(1)})_{(0)}(\gamma^\sigma)^{-1}(h_{(2)})_{(0)}\cdot\\ &\qquad\cdot\sigma^{-1}(\gamma(h_{(1)})_{(1)},(\gamma^\sigma)^{-1}(h_{(2)})_{(1)})\\ &= \gamma(h_{(1)(1)})\gamma^{-1}(h_{(2)(4)})\sigma^{-1}(h_{(1)(2)},Sh_{(2)(3)})\cdot\\ &\qquad\cdot\sigma(h_{(2)(1)},Sh_{(2)(2)}))\\ &= \gamma(h_{(1)})\gamma^{-1}(h_{(6)})\sigma^{-1}(h_{(2)},Sh_{(5)})\sigma(h_{(3)},Sh_{(4)})\\ &= \gamma(h_{(1)})\gamma^{-1}(h_{(2)})=\varepsilon(h)1.\end{aligned}$$

Since H is finite-dimensional, also $(\gamma^\sigma)^{-1}$ is the left inverse of γ, and so A_σ is H_σ-cleft.

To see that the new cocycle τ^σ is as described, first recall that cleft extensions are always crossed products. Thus $A_\sigma \cong R\#_{\tau^\sigma} H_\sigma$ for some Hopf 2-cocycle $\tau^\sigma : H_\sigma \otimes H_\sigma \longrightarrow R$, where the H_σ-comodule structure on $R\#_{\tau^\sigma} H_\sigma$ is given by $id \otimes \triangle_{H_\sigma} = id \otimes \triangle_H$.

Choose $r\#g$ and $s\#h$ in $A = R\#_\tau H$, and consider their multiplication in A_σ:

$$\begin{aligned}(r\#g) \cdot_\sigma (s\#h) &= (r\#g_{(1)})(s\#h_{(1)})\sigma^{-1}(g_{(2)}, h_{(2)}) \\ &= r(g_{(1)} \cdot s)\tau(g_{(2)}, h_{(1)})\#g_{(3)}h_{(2)}\sigma^{-1}(g_{(4)}, h_{(3)}) \\ &= r(g_{(1)} \cdot s)\tau(g_{(2)}, h_{(1)})\#\sigma^{-1}(g_{(3)}, h_{(2)})\sigma(g_{(4)}, h_{(3)})g_{(5)}h_{(4)} \\ &\qquad \sigma^{-1}(g_{(6)}, h_{(5)}) \\ &= r(g_{(1)} \cdot s)\tau(g_{(2)}, h_{(1)})\sigma^{-1}(g_{(3)}, h_{(2)})\#g_{(4)} \cdot_\sigma h_{(3)}.\end{aligned}$$

But considered as elements in $R\#_{\tau^\sigma} H_\sigma$, their product is

$$(r\#g)(s\#h) = r(g_{(1)} \cdot s)\tau^\sigma(g_{(2)}, h_{(1)})\#g_{(3)} \cdot_\sigma h_{(2)}.$$

Thus $\tau^\sigma(g, h) = \tau(g_{(1)}, h_{(1)})\sigma^{-1}(g_{(2)}, h_{(2)})$.

Alternatively the cocycle can be expressed in terms of the cleft map γ (respectively γ^σ).

(3) The identification of the stable parts of Spec follows from the remarks at the beginning of the proof, once we know (1). □

Remark 4.6. In the terminology of [MS, Definition 8.8], H is called *strongly cosemisimple* if for all right H-comodule algebras A with ring of coinvariants R and any $P \in \mathrm{Spec}A$, $P \cap R$ is a semiprime ideal of R. By [MS, Theorem 8.11], H is strongly cosemisimple if and only if for all faithfully flat H-Galois extensions $R \subset A$ with R being H-prime, R is semiprime. Hence Theorem 4.3(3) implies that H is strongly cosemisimple if and only if H_σ is strongly cosemisimple.

Theorem 4.7. *Let H_σ be any cocycle twist of H. Then*

(1) any one of the Krull relations coINC, t-coLO, and strong coGU are true for $H \iff$ they are true for H_σ;

(2) if H has coGU and t-LO, then H_σ also has coGU.

Proof. (1) follows from Theorem 4.3(3), because the three dual Krull relations coINC, t-coLO, and strong coGU are defined only in terms of ideals of R.

(2) follows from (1) and Lemma 2.5. □

Corollary 4.8. *If H has the six Krull relations INC, s-LO, GU, coINC, t-coLO, and coGU, then the double $D(H)$ has coINC, st-coLO, and coGU.*

Proof. By Lemma 2.5, H also has strong GU and strong coGU. Now it is known that $D(H) = (H^{*cop} \otimes H)_\sigma$ for some cocycle σ on $(H^{*cop} \otimes H)$ (see [DT]). The result now follows by Theorem 4 since the tensor product $H^{*cop} \otimes H$ has all the Krull relations by Remark 2.4 and [MS, 6.7]. □

5. Twisting the comultiplication

In this section we consider dual cocycle twists, as in [Dr]. That is, let $\Omega \in H \otimes H$ be a dual cocycle for H. Then H^Ω has the same multiplication as H but has new comultiplication $\Delta_\Omega(h) = \Omega\Delta_H(h)\Omega^{-1}$. This construction is the formal dual of the construction of the cocycle twists in Section 4, in the following sense: if H is finite-dimensional, then $(H^*)^\Omega = (H_\sigma)^*$. For if σ is a 2-cocycle on H, then σ corresponds to an invertible element $\Omega \in H^* \otimes H^* \cong (H \otimes H)^*$, and we may twist the comultiplication of H^* by Ω. The explicit correspondence between σ and Ω is given by

$$\sigma(h, l) = \sum \Omega^{(1)}(h)\Omega^{(2)}(l),$$

for all $h, l \in H$, where we use the formal notation $\Omega = \sum \Omega^{(1)} \otimes \Omega^{(2)}$.

Analogously if A is an H-comodule algebra, then A is an H^*-module algebra, and we can consider it either as twisted by σ or by Ω.

Using this reformulation we may state the dual version of Theorem 4.7.

Theorem 5.1. *Let H^Ω be any dual cocycle twist of H. Then*

(1) any one of the Krull relations INC, t-LO, and strong GU are true for $H \iff$ they are true for H^Ω;

(2) if H has GU and t-coLO, then H^Ω also has GU.

We first consider triangular Hopf algebras.

Theorem 5.2. *Let k be an algebraically closed field of characteristic 0, and let H be a (finite-dimensional) triangular Hopf algebra. Then H has the Krull relations INC, t-LO, and strong GU.*

Proof. By [EG2], for any triangular Hopf algebra, the Jacobson radical $J(H)$ is a Hopf ideal. Thus $\overline{H} = H/J(H)$ is a semisimple triangular Hopf algebra. By [EG1], it follows that $\overline{H} = kG^\Omega$, the twist of a group algebra by a dual cocycle $\Omega \in kG \otimes kG$. Applying Theorem 5.1, we see that $\overline{H}$ has INC, t-LO, and strong GU, since kG has these three properties. The theorem now follows from Corollary 3.5. □

Another approach to Theorem 5.2 is by using supergroups. For, in [EG2], it is shown (using [AEG]) that if H is triangular then it is a dual cocycle twist of a modified supergroup algebra. Thus Theorem 5.2 would follow immediately from Theorem 5.1 and the next Corollary.

More precisely, we consider *modified supergroup algebras* as described in [AEG]; these are based on the definition of supergroups due to Kostant [Ko]. First, a (finite-dimensional) supergroup is constructed from a finite group G and a finite-dimensional representation V of G. Let $\wedge V$ be the exterior algebra of V and let $\mathcal{H} = \wedge V \# kG$. Then $\mathcal{H}$ becomes a cocommutative Hopf superalgebra by letting V be odd, G be even, and each $x \in V$ be (graded) primitive. $\mathcal{H}$ is called a *supergroup* in [Ko], although in his formulation $\wedge V$ is viewed as $U(\mathfrak{g})$, where $\mathfrak{g} = V$ is an odd Lie superalgebra. Moreover, every finite-dimensional cocommutative Hopf superalgebra over $\mathbb{C}$ is of this form.

To describe the modified supergroup algebra H, consider $\mathcal{H}$ as above and assume in addition that G contains a central group-like element g such that $g^2 = 1$ and $gxg = -x$ for all $x \in V$. We define H by letting $H = \mathcal{H}$ as an algebra, but changing the comultiplication on $\mathcal{H}$ by defining $\Delta_H(x) := x \otimes 1 + g \otimes x$ for all $x \in V$, and letting $\Delta_H(y) = \Delta_{\mathcal{H}}(y)$ for all $y \in G$. With this definition, (H, Δ_H) becomes an ordinary Hopf algebra, the *modified supergroup algebra.*

Alternatively, H can be described as follows: note that $k\mathbb{Z}_2 = k\langle u\rangle$ acts on $\mathcal{H}$ via $u \cdot x = -x$ for all $x \in V$ and $u \cdot y = y$ for all $y \in G$. We may thus form the Radford biproduct $\tilde{\mathcal{H}} = \mathcal{H} * k\mathbb{Z}_2$; it is an ordinary Hopf algebra, and H may be identified with the quotient $\tilde{\mathcal{H}}/(gu - 1)$.

Now $K := \wedge V \# k\langle g\rangle$ is a normal Hopf subalgebra of H, with quotient Hopf algebra $H/HK^+ \cong k(G/\langle g\rangle)$.

We recall the Transitivity Theorem in our previous paper [MS, Theorem 6.7]: assume that H has a normal Hopf subalgebra K with quotient Hopf algebra $\overline{H}$, and assume that both K and $\overline{H}$ have all the Krull relations. Then H has all the Krull relations.

Corollary 5.3. *Let H be a modified supergroup algebra as above. Then H has all of the Krull relations.*

Proof. Let V, G, and K be as above. We have $\overline{H} = H/HK^+ \cong k(G/\langle g\rangle)$. Thus Corollary 3.7 and the Transitivity Theorem apply to give that H has all the Krull relations. □

Finally we consider factorizable Hopf algebras. We recall a theorem of [RS]: if H is factorizable, then for some dual cocycle Ω, $D(H) \cong (H \otimes H)^{\Omega}$; for another proof, see [S2].

Corollary 5.4. *Assume that H is factorizable and satisfies all of the Krull relations. Then $D(H)$ satisfies INC, t-LO, and strong GU.*

Proof. By the Transitivity Theorem, $H \otimes H$ also satisfies all of the Krull relations. Now apply [RS] and Theorem 5.1. □

Corollary 5.5. *Let G be any finite group. Then $D(G)$ and $D(D(G))$ satisfy all of the Krull relations.*

Proof. First note that since the action of $(kG)^* = k^G$ on kG is trivial, in fact $D(G) = k^G \# kG$, an ordinary smash product. Thus $K = k^G \# 1$ is a normal Hopf subalgebra of $D(G)$ with Hopf quotient $\overline{H} \cong kG$. Thus since both k^G and kG have all the Krull relations, so does $D(G)$ by transitivity. Now apply Theorem 4.7 to see that $D(D(G))$ has all of the co-Krull relations, and Corollary 5.4 to see that $D(D(G))$ has all of the basic Krull relations. □

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University of Southern California, Los Angeles, CA 90089-1113
E-mail address: `smontgom@math.usc.edu`

Mathematisches Institut, Universität München, D-80333 Munich, Germany
E-mail address: `hanssch@rz.mathmatik.uni-muenchen.de`