On the Even Powers of the Antipode of a Finite-Dimensional Hopf Algebra *

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Abstract

The trace of powers of the square of the antipode s^2 of a finite-dimensional Hopf algebra A over a field k is studied. It is shown in many cases that the trace function vanishes on s^{2m} when $s^{2m} \neq 1_A$. Finer properties of the antipode are related to this phenomenon.

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Introduction

Let A be a finite-dimensional Hopf algebra with antipode s over a field k. By an earlier result [14, Theorem 1] of the first author the order of s^2 is finite. In this paper we study some of the finer properties of s^2 which are related to the traces $\mathsf{tr}(s^{2m})$ of powers of s^2.

Suppose that $t : V \longrightarrow V$ is a linear endomorphism of a finite-dimensional vector space V over k. We say that t satisfies the *vanishing trace condition*

$$\text{if whenever } m \geq 0 \text{ and } t^m \neq 1_V \text{ then } \mathsf{tr}(t^m) = 0.$$

Now suppose that V is an algebra over k. We say that t satisfies the *strong vanishing trace condition*

$$\text{if whenever } m \geq 0 \text{ and } t^m \neq 1_V \text{ then } \mathsf{tr}(t^m \circ r(a)) = 0 \text{ for all } a \in V,$$

where $r(a) : V \longrightarrow V$ is right multiplication by a which is therefore defined by $r(a)(x) = xa$ for all $x \in V$. If t satisfies the strong vanishing trace condition then t satisfies the vanishing trace condition. We show that s^2 satisfies the strong vanishing trace condition in a number of interesting cases.

Research for this paper began with a question raised by Pavel Etingof to a number of Hopf algebraists at a conference held in Durham, England, in the summer of 1999. In our terminology the question was whether or not s^2 satisfies the vanishing trace condition. Investigation of this question led us to formulate the strong vanishing trace condition and to study it connection with Hopf algebra automorphisms of A, in particular with s^2. When the characteristic of k is zero and s^2 satisfies the strong vanishing trace condition, much can be said about s^2, the eigenvalues and eigenspaces of s^2, and there are implications for the algebra structure of A.

The strong vanishing trace condition has implications for classification of finite-dimensional Hopf algebras. For example, under the assumption that s^2 satisfies the strong vanishing trace condition, the classification of all Hopf algebras of dimension p^2, where p is an odd prime, over an algebraically closed field of characteristic zero is complete. See the discussion following Proposition 6.

The strong vanishing trace condition for a linear endomorphism t of A can be expressed in a way which is very useful for us in terms of integrals.

Let $\Lambda \in A$ and $\lambda \in A^*$ be left and right integrals for A and A^* respectively which satisfy $\lambda(\Lambda) = 1$ and set

$$\nu_A(t) = s(\Lambda_{(2)})t(\Lambda_{(1)}).$$

Then we show in Lemma 3 b) that $\mathtt{tr}(t \circ r(a)) = 0$ for all $a \in A$ if and only if $\nu_A(t) = 0$. Thus t satisfies the strong vanishing trace condition if and only if whenever $m \geq 0$ and $t^m \neq 1_A$ then $\nu_A(t^m) = 0$.

Our study of the strong vanishing trace condition for s^2 splits very naturally into two cases: s^2 semisimple, and s^2 not semisimple, as a linear endomorphism of A. Suppose that s^2 is not semisimple and that the characteristic of k is not 2. We show in Proposition 3 that either s^2 or s^{*2} satisfies the strong vanishing trace condition; hence s^2 (and thus s^{*2}) satisfies the vanishing trace condition.

The case s^2 semisimple is far more interesting. If $s^2 = 1_A$ then s^2 is semisimple and satisfies the strong vanishing trace condition vacuously. We note that $s^2 = 1_A$ if A is either commutative or cocommutative, or if A is semisimple or cosemisimple and the characteristic of k is zero. Our main results when s^2 is semisimple are that s^2 satisfies the strong vanishing trace condition if A is pointed or if A^* is pointed. These results follow from Theorem 1 and Theorem 4 respectively.

The ideas developed in this paper enable us to prove Theorem 6 which together with Theorem 1 implies when A is pointed and k is an algebraically closed field of characterictic zero that:

a) The order r of s^2 divides $\mathrm{Dim}\, A/\mathrm{Dim}\, B$, where $B = A_0$ is the coradical of A, and the minimal polynomial of s^2 is $X^r - 1$;

b) The eigenvalues of s^2 are the r^{th} roots of unity in k;

c) The eigenspaces for s^2 all have the same dimension which is $\mathrm{Dim}\, A/r$;

d) If e is an idempotent of A which satisfies $s^2(e) = e$ then r divides $\mathrm{Dim}\, Ae$.

The bound described in a) is sharp. When A is pointed it follows from a) that $r = \mathrm{Dim}\, A$ if and only if $\mathrm{Dim}\, A = 1$.

More generally, Theorem 6 holds for a Hopf algebra automorphism t of A which satisfies the strong vanishing trace condition when k is an algebraically closed field of characteristic zero. Here t replaces s^2 and B is the unique maximal semisimple (and hence the maximal cosemisimple) sub-Hopf algebra of A.

The paper is organized as follows. In Section 1 results are derived for certain linear automorphisms of V when k is algebraically closed, has characteristic zero and t satisfies certain trace conditions. In the next section we study the element $\nu_A(t)$ for linear endomorphisms t of A and prove that either $\nu_A(s^{2m}) = 0$ for all $m \geq 0$ or $\nu_{A^*}(s^{*2m}) = 0$ for all $m \geq 0$ when s^2 is not semisimple and the characteristic of k is not 2. A key step in the proof is the observation that if t is an algebra endomorphism of A and B is a sub-Hopf algebra of A which satisfies $t(B) \subseteq B$, then $\nu_B(t|_B) = 0$ implies $\nu_A(t) = 0$. Thereby the proof reduces to the case A or A^* is the group algebra $k[Z_p]$ of the cyclic group of order p and the field k has characteristic $p > 0$.

In Section 3 we show that s^2 satisfies the strong vanishing trace condition when s^2 is semisimple and the coradical A_0 of A is cocommutative. We reduce to the case where A is generated by a, x as an algebra which satisfy the relation $xa = qax$, where $q \in k$ is a root of unity, and whose coproducts are given by $\Delta(a) = a \otimes a$ and $\Delta(x) = x \otimes a + 1 \otimes x$ respectively.

We continue the study of s^2 when it is semisimple in Section 4 and prove that s^2 satisfies the strong vanishing trace condition when the coradical of A^* is cocommutative. Our proof is based on the stucture theory of finite-dimensional graded Hopf algebras. A key step in the proof is the following observation. Suppose that t is a Hopf algebra endomorphism of A and that I is a nilpotent Hopf ideal of A which satisfies $t(I) \subseteq I$. Then $\nu_{A/I}(\bar{t}) = 0$ implies $\nu_A(t) = 0$, where $\bar{t}$ is the Hopf algebra automorphism of A/I induced by t.

In Section 5 we discuss the characters $\chi_A \in A^*$ and $x_A \in A$ of the right regular representations of A and A^* respectively using the properties of A as a Frobenius algebra. We deduce the trace formula

$$(\mathrm{Dim}\, A)\mathsf{tr}(t^*|_{\chi_A A^*}) = <\nu_{A^*}(t^*), \nu_A(1_A)>$$

for Hopf algebra automorphisms t of A. This formula generalizes [6, Equation (6)]. Using the results of Section 4 we show that

$$x_A = \left(\frac{\mathrm{Dim}\, A}{\mathrm{Dim}\, B}\right) x_B,$$

where B is the sub-Hopf algebra of A generated by the coradical of A.

When the characteristic of k does not divide $\mathrm{Dim}\, A$ then $\chi_A/\mathrm{Dim}\, A$ and $x_A/\mathrm{Dim}\, A$ are important non-zero idempotents. For example, when the characteristic of k is zero and A is semisimple then $\chi_A/\mathrm{Dim}\, A$ and $x_A/\mathrm{Dim}\, A$ are non-zero two-sided integrals of A^* and A respectively.

In Section 6 we examine implications of the ideas developed in this paper for finite-dimensional pointed Hopf algebras over an algebraically closed field of charcteristic zero and we examine implications for the Drinfel'd double. Our discussion of pointed Hopf algebras is based on certain idempotents.

We want to point out that it may very well be true that the square of the antipode of any finite-dimensional Hopf algebra over k satisfies the strong vanishing trace condition. Needless to say we know of no counterexample.

We shall assume that the reader has some familiarity with Hopf algebras [9, 18]. Throughout this paper k is a field.

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1 Preliminaries

We begin with several observations about a linear endomorphism t of a finite-dimensional vector space V over k. The endomorphism t is *semisimple* [3, A VII, 8] if each t-stable subspace of V has a t-stable direct summand, or equivalently if the minimal polynomial of t over k has no multiple factors. If k is algebraically closed, then t is semisimple if and only if t is diagonalizable.

If $t^r = 1_V$ for some positive integer r, then the smallest such r is called the *order of* t; otherwise the order of t is said to be infinite. We denote the order of t by $\mathrm{order}(t)$ when it is finite. We have remarked that the order of the antipode of a finite-dimensional Hopf algebra over k is finite. If t has finite order then t is semisimple if and only if the characteristic of k does not divide the order of t.

Lemma 1 *Suppose that the field k is algebraically closed, has characteristic 0 and that V is a finite-dimensional vector space over k. Assume further that t is a linear automorphism of V with finite order r and that t satisfies the vanishing trace condition. Then:*

a) *The minimal polynomial for t is $X^r - 1$.*

b) *r divides* $\mathrm{Dim}\, V$.

c) *The eigenvalues of t are the r^{th} roots of unity in k and each eigenspace has dimension* $(\mathrm{Dim}\, V)/r$.

PROOF: We can assume that $r > 1$. Let $p(X) = a_0 + a_1X + \cdots + X^\ell$ be the minimal polynomial of t. Since t is an automorphism of V necessarily $a_0 \neq 0$. Since $t^r - 1_V = 0$ it follows that $\ell \leq r$. Taking the trace of both sides of the equation $0 = a_0 1_V + a_1 t + \cdots + t^\ell$ gives $\mathsf{tr}(t^\ell) = -a_0(\mathrm{Dim}\, V) \neq 0$. Therefore $\ell \geq r$. Consequently $\ell = r$ and $X^r - 1$ is the minimal polynomial of t.

To complete the proof we first note that the eigenvalues of t are among the r^{th} roots of unity $\omega_1, \ldots, \omega_r \in k$. Let $V_\imath = \mathrm{Ker}\,(t - \omega_\imath 1_V)$ and $n_\imath = \mathrm{Dim}\, V_\imath$ for all $1 \leq \imath \leq r$. Consider the invertible matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_1 & \omega_2 & \cdots & \omega_r \\ \vdots & \vdots & & \vdots \\ \omega_1^{r-1} & \omega_2^{r-1} & \cdots & \omega_r^{r-1} \end{pmatrix}.$$

Then

$$\mathcal{A}\begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{pmatrix} = \begin{pmatrix} \mathrm{Dim}\, V \\ \mathsf{tr}(t) \\ \vdots \\ \mathsf{tr}(t^{r-1}) \end{pmatrix} = \begin{pmatrix} \mathrm{Dim}\, V \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathcal{A}\begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix}$$

for some $a \in k$. The last equation follows from the fact that if G is a finite subgroup of k^* of order $m > 1$ then $\sum_{g \in G} g^\ell = 0$ for all $1 \leq \ell < m$. Since $\mathcal{A}$ is invertible $n_1 = \cdots = n_r = a$; thus $ra = \mathrm{Dim}\, V$. $\square$

When t has finite order and k is an algebraically closed field of characteristic 0 the dimension of the subspace of fixed points of t divides $\mathrm{Dim}\, V$ by part b) of the previous lemma. The full hypothesis of the lemma is not required for part b).

Lemma 2 *Let V be a finite-dimensional vector space over the field k and suppose that k has characteristic* 0. *Let t be a diagonalizable linear automorphism of V which has finite order r and suppose that* $\mathsf{tr}(t + t^2 + \cdots + t^{r-1}) = 0$. *Then* $\mathrm{Dim}\, V = r(\mathrm{Dim}\, U)$, *where U is the subspace of fixed points of t.*

PROOF: Since t has finite order r and is diagonalizable, V has a basis of eigenvectors for t belonging to r^{th} roots of unity $\omega_1, \ldots, \omega_n$, where $n = \mathrm{Dim}V$. We may assume that $\omega_1, \ldots, \omega_s \neq 1$ and that $\omega_{s+1} = \cdots = \omega_n = 1$. Since

$$\mathsf{tr}(t^\ell) = \omega_1^\ell + \cdots + \omega_s^\ell + (n-s)$$

for all $1 \leq \ell \leq r-1$, and $\omega_\imath + \cdots + \omega_\imath^{r-1} = -1$ for all $1 \leq \imath \leq s$, it follows that

$$-s + (r-1)(n-s) = \mathsf{tr}(t + t^2 + \cdots + t^{r-1}) = 0.$$

Thus $n = r(n-s)$ which is the conclusion of the lemma. $\square$

Now let A be a finite-dimensional Hopf algebra over k. Recall that $a \in A$ is *grouplike* if $\Delta(a) = a \otimes a$ and $\epsilon(a) = 1$. The set of grouplike elements $G(A)$ of A is linearly independent and is a multiplicative subgroup of A. Thus the span $k[G(A)]$ of $G(A)$ is the group algebra of G over k. By [10, Theorem 7] the order of G divides $\mathrm{Dim}\, A$. In particular the order of any $a \in G(A)$, denoted by $\mathrm{order}(a)$, divides $\mathrm{Dim}\, A$.

Let $g \in G(A)$ and $\alpha \in G(A^*)$ be the distinguished grouplike elements of A and A^* respectively. Then

$$s^4 = \imath_g \circ (\imath_{\alpha^{-1}})^* \tag{1}$$

is the composition of two commuting automorphisms of A, where $\imath_g$ is defined by $\imath_g(x) = gxg^{-1}$ for all $x \in A$ and the automorphism $\imath_{\alpha^{-1}}$ of A^* is defined by $\imath_{\alpha^{-1}}(p) = \alpha^{-1}p\alpha$ for all $p \in A^*$. See [14, 16] for example. In particular the order of s^2 is finite and

$$\mathrm{order}(s^2) \ \text{ divides } \ 2\Big(\mathrm{lcm}(\mathrm{order}(g), \mathrm{order}(\alpha))\Big) \tag{2}$$

by (1). By (2) it follows that s^2 is semisimple if the characteristic of k is not 2 and does not divide $\mathrm{order}(g)$, $\mathrm{order}(\alpha)$.

2 General Results for $\nu_A(t)$ and $\nu_A(s^{2m})$ in the Special Case when s^2 Is not Semisimple

Let A be a finite-dimensional Hopf algebra with antipode s over the field k and let Λ be a fixed non-zero left integral for A. For a linear endomorphism t of A we define

$$\nu_A(t) = s(\Lambda_{(2)})t(\Lambda_{(1)}). \tag{3}$$

Since the subspace of left integrals for A is one-dimensional, choosing a different non-zero left integral to define $\nu_A(t)$ results in a non-zero scalar multiple of the original.

If $\nu_A(t) = 0$ then $\mathsf{tr}(t) = 0$ by part a) of the next lemma. We are interested in determining when $\mathsf{tr}(t) = 0$ for certain t, specifically when $t = s^{2m}$ for some integer m. For $a \in A$ let $\ell(a), r(a)$ be the endomorphisms of A defined by

$$\ell(a)(x) = ax \quad \text{and} \quad r(a)(x) = xa$$

for all $x \in A$.

First we will see that $\nu_A(t) = 0$ if and only if the traces of certain related endomorphisms vanish. This result has a symmetric form if t is an algebra automorphism which commutes with s. We note that Hopf algebra endomorphisms commute with the antipode.

Lemma 3 *Let A be a finite-dimensional Hopf algebra with antipode s over the field k and suppose that t is a linear endomorphism of A. Then:*

a) $\mathsf{tr}(t) = \lambda(\nu_A(t)) = \lambda(s(\Lambda_{(2)})t(\Lambda_{(1)}))$*, where Λ is a left integral for A and λ is a right integral for A^* which satisfy $\lambda(\Lambda) = 1$.*

b) $\nu_A(t) = 0$ *if and only if* $\mathsf{tr}(r(a) \circ t) = 0$ *for all $a \in A$.*

c) *If t is an algebra automorphism of A then $\nu_A(t^{-1}) = 0$ if and only if* $\mathsf{tr}(l(a) \circ t) = 0$ *for all $a \in A$.*

Proof: Let Λ be a left integral for A and λ be a right integral for A^* such that $\lambda(\Lambda) = 1$. Then

$$\mathsf{tr}(t) = \lambda(s(\Lambda_{(2)})t(\Lambda_{(1)})) = \lambda(s(t(\Lambda_{(2)}))\Lambda_{(1)}) = \lambda(t(s(\Lambda_{(2)}))\Lambda_{(1)})$$

for all linear endomorphisms t of A by [16, Theorem 2]. Thus part a) follows.

To show part b) we recall that A^* is a free left A-module with basis $\{\lambda\}$, where $(a{\cdot}\lambda)(x) = \lambda(xa)$ for all $a, x \in A$. Therefore

$$\begin{aligned}
(a{\cdot}\lambda)(\nu_A(t)) &= \lambda(\nu_A(t)a) \\
&= \lambda(s(\Lambda_{(2)})t(\Lambda_{(1)})a) \\
&= \lambda(s(\Lambda_{(2)})(r(a) \circ t)(\Lambda_{(1)})) \\
&= \mathsf{tr}(r(a) \circ t)
\end{aligned}$$

for all $a \in A$. Since $A^* = A{\cdot}\lambda$ it follows that $\nu_A(t) = 0$ if and only if $(a{\cdot}\lambda)(\nu_A(t)) = 0$ for all $a \in A$, or equivalently if and only if $\mathsf{tr}(r(a) \circ t) = 0$ for all $a \in A$.

The proof of part c) is similar to that of part b). We note that A^* is also a free right A-module with basis $\{\lambda\}$, where $(\lambda{\cdot}a)(x) = \lambda(ax)$ for all $a, x \in A$. Thus using [16, Theorem 2] again we see that $\mathsf{tr}(l(a) \circ t) = \lambda(at(s(\Lambda_{(2)}))\Lambda_{(1)}) = 0$ for all $a \in A$ if and only if $t(s(\Lambda_{(2)}))\Lambda_{(1)} = 0$, or equivalently, since t is an algebra automorphism of A, $0 = s(\Lambda_{(2)})t^{-1}(\Lambda_{(1)}) = \nu_A(t^{-1})$. $\Box$

Since $r(ab) = r(b) \circ r(a)$ for all $a, b \in A$, we have as a consequence of part b) of the preceding lemma:

Corollary 1 *Let A be a finite-dimensional Hopf algebra over the field k and suppose that t is a linear automorphism of A. If $\nu_A(t) = 0$ then $\nu_A(r(a) \circ t) = 0$ for all $a \in A$.* $\Box$

Suppose t is an algebra automorphism which commutes with s. Since s^{-1} is the antipode of A^{cop}, taking Λ as the non-zero left integral for A^{cop} to define $\nu_{A^{cop}}(t)$ it follows that

$$\nu_{A^{cop}}(t) = s^{-1}(t(\nu_A(t^{-1}))). \tag{4}$$

Proposition 1 *Let A be a finite-dimensional Hopf algebra with antipode s and t be an algebra automorphism of A which commutes with s. Then the following are equivalent:*

a) $\nu_A(t) = 0$.

b) $\nu_A(t^{-1}) = 0$.

c) $\mathsf{tr}(r(a) \circ t) = 0$ *for all* $a \in A$.

d) $\mathsf{tr}(l(a) \circ t) = 0$ *for all* $a \in A$.

PROOF: By Lemma 3 it follows that a) and c) are equivalent, b) and d) are equivalent. To show that a) and b) are equivalent we apply the equivalence of a) and c) to A^{cop}. Hence $\nu_A(t) = 0$ if and only if $\nu_{A^{cop}}(t) = 0$. Thus $\nu_{A^{cop}}(t) = 0$ if and only if $\nu_A(t^{-1}) = 0$ by (4). $\Box$

When $t = s^2$ much more can be said about $\nu_A(s^2)$.

Proposition 2 *Let A be a finite-dimensional Hopf algebra with antipode s over the field k. Then:*

a) $\nu_A(s^2) \neq 0$ *if and only if A is semisimple.*

b) *If the characteristic of k is 0 and m is an integer relatively prime to the order of s^2 then* $\mathsf{tr}(s^{2m}) = 0$.

c) *Suppose B is a sub-Hopf algebra of A and t is an algebra endomorphism of A which satisfies $t(B) \subseteq B$. Then $\nu_B(t|_B) = 0$ implies $\nu_A(t) = 0$.*

PROOF: Since $\nu_A(s^2) = s(\Lambda_{(2)})s^2(\Lambda_{(1)}) = s(s(\Lambda_{(1)})\Lambda_{(2)}) = \epsilon(\Lambda)1$, and A is semisimple if and only if $\epsilon(\Lambda)1 \neq 0$, part a) follows.

To prove b) we first consider the case when $\mathsf{tr}(s^2) \neq 0$. Then A is semisimple and cosemisimple by [5, Theorem 2.5], and $s^2 = 1_A$ by [6, Theorem 3]. Hence b) is vacuous in this case. Now we assume that $\mathsf{tr}(s^2) = 0$. We may also assume that k is algebraically closed. Thus s^2 is diagonalizable.

Let r be the order of s^2. Let $\{x_\imath\}_{1 \leq \imath \leq n}$ be a basis of A of eigenvectors of s^2. Write $s^2(x_\imath) = \omega^{a_\imath} x_\imath$ for all $\imath$, where ω is a root of unity of order r, and $0 \leq a_\imath < r$ for all i. Since $\mathsf{tr}(s^2) = 0$ it follows that ω is a root of the polynomial $X^{a_1} + \cdots + X^{a_n}$. Let Φ_r be the minimal polynomial over the rational numbers of a root of unity of order r. Then $X^{a_1} + \cdots + X^{a_n} = f\Phi_r$ for some polynomial f. Since m is relatively prime to r necessarily ω^m is a root of Φ_r. Thus $0 = \omega^{ma_1} + \cdots + \omega^{ma_n} = \mathsf{tr}(s^{2m})$.

To show part c) we use the fact that A is a free left B-module by [10, Theorem 7]. Choose a basis $\{m_1, \ldots, m_r\}$ for A as a left B-module and write $\Lambda = b_1 m_1 + \cdots + b_s m_s$ for some $b_1, \ldots, b_r \in B$. Let Λ_B be a non-zero left integral for B. From the equation $b\Lambda = \epsilon(b)\Lambda$ for all $b \in B$ we deduce that $b_\imath$ is a left integral for all $1 \leq \imath \leq r$. Thus for each $1 \leq \imath \leq r$ we have $b_\imath = \alpha_\imath \Lambda_B$ for some $\alpha_\imath \in k$. We have shown $\Lambda = \Lambda_B a$, where $a = \alpha_1 m_1 + \cdots + \alpha_r m_r$.

Suppose t is an algebra endomorphism of A which satisfies $t(B) \subseteq B$. The calculation

$$s(\Lambda_{(2)})t(\Lambda_{(1)}) = s(\Lambda_{B\,(2)}a_{(2)})t(\Lambda_{B\,(1)}a_{(1)}) = s(a_{(2)})s(\Lambda_{B\,(2)})t(\Lambda_{B\,(1)})t(a_{(1)})$$

shows that $\nu_A(t) = s(a_{(2)})\nu_B(t|_B)t(a_{(1)})$. Part c) follows from this equation. $\square$

Part b) of the preceding proposition implies that if the order of s^2 is prime and k has characteristic zero then s^2 satisfies the strong vanishing trace condition.

By part c) of the preceding proposition in many cases $\nu_A(s^{2m}) = 0$ when the characteristic of k is positive.

Corollary 2 *Let A be a finite-dimensional Hopf algebra with antipode s over the field k which contains non-semisimple sub-Hopf algebra which is either commutative or cocommutative. Then $\nu_A(s^{2m}) = 0$ for all integers m.*

PROOF: Let B be a sub-Hopf algebra of A which satisfies the hypothesis of the lemma. Since B is commutative or cocommutative $s^2|_B = 1_B$. Since B is not semisimple $\nu_B(s^2|_B) = 0$ by part a) of Proposition 2. For any integer m note that B is invariant under the algebra map s^{2m} and that $\nu_B(s^{2m}|_B) = \nu_B(1|_B) = \nu_B(s^2|_B) = 0$. Hence the Corollary follows from part c) of Proposition 2. □

When s^2 is not a semisimple endomorphism of A, and the characteristic of k is not 2, it is always the case that $\mathsf{tr}(s^{2m}) = 0$.

Proposition 3 *Let A be a finite-dimensional Hopf algebra with antipode s over the field k. Suppose that s^2 is not a semisimple endomorphism of A and that the characteristic of k is not 2. Then:*

a) *Either $\nu_A(s^{2m}) = 0$ for all integers m or $\nu_{A^*}({s^*}^{2m}) = 0$ for all integers m.*

b) $\mathsf{tr}(s^{2m}) = 0$ *for all integers m.*

PROOF: We may assume that k is algebraically closed. Since the characteristic of k is not 2 and s^2 is not semisimple, it follows that the characteristic of k is $p > 2$ and p divides order(g) or p divides order(α) by (2). Thus the group algebra $k[Z_p]$ of the cyclic group Z_p of order p over k is a sub-Hopf algebra of A or A^*. Consequently part a) follows by Corollary 2. Since $\mathsf{tr}(t^*) = \mathsf{tr}(t)$ for all linear endomorphisms t of A, part b) follows by part a) and part a) of Lemma 3. □

We end this section with a result on the order of a semisimple coalgebra automorphism.

Lemma 4 *Let C be a finite-dimensional coalgebra over the field k and suppose that t is a semisimple coalgebra automorphism of C. Then the order of t is the order of the coalgebra automorphism $t|_{C_1}$ of C_1.*

PROOF: Since t is a coalgebra automorphism of the finite-dimensional coalgebra C, the restriction $t|_{C_n}$ is a coalgebra automorphism of C_n for all $n \geq 0$. We may as well assume that $t|_{C_1}$ has finite order r. To prove the lemma we need only show that $t^r|_C = 1_C$, or equivalently that $t^r|_{C_n} = 1_{C_n}$ for all $n \geq 1$. The latter we do by induction on n. The case $n = 1$ follows by the definition of r.

Suppose that $n > 1$ and $t^r|_{C_{n-1}} = 1_{C_{n-1}}$. Then $t^r - 1_C$ vanishes on C_{n-1}. Therefore

$$\begin{aligned}
\Delta(t^r - 1_C)(C_n) &= (t^r \otimes t^r - 1_C \otimes 1_C)(\Delta(C_n)) \\
&\subseteq ((t^r - 1_C)\otimes t^r + 1_C \otimes (t^r - 1_C)) \left(\sum_{\imath=0}^{n} C_{n-\imath} \otimes C_\imath\right) \\
&\subseteq C_n \otimes C_0 + C_0 \otimes C_n.
\end{aligned}$$

Thus $(t^r - 1_C)(C_n) \subseteq C_1$, and consequently $(t^r - 1_C)^2(C_n) = (0)$. The last equation implies that the minimal polynomial $m(X)$ of $t|_{C_n}$ divides $(X^r - 1)^2$. Since t is semisimple the restriction $t|_{C_n}$ is semisimple. Therefore $m(X)$ divides $X^r - 1$; consequently $t^r|_{C_n} - 1_{C_n} = 0$ as required. □

3 The Element $\nu_A(s^{2m})$ when s^2 Is Semisimple and A is Pointed

Let A be a finite-dimensional pointed Hopf algebra over the field k and suppose that s^2 is a semisimple endomorphism of A. The purpose of this section is to show that s^2 satisfies the strong vanishing trace condition. To prove this result we will use the following two technical lemmas.

The first lemma involves q-binomial coefficients. Let q be a non-zero element of k. To describe q-binomial coefficients we consider the algebra A over k generated by symbols a, x subject to the relation $xa = qax$. The $a^\imath x^\jmath$'s where $0 \leq \imath, \jmath$ form a linear basis for A. For $0 \leq \imath \leq n$ the q-binomial coefficient $\binom{n}{\imath}_q$ is the coefficient of $a^{n-\imath}x^{\imath}$ in the expansion of $(a+x)^n$; otherwise $\binom{n}{\imath}_q = 0$ by definition.

Lemma 5 *Let $N > 1$ and suppose that the field k contains a primitive N^{th} root of unity ω. Then:*

a) $\binom{N-1}{\ell}_\omega = (-1)^\ell \omega^{-\ell(\ell+1)/2}$ *for all* $0 \leq \ell \leq N-1$.

b) *If the characteristic of k is $p > 0$ then* $\binom{Np^s-1}{\ell}_\omega = (-1)^\ell \omega^{-\ell(\ell+1)/2}$ *for all $s \geq 0$ and $0 \leq \ell \leq Np^s - 1$.*

Proof: Part a) follows from the equations

$$0 = \binom{N}{\ell}_\omega = \omega^\ell \binom{N-1}{\ell}_\omega + \binom{N-1}{\ell-1}_\omega$$

for all $1 \leq \ell \leq N-1$. See [11, pages 691–692] for example. To show part b) we first note that any non-negative integer m has a unique decomposition $m = m_D N + m_R$, where m_D, m_R are integers and $0 \leq m_R < N$. By part b) of [11, Proposition 1], for example,

$$\binom{m}{\ell}_\omega = \binom{m_R}{\ell_R}_\omega \binom{m_D}{\ell_D}$$

for all $0 \leq \ell < m$, where $\binom{m_D}{\ell_D}$ is the ordinary binomial symbol. Suppose that the characteristic of k is $p > 0$, $s \geq 0$ and $0 \leq \ell < Np^s$. When $s = 0$ part b) is part a). Thus we may assume $s > 0$. In this case

$$\binom{Np^s-1}{\ell}_\omega = \binom{N-1}{\ell_R}_\omega \binom{p^s-1}{\ell_D} = (-1)^{\ell_R} \omega^{-\ell_R(\ell_R+1)/2} \binom{p^s-1}{\ell_D}$$

by part a). Since $0 = \binom{p^s}{m} = \binom{p^s-1}{m} + \binom{p^s-1}{m-1}$ for all $1 \le m < p^s$, it follows that $\binom{p^s-1}{m} = (-1)^m$ for all $0 \le m < p^s$. Therefore

$$\binom{Np^s-1}{\ell}_\omega = (-1)^{\ell_R}\omega^{-\ell_R(\ell_R+1)/2}(-1)^{\ell_D}.$$

By considering the parities of N and ℓ_D one can show that this last expression is $(-1)^\ell\omega^{-\ell(\ell+1)/2}$. $\square$

The proof of the main result of this section reduces to the following special case.

Lemma 6 *Let A be a finite-dimensional Hopf algebra with antipode s over the field k generated by non-zero elements a, x which satisfy*

$$\Delta(a) = a\otimes a, \quad \Delta(x) = x\otimes a + 1\otimes x$$

and $xa = \omega ax$ for some primitive N^{th} root of unity $\omega \in k$. Then s^2 satisfies the strong vanishing trace condition.

PROOF: Our proof relies on a characterization of A in terms of generators and relations. The details which we do not include here can be filled in using the ideas developed in §1.2–§3 of [11].

Since $s^2(a) = a$ and $s^2(x) = \omega^{-1}x$ it follows that $s^2(a^\imath x^\jmath) = \omega^{-\jmath}a^\imath x^\jmath$, and hence

$$s^{2m}(a^\imath x^\jmath) = \omega^{-m\jmath}a^\imath x^\jmath, \tag{5}$$

for all $0 \le \imath, \jmath$. The equation $s(x) = -xa^{-1}$ implies that

$$s(x^\ell) = (-1)^\ell\omega^{(\ell-1)\ell/2}x^\ell a^{-\ell} \tag{6}$$

for all $\ell \ge 0$.

Now suppose that $1 \le m < \mathrm{order}(s^2)$. Since $s^{2m}(x) = \omega^{-m}x$ and $x \ne 0$ it follows that $N > 1$.

Let n be the order of a. From the relation $xa = \omega ax$ we deduce $x = xa^n = \omega^n a^n x$. Thus $\omega^n = 1$ since $x \ne 0$. This means $n = NL$ for some $L \ge 1$. The relation also implies

$$\Delta(x^r) = \sum_{\ell=0}^{r}\binom{r}{\ell}_\omega x^{r-\ell}\otimes a^{r-\ell}x^\ell \tag{7}$$

for all $r \geq 0$.

For some $M > 1$ the set of monomials $\{a^\imath x^\jmath \mid 0 \leq \imath < n, 0 \leq \jmath < M\}$ is a linear basis for A. Thus $\mathrm{Dim}\, A = nM$. Let $\{\overline{a^\imath x^\jmath} \mid 0 \leq \imath < n, 0 \leq \jmath < M\}$ be the dual basis for A^*. Since $\lambda \in A^*$ is a right integral if and only if $\lambda(c_{(1)})c_{(2)} = \lambda(c)1$ for all $c \in A$, using (7) it is easy to see that

$$\lambda = \overline{a^{1-M}x^{M-1}}$$

is a non-zero right integral for A^*.

Case 1: The characteristic of k is 0.

In this case $M = N$, so $\mathrm{Dim}\, A = nN$. The relations

$$a^n = 1, \qquad xa = \omega ax, \qquad \text{and } \; x^N = \alpha(a^N - 1)$$

for some $\alpha \in k$ hold. Notice that $\Lambda = (\sum_{\imath=0}^{n-1} a^\imath)x^{N-1}$ is a non-zero left integral for A and that $\lambda(\Lambda) = 1$.

We next compute $\mathsf{tr}(s^{2m})$. Since $s^{2m} \neq 1_A$, it follows by (5) that $\omega^{-m} \neq 1$ and therefore $\mathsf{tr}(s^{2m}) = n(\sum_{\jmath=0}^{N-1} \omega^{-\jmath m}) = 0$.

Now let t be an algebra endomorphism of A which satisfies $t(a) = a$ and $t(x) = \beta x$ for some $\beta \in k$. We will show that $\nu_A(t) = \mathsf{tr}(t)a^{1-N}x^{N-1}$, and thus $\nu_A(s^{2m}) = 0$. Using (7) and part a) of Lemma 5 we calculate

$$\begin{aligned}
\nu_A(t) &= s(\Lambda_{(2)})t(\Lambda_{(1)}) \\
&= \sum_{\imath=0}^{n-1}\sum_{\ell=0}^{N-1} \binom{N-1}{\ell}_\omega s(a^{\imath+N-1-\ell}x^\ell)t(a^\imath x^{N-1-\ell}) \\
&= \sum_{\imath=0}^{n-1}\sum_{\ell=0}^{N-1} (-1)^\ell \omega^{-\ell(\ell+1)/2} s(x^\ell)s(a^{\imath+N-1-\ell})t(a^\imath)t(x^{N-1-\ell}).
\end{aligned}$$

Now using (6) for each $0 \leq \imath \leq n-1$ we see that

$$\begin{aligned}
&\sum_{\ell=0}^{N-1} (-1)^\ell \omega^{-\ell(\ell+1)/2} s(x^\ell)s(a^{\imath+N-1-\ell})t(a^\imath)t(x^{N-1-\ell}) \\
&= \sum_{\ell=0}^{N-1} (-1)^\ell \omega^{-\ell(\ell+1)/2} (-1)^\ell \omega^{(\ell-1)\ell/2} \beta^{N-1-\ell} x^\ell a^{-\ell} a^{-\imath-N+1+\ell} a^\imath x^{N-1-\ell} \\
&= \sum_{\ell=0}^{N-1} \omega^{-\ell} \beta^{N-1-\ell} x^\ell a^{-N+1} x^{N-1-\ell}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\ell=0}^{N-1} \omega^{-\ell}\omega^{\ell(-N+1)}\beta^{N-1-\ell}a^{-N+1}x^{N-1} \\
&= \left(\sum_{\ell=0}^{N-1} \beta^{N-1-\ell}\right) a^{-N+1}x^{N-1}.
\end{aligned}$$

Thus $\nu_A(t) = \mathsf{tr}(t)a^{1-N}x^{N-1}$.

We note that the formula $\nu_A(t) = \mathsf{tr}(t)a^{1-N}x^{N-1}$ can be derived in a simpler fashion. It is not hard to see that $s(x^{\ell})s(a^{\imath+N-1-\ell})t(a^{\imath})t(x^{N-1-\ell})$ is some scalar multiple of $a^{1-N}x^{N-1}$. Therefore $\nu_A(t) = ca^{1-N}x^{N-1}$ for some $c \in k$. Using part a) of Lemma 3 we compute $\mathsf{tr}(t) = \lambda(\nu_A(t)) = c\lambda(a^{1-N}x^{n-1}) = c$.

The positive characteristic case is rather interesting.

Case 2: The characteristic of k is $p > 0$.

In this case $M = Np^r$ for some $r \geq 0$, so $\mathrm{Dim}\, A = nNp^r$. The relations

$$a^n = 1, \quad xa = \omega ax, \quad \text{and } x^{Np^r} = \alpha(a^{Np^r} - 1) + \alpha_0 x^N + \cdots + \alpha_{r-1}x^{Np^{r-1}}$$

for some $\alpha, \alpha_0, \ldots, \alpha_{r-1} \in k$ hold with the restriction that $\alpha_\imath \neq 0$ implies that $a^{Np^r} = a^{Np^\imath}$.

We have noted that $\lambda = \overline{a^{1-Np^r}x^{Np^r-1}}$ is a non-zero right integral for A^*. It is not hard to see that

$$\Lambda = (\sum_{\imath=0}^{n-1} a^\imath)(x^{Np^r-1} - \alpha_0 x^{N-1} - \cdots - \alpha_{r-1}x^{Np^{r-1}-1})$$

is a non-zero left integral for A and that $\lambda(\Lambda) = 1$. The essential calculation is

$$\begin{aligned}
x\Lambda &= \left(\sum_{\imath=0}^{n-1} \omega^\imath a^\imath\right)\left(x^{Np^r} - \alpha_0 x^N - \cdots - \alpha_{r-1}x^{Np^{r-1}}\right) \\
&= \left(\sum_{\imath=0}^{n-1} \omega^\imath a^\imath\right)\alpha\left(a^{Np^r} - 1)\right) \\
&= \alpha\left(\sum_{\imath=0}^{n-1} \omega^\imath a^{Np^r+\imath} - \sum_{\imath=0}^{n-1} \omega^\imath a^\imath\right) \\
&= \alpha\left(\sum_{\imath=0}^{n-1} \omega^{Np^r+\imath} a^{Np^r+\imath} - \sum_{\imath=0}^{n-1} \omega^\imath a^\imath\right) \\
&= 0
\end{aligned}$$

since $\omega^N = 1$ and both sums in the next to the last expression are $\sum_{\imath \in Z_n} \omega^\imath a^\imath$.

Let t be an algebra endomorphism of A which satisfies $t(a) = a$ and $t(x) = \beta x$ for some $\beta \in k$. For $\imath \geq 0$ let $\beta_\imath = \sum_{\jmath=0}^{Np^\imath - 1} \beta^\jmath$. We will show that

$$\nu_A(t) = a^{1-Np^r}\left(\mathsf{tr}(t)x^{Np^r-1} - \sum_{\imath=0}^{r-1} n\alpha_\imath\beta_\imath x^{Np^\imath-1}\right) \tag{8}$$

which implies $\nu_A(s^{2m}) = 0$. For let $t = s^{2m}$. Since $\omega^{-m} \neq 1$ it follows that $\beta_\imath = (1-\omega^{-mNp^\imath})/(1-\omega^{-m}) = 0$ for all $0 \leq \imath$ and consequently $\nu_A(s^{2m}) = 0$.

Let $g \in G(A)$, $0 \leq \imath < r$ and let $c = gx^{Np^\imath-1}$. To calculate $\nu_A(t)$ we first determine $s(c_{(2)})t(c_{(1)})$. Following the steps of the calculation of $\nu_A(t)$ in the characteristic 0 case, and using part b) of Lemma 5, one can easily deduce that $s(c_{(2)})t(c_{(1)}) = \beta_\imath a^{1-Np^\imath}x^{Np^\imath-1}$. Since $\alpha_\imath \neq 0$ implies $a^{Np^\imath} = a^{Np^r}$ we have established (8). Our proof of the lemma is complete. $\square$

Lemma 7 *Let k be algebraically closed and A be a finite-dimensional pointed Hopf algebra with antipode s over k. Assume that s^2 is semisimple and $s^{2m} \neq 1_A$ for some m. Then there is a group-like element $a \in G(A)$, a non-zero element $x \in A$ and a root of unity $\rho \in k$ such that*

$$\Delta(x) = x \otimes a + 1 \otimes x, \quad s^2(x) = \rho x \quad \textit{and} \quad \rho^m \neq 1.$$

Proof: By Lemma 4 we have that order(s^2) = order $(s^2|_{A_1})$. Therefore $s^{2m}|_{A_1} \neq 1_{A_1}$.

For $a \in G(A)$ let $V_a = \{x \in A \mid \Delta(x) = x \otimes a + 1 \otimes x\} \subseteq A_1$. Since A is pointed, A_1 is generated as a left A_0-module by 1 and $\sum_{a \in G(A)} V_a$.

By assumption s^2 is a semisimple endomorphism of A. Therefore s^2 is diagonalizable since k is algebraically closed. Suppose that $a \in G(A)$. Since s^2 is a coalgebra endomorphism of A and $s^2(g) = g$ for all $g \in G(A)$ it follows that $s^2(V_a) \subseteq V_a$.

Now $s^{2m}|_{A_1} \neq 1_{A_1}$. Since s^2 is an algebra endomorphism, $s^{2m}|_{V_a} \neq 1_{V_a}$ for some $a \in G(A)$. Since $s^2|_{V_a}$ is diagonalizable it follows that $s^{2m}(x) \neq x$ where $x \in V_a$ is an eigenvector for s^2 belonging to some $\rho \in k$. Since s^2 has finite order, ρ is a primitive N^{th} root of unity for some $N \geq 1$. Since $s^{2m}(x) \neq x$ it follows that $N > 1$ and $\rho^m \neq 1$. $\square$

Theorem 1 *Let A be a finite-dimensional Hopf algebra with antipode s over the field k. Suppose that s^2 is a semisimple endomorphism of A and that the* *coradical A_0 of A is cocommutative. Then s^2 satisfies the strong vanishing trace condition.*

PROOF: Let K be an algebraic closure of k. Since $(A \otimes K)_0 \subseteq A_0 \otimes K$ it follows that the coradical of $A \otimes K$ is cocommutative. Thus since K is algebraically closed $A \otimes K$ is pointed. To prove the theorem, it is easy to see that we may replace A with $A \otimes K$; thus we will assume that A is pointed.

Suppose that $s^{2m} \neq 1_A$. Pick elements $a, x \in A$ and ρ according to Lemma 7 and let $B = k\{a, x\}$ be the subalgebra of A generated by a and x. Then B is a sub-bialgebra of A. The calculations $s(a) = a^{-1}$ and $s(x) = -xa^{-1}$ show that B is a sub-Hopf algebra of A. Using the last equation we calculate $s^2(x) = axa^{-1} = \rho x$. In particular $xa = \omega ax$, where $\omega = \rho^{-1}$. Since $s^{2m}(x) \neq x$ it follows that $s^{2m}|_B \neq 1_B$. Therefore $\nu_B(s^{2m}|_B) = 0$ by Lemma 6, and consequently $\nu_A(s^{2m}) = 0$ by part c) of Proposition 2. □

When s^2 is conjugation by a grouplike element, part a) of Proposition 3 can be strengthened a bit in the pointed case.

Corollary 3 *Let A be a finite-dimensional Hopf algebra with antipode s over the field k. Suppose that the coradical A_0 of A is cocommutative and that $s^2(x) = axa^{-1}$ for all $x \in A$, where $a \in G(A)$. Then s^2 satisfies the strong vanishing trace condition.*

PROOF: We may assume that k is algebraically closed. If the characteristic of k does not divide order(a) then s^2 is semisimple and Theorem 1 applies. Suppose that the characteristic of k divides order(s^2). Then k has characteristic $p \geq 2$ and the group algebra $k[Z_p]$ of the cyclic group Z_p is a sub-Hopf algebra of A; thus Corollary 2 applies. □

A very basic example to which Corollary 3 applies is the Taft (Hopf) algebra $\mathrm{T}_{n,q}(k)$, where n is a positive integer and $q \in k$ is a primitive n^{th} root of unity [19]. As an algebra $\mathrm{T}_{n,q}(k)$ is generated by a, x subject to the relations $a^n = 1$, $x^n = 0$ and $xa = qax$. Thus $\mathrm{Dim}\, \mathrm{T}_{n,q}(k) = n^2$. The coalgebra structure of $\mathrm{T}_{n,q}(k)$ is determined by $\Delta(a) = a \otimes a$ and $\Delta(x) = x \otimes a + 1 \otimes x$.

4 The Element $\nu_A(s^{2m})$ when s^2 is Semisimple and A^* is Pointed

Let A be a finite-dimensional Hopf algebra over the field k and let $B \subseteq A$ be a sub-Hopf algebra of A containing the coradical A_0 of A. We begin our discussion with an analysis of the associated graded Hopf algebra $\mathrm{gr}_B(A)$.

Recall that if $U, V \subseteq A$ are subspaces, the wedge $U \wedge V \subseteq A$ is the inverse image under the comultiplication of $U \otimes A + A \otimes V$. Note that

$$F_0(A) = B \subseteq F_1(A) = B \wedge B \subseteq F_2(A) = B \wedge B \wedge B \subseteq \dots$$

is a Hopf algebra filtration of A with $\bigcup_{n \geq 0} F_n(A) = A$.

Define $F_{-1}(A) = 0$ and let

$$\mathrm{gr}_B(A) = \bigoplus_{n \geq 0} \mathrm{gr}_B(A)(n), \text{ where } \mathrm{gr}_B(A)(n) = F_n(A)/F_{n-1}(A) \text{ for all } n \geq 0,$$

be the associated graded Hopf algebra.

The projection $\pi : \mathrm{gr}_B(A) \to \mathrm{gr}_B(A)(0) = B$ is a Hopf algebra map, and the inclusion map $B \subseteq \mathrm{gr}_B(A)$ is a Hopf algebra section of π. Let $R = \mathrm{gr}_B(A)^{coB}$ be the right coinvariant elements with respect to the projection π. By definition $R = \{x \in \mathrm{gr}_B(A) \mid (1 \otimes \pi)(\Delta(x)) = x \otimes 1\}$. By [15, Theorem 3] there is an isomorphism of Hopf algebras

$$\mathrm{gr}_B(A) \cong R \# B,$$

defined by multiplication in $\mathrm{gr}_B(A)$, where R has the structure of a braided Hopf algebra in the category of left Yetter-Drinfeld modules over B. See [9] and the discussion in [2, Section 2].

Lemma 8 *Let A be a finite-dimensional Hopf algebra over the field k and suppose that B is a sub-Hopf algebra of A which contains the coradical A_0 of A. Then:*

a) *$R = \bigoplus_n R(n)$ with $R(n) = R \cap gr_B(A)(n)$ for all $n \geq 0$ is a graded Hopf algebra in the braided sense and $gr_B(A)(n) \cong R(n) \# B$ for all $n \geq 0$.*

b) *$R(n) = \{\bar{a} \in F_n(A)/F_{n-1}(A) \mid a \in F_n(A), \Delta(a) - a \otimes 1 \in F_{n-1}(A) \otimes F_n(A)\}$ for all $n \geq 0$.*

PROOF: Part a) is explained in [2, Lemma 2.1]. To see part b) let $a \in F_n(A)$. The residue class $\bar{a} \in F_n(A)/F_{n-1}(A)$ is in R if and only if $\bar{a}_{(1)} \otimes \bar{a}_{(2)} - \bar{a} \otimes 1 \in \mathrm{gr}_B(A) \otimes \left(\bigoplus_{i \geq 1} \mathrm{gr}_B(A)(i)\right)$. The last condition means exactly that $\Delta(a) - a \otimes 1$ is an element in $F_{n-1}(A) \otimes F_n(A)$. □

We will need the dual of Proposition 1. For all $p \in A^*$, let $R(p)$ and $L(p)$ be the linear endomorphisms of A defined by $R(p)(a) = p(a_{(1)})a_{(2)}$ and $L(p)(a) = a_{(1)}p(a_{(2)})$ respectively for all $a \in A$. Note that $R(p)$ and $L(p)$ are the transposes of left and right multiplication respectively by p on A^*. Since $\mathsf{tr}(t) = \mathsf{tr}(t^*)$ for all linear endomorphisms t of A, the dual version of Proposition 1 can be formulated:

Proposition 4 *Let A be a finite-dimensional Hopf algebra with antipode s and t be a coalgebra automorphism of A which commutes with s. Then the following are equivalent:*

a) $\nu_{A^*}(t^*) = 0$.

b) $\nu_{A^*}(t^{*-1}) = 0$.

c) $\mathsf{tr}(L(p) \circ t) = 0$ *for all* $p \in A^*$.

d) $\mathsf{tr}(R(p) \circ t) = 0$ *for all* $p \in A^*$.

□

Theorem 2 *Let A be a finite-dimensional Hopf algebra over the field k and suppose that B is a sub-Hopf algebra of A containing the coradical of A. Let t a Hopf algebra automorphism of A. Assume $t(B) = B$, let $gr(t)$ denote the induced Hopf algebra automorphism on $gr_B(A)$ and let $R = gr_B(A)^{coB}$. Then*

$$\mathsf{tr}(L(p) \circ t) = \mathsf{tr}(gr(t)|_R) \cdot \mathsf{tr}((L(p) \circ t)|_B)$$

for all $p \in A^*$.

PROOF: Let $n \geq 1$. By part a) of Lemma 8 the multiplication map $R(n)\#B \to \mathrm{gr}_B(A)(n)$ is bijective. Now let $\bar{a} \in R(n)$, $a \in F_n(A)$ and $b \in B$. Then $\Delta(a) - a \otimes 1 \in F_{n-1}(A) \otimes F_n(A)$ by part b) of Lemma 8. Therefore $a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} - ab_{(1)} \otimes b_{(2)} \in F_{n-1}(A) \otimes F_n(A)$. Hence we compute

$$\mathrm{gr}(L(p))(\bar{a}b) = \overline{a_{(1)}b_{(1)}p(a_{(2)}b_{(2)})} = \bar{a}b_{(1)}p(b_{(2)})$$

for all $p \in A^*$, where $\mathrm{gr}(L(p))$ denotes the induced map on $\mathrm{gr}_B(A)$.

We have shown that the induced map $\mathrm{gr}(L(p) \circ t)$ can be identified with the map $(\mathrm{gr}(t)|_R) \otimes (L(p) \circ t)|_B)$. This proves $\mathsf{tr}(\mathrm{gr}(L(p) \circ t)) = \mathsf{tr}(\mathrm{gr}(t)|_R) \cdot \mathsf{tr}((L(p) \circ t)|_B)$, and the theorem follows since $\mathsf{tr}(L(p) \circ t) = \mathsf{tr}(\mathrm{gr}(L(p) \circ t))$. $\square$

Theorem 2 implies an analog of part c) of Proposition 2 under an additional assumption.

Corollary 4 *Let A be a finite-dimensional Hopf algebra over the field k. Suppose that $I \subseteq A$ a nilpotent Hopf ideal. Let t be a Hopf algebra automorphism of A such that $t(I) \subseteq I$ and let $\bar{t}$ denote the induced Hopf algebra automorphism on A/I. If $\nu_{A/I}(\bar{t}) = 0$ then $\nu_A(t) = 0$.*

PROOF: Let $\pi : A \longrightarrow A/I = B$ be the projection and identify B^* with $\pi^*(B^*)$. Since $I \subseteq \mathrm{Rad}\, A$ it follows that the sub-Hopf algebra B^* of A^* contains the coradical of A^*. Observe that $\bar{t}^* = t^*|_{B^*}$ and $L(\pi(a)) = L(a)|_{B^*}$ for all $a \in A$.

Suppose that $\nu_{A/I}(\bar{t}) = 0$. Then

$$\mathsf{tr}((L(a) \circ t^*)|_{B^*}) = \mathsf{tr}(L(\pi(a)) \circ \bar{t}^*) = \mathsf{tr}(\bar{t} \circ r(\pi(a))) = 0$$

for all $a \in A$ by Proposition 4. Now we use Theorem 2 to conclude that $\mathsf{tr}(t \circ r(a)) = \mathsf{tr}(L(a) \circ t^*) = 0$ for all $a \in A$, whence $\nu_A(t) = 0$ by Proposition 4 again. $\square$

Now we assume that A is pointed and $B = A_0$. The graded Hopf algebra associated to the coradical filtration will be denoted by $\mathrm{gr}(A)$. Our goal is to show in Theorem 2 that $\mathsf{tr}(\mathrm{gr}(t)|_R) = 0$ under additional assumptions. We will need the next lemma to reduce to a subalgebra where the trace can easily be computed.

Lemma 9 *Let A be a finite-dimensional algebra over the field k and suppose that B a local subalgebra of A with maximal ideal B^+ and $B/B^+ \cong k$. Let $\sigma : A \to A$ be an algebra homomorphism such that $\sigma(B^+) \subseteq B^+$. Assume that A is free as a left B-module by restriction. If σ is a diagonalizable endomorphism of A, then there exists a σ-stable linear subspace $X \subseteq A$ such that the multiplication in A defines an isomorphism $B \otimes X \to A$. In particular*

$$\mathsf{tr}(\sigma|_B)\mathsf{tr}(\sigma|_X) = \mathsf{tr}(\sigma).$$

Proof: Let $\{a_\imath\}_{1 \le \imath \le n}$ be a left B-basis of A, and let V be the k-span of the a_i's. Then $A = B^+V \oplus V$, and $B^+A = (B^+)^2V + B^+V = B^+V$ is σ-stable. Since σ is diagonalizable, there exists a σ-stable subspace $X \subseteq A$ such that $A = B^+V \oplus X$. Since $B^+A = B^+V$, we deduce form the last equation that $B^+V = (B^+)^2V + B^+X$ from which $B^+V \subseteq (B^+)^mV + B^+X$ follows for all natural numbers $m \ge 1$. Since B^+ is a nilpotent ideal of A, we conclude that $B^+A = B^+V = B^+X$. Therefore $A = B^+X \oplus X$, and the multiplication map $B \otimes X \to A$ is surjective, hence bijective since $B \otimes X$ and A have the same dimension.

The claim about the traces follows easily. □

Theorem 3 *Let A be a finite-dimensional pointed Hopf algebra with antipode s over the field k. Assume that s^2 is semisimple and set $R = gr(A)^{coA_0}$. If m is an integer such that $s^{2m} \neq 1_A$ then $\mathsf{tr}(gr(s^{2m})|_R) = 0$.*

Proof: We may assume that k is algebraically closed. By Lemma 7, there are a group-like element $a \in G(A), 0 \neq x \in A$ and a root of unity $\rho \in k$ of order $N > 1$ such that $\Delta(x) = x \otimes a + 1 \otimes x, s^2(x) = \rho x, \rho^m \neq 1$. Let $g = a^{-1}$ and $y = xa^{-1}$. Then $\Delta(y) = y \otimes 1 + g \otimes y$ and $s^2(y) = \rho y$.

Let z be the residue class of y in $\mathrm{gr}(A)(1) = A_1/A_0$. Then z is non-zero and $\Delta_{\mathrm{gr}(A)}(z) = g \otimes z + z \otimes 1$. Hence $z \in R = \mathrm{gr}(A)^{coA_0}$, $\Delta_R(z) = 1 \otimes z + z \otimes 1$ and $\delta(z) = g \otimes z$. Here, Δ_R and δ are the comultiplication and the coaction of the Yetter-Drinfeld Hopf algebra R over $kG(A)$. Since $\mathrm{gr}(s)(z) = -g^{-1}z$, we compute $(\mathrm{gr}(s))^2(z) = g^{-1}zg$.

Thus $z \in P(R)$ is g-homogeneous and the action of g on z is given by $g \cdot z = gzg^{-1} = qz$ with $q = \rho^{-1}$ a root of unity of order N.

Since $\mathrm{gr}(A)$ is finite-dimensional, $z \in \mathrm{gr}(A)(1)$ is nilpotent. Hence $z^n = 0, z^{n-1} \neq 0$ for some $n \ge 2$. Then $1, z, \dots, z^{n-1}$ are linearly independent, and by the q-binomial formula,

$$0 = \Delta_R(z^n) = (z \otimes 1 + 1 \otimes z)^n = \sum_{\imath=1}^{n-1} \binom{n}{\imath}_q z^\imath \otimes z^{n-\imath}.$$

Hence $\binom{n}{\imath}_q = 0$ for all $1 \le \imath \le n-1$. Then we know from [11, Corollary 2] that N divides n, and we obtain

$$\mathsf{tr}(\mathrm{gr}(s^{2m})|_{k[z]}) = \sum_{\imath=0}^{n-1} q^{m\imath} = 0 \text{ since } q^m \neq 1 \text{ and } q^n = 1.$$

Since $k[z]$ is a local subalgebra of R, $\mathsf{tr}(\mathrm{gr}(s^{2m})|_R) = 0$ by Lemma 9. $\Box$

Theorem 4 *Let A be a finite-dimensional Hopf algebra with antipode s over the field k. Assume that s^2 is semisimple and the coradical of A^* is cocommutative. Then s^2 satisfies the strong vanishing trace condition.*

PROOF: We may assume that k is algebraically closed. Then the theorem follows by duality from Theorem 3, Theorem 2 and Proposition 4. $\Box$

5 A Generalization of a Basic Trace Formula

We want to generalize an important trace formula concerning the square of the antipode in [6, Equation (6)] to higher even powers; indeed to Hopf algebra automorphisms of A. It turns out that the elements $\nu_A(t)$ are an ingredient of this generalization. Our proof follows the method given in [17, Theorem 3.13], and we formulate the result dually to [5, 6].

Recall that a finite-dimensional algebra A over a field k is a *Frobenius algebra* if there is a linear function $\varphi : A \to k$, a natural number n, and there are elements $a_\imath, b_\imath \in A$ for all $1 \leq \imath \leq n$ such that

$$x = \sum_{\imath=1}^{n} \varphi(x a_\imath) b_\imath \text{ for all } x \in A. \tag{9}$$

We say that φ is a *Frobenius homomorphism* and, with an abuse of terminology, that $\{a_\imath\}, \{b_\imath\}$ are *dual bases of* A.

We note that if (9) holds then the linear map $A \longrightarrow A^*$ defined by $a \mapsto \varphi{\cdot}a$ is an isomorphism of right A-modules. In this case $A \longrightarrow A^*$ defined by $a \mapsto a{\cdot}\varphi$ is an isomorphism of left A-modules which is also a consequence of

$$x = \sum_{\imath=1}^{n} a_\imath \varphi(b_\imath x) \text{ for all } x \in A. \tag{10}$$

Observe that (9) and (10) are logically equivalent. To see this note that (9) holds if and only if

$$(y{\cdot}\varphi)(x) = (y{\cdot}\varphi)(\sum_{\imath=1}^{n} \varphi(x a_\imath) b_\imath),$$

or

$$(\varphi{\cdot}x)(y) = (\varphi{\cdot}x)(\sum_{\imath=1}^{n} a_\imath \varphi(b_\imath y)),$$

for all $x, y \in A$. The latter holds if and only if (10) holds.

Lemma 10 *Let A be a Frobenius algebra over the field k with Frobenius homomorphism φ and dual bases $\{a_\imath\}, \{b_\imath\}$. Let $\alpha \in k$ and $v \in A$ such that $v^2 = \alpha v$, and let $t : vA \to vA$ be a linear map. Then*

$$\alpha \mathsf{tr}(t) = \sum_{\imath=1}^{n} \varphi(t(vb_\imath)a_\imath) = \sum_{\imath=1}^{n} \varphi(b_\imath t(va_\imath)).$$

PROOF: For a finite-dimensional vector space V over k we identify $V^*{\otimes}V$ with $\mathrm{End}\,(V)$ by $(v^*{\otimes}v)(u) = v^*(u)v$ for all $v^* \in V^*$ and $v, u \in V$. It is easy to see that $\mathsf{tr}(v^*{\otimes}v) = v^*(v)$.

For all $x \in A$ we have

$$\alpha vx = v^2 x = \sum_{\imath=1}^{n} \varphi(vxa_\imath)vb_\imath = \sum_{\imath=1}^{n} va_\imath \varphi(b_\imath vx)$$

by (9) and (10) respectively. Thus

$$\alpha t = \sum_{\imath=1}^{n} a_\imath{\cdot}\varphi{\otimes}t(vb_\imath) = \sum_{\imath=1}^{n} \varphi{\cdot}b_\imath{\otimes}t(va_\imath)$$

from which the trace formulas follow. $\square$

Let A be a finite-dimensional Hopf algebra with antipode s, let $\Lambda \in A$ be a left integral of A and let $\lambda \in A^*$ be a right integral of A^* such that $\lambda(\Lambda) = 1$. Then

$$x = \lambda(x\Lambda_{(1)})s(\Lambda_{(2)}). \tag{11}$$

We recall the standard proof: For all $x \in A$ we compute $\lambda(x\Lambda_{(1)})s(\Lambda_{(2)}) = \lambda(x_{(1)}\Lambda_{(1)})x_{(2)}\Lambda_{(2)}s(\Lambda_{(3)}) = \lambda(x_{(1)}\Lambda)x_{(2)} = x$, where the first equality holds since λ is a right integral and the last equality follows since Λ is a left integral and $\lambda(\Lambda) = 1$. Thus A is a Frobenius algebra with Frobenius homomorphism λ and dual bases $\{\Lambda_{(1)}\}, \{s(\Lambda_{(2)})\}$.

Let $\chi_A \in A^*$ be the character of the regular representation, that is $\chi_A(x) = \mathsf{tr}(l(x))$ for all $x \in A$. Since A is a Frobenius algebra Lemma

10 implies that $\chi_A(x) = \mathsf{tr}(r(x))$ for all $x \in A$ as well. Using this latter formulation of χ_A we see that

$$\chi_A(x) = \lambda(\nu_A(1_A)x) \text{ for all } x \in A \tag{12}$$

by part a) of Lemma 3, where Λ is used to define $\nu_A(1_A)$.

The isomorphism $A \otimes A \to A \otimes A, x \otimes y \mapsto x_{(1)} \otimes x_{(2)}y$, implies the following well-known identity in A^*

$$\chi_A^2 = (\mathrm{Dim}\, A)\chi_A. \tag{13}$$

Now $\chi_A \circ t = \chi_A$ for any algebra automorphism t of A, or equivalently

$$T(\chi_A) = \chi_A, \tag{14}$$

where $T = t^*$. In particular $S^2(\chi_A) = \chi_A$ where S is the antipode of A^*.

By the same argument, for any grouplike element $\phi \in A^*$,

$$\chi_A = \chi_A\phi, \tag{15}$$

since for all $a \in A$, $(\chi_A\phi)(a) = \chi_A(a_{(1)})\phi(a_{(2)}) = \chi_A(a_{(1)}\phi(a_{(2)}))$, and $a \mapsto a_{(1)}\phi(a_{(2)})$ is an algebra automorphism of A.

For clarity we will use the notation $\langle p, x\rangle = p(x)$ for all $p \in A^*$ and $x \in A$.

Theorem 5 *Let A be a finite-dimensional Hopf algebra over the field k, let $\Lambda \in A$ be a left integral of A and let $\lambda \in A^*$ be a right integral of A^* such that $\lambda(\Lambda) = 1$. Let T be a Hopf algebra automorphism of A^*. Then*

$$(\mathrm{Dim}\, A)\, \mathsf{tr}(T|_{\chi_A A^*}) = \langle \nu_{A^*}(T), \nu_A(1_A)\rangle,$$

where Λ is used to define $\nu_A(1_A)$ and $S^{-1}(\lambda)$ to define $\nu_{A^*}(T)$.

Proof: We first note that

$$S(\nu_{A^*}(T)) = T(\lambda_{(2)})S(\lambda_{(1)}) \tag{16}$$

since S and T commute. It follows that $\widetilde{\Lambda} : A^* \to k$ defined by $\widetilde{\Lambda}(p) = p(\Lambda)$ for all $p \in A^*$ is a Frobenius homomorphism of A^* with dual bases $\{S(\lambda_{(1)})\}, \{\lambda_{(2)}\}$; verify (10) using the fact that $\widetilde{\Lambda}$ is a left integral for A^{**}

and that λ is a right integral for A^*. Using (14) we observe that T defines an endomorphism of $\chi_A A^*$. Hence we get from Lemma 10 and (13) with $v = \chi_A$

$$\begin{aligned}
(\mathrm{Dim}\, A)\, \mathsf{tr}(T|_{\chi_A A^*}) &= \widetilde{\Lambda}(T(\chi_A \lambda_{(2)})S(\lambda_{(1)})) \\
&= \langle \chi_A T(\lambda_{(2)})S(\lambda_{(1)}), \Lambda\rangle \\
&= \langle \chi_A S(\nu_{A^*}(T)), \Lambda\rangle && \text{by (16)} \\
&= \chi_A(\Lambda_{(1)})\langle \nu_{A^*}(T), s(\Lambda_{(2)})\rangle \\
&= \langle \nu_{A^*}(T), \chi_A(\Lambda_{(1)})s(\Lambda_{(2)})\rangle \\
&= \langle \nu_{A^*}(T), \lambda(\nu_A(1_A)\Lambda_{(1)})s(\Lambda_{(2)})\rangle && \text{by (12)} \\
&= \langle \nu_{A^*}(T), \nu_A(1_A)\rangle && \text{by (11).}
\end{aligned}$$

□

If $T = S^2$ then the trace formula of Theorem 5 is the trace formula in [6, Equation (6)] since

$$\langle \nu_{A^*}(S^2), \nu_A(1_A)\rangle = \langle \lambda(1)\varepsilon, S(\Lambda_{(2)})\Lambda_{(1)}\rangle = \lambda(1)\varepsilon(\Lambda).$$

For a finite-dimensional Hopf algebra A over the field k let $x_A = \chi_{A^{**}} \in A^{**} = A$. Thus x_A is determined by $p(x_A) = \mathsf{tr}(R(p))$ for all $p \in A^*$, or by $p(x_A) = \mathsf{tr}(L(p))$ for all $p \in A^*$. Suppose that B is a sub-Hopf algebra of A. Then A is a free left (or right) B-module by [10, Theorem 7]. Thus $\mathrm{Dim}\, B$ divides $\mathrm{Dim}\, A$ which means that $\mathrm{Dim}\, A/\mathrm{Dim}\, B$ is an integer.

We end this section by describing the relationship between x_A and x_B, where B is the sub-Hopf algebra of A generated by the coradical of A.

Proposition 5 *Let A be a finite-dimensional Hopf algebra over the field k and suppose that B is the sub-Hopf algebra of A generated by the coradical of A. Then*

$$x_A = \left(\frac{\mathrm{Dim}\, A}{\mathrm{Dim}\, B}\right) x_B.$$

PROOF: By Theorem 2 we have $p(x_A) = \mathsf{tr}(\mathrm{gr}(1_A)|_R)p(x_B)$ for all $p \in A^*$. Since $R \otimes B \simeq A$ as vector spaces and $\mathrm{gr}(1_A|_R) = 1_R$ the proposition follows. □

6 Applications to Hopf Algebra Automorphisms t and Further Results on $\nu_A(t)$

Suppose that t is a Hopf algebra automorphism of A which has finite order and assume that t satisfies the strong vanishing trace condition. In this section we study t with particular emphasis on the case $t = s^2$. Our discussion is based on the following general lemma.

Lemma 11 *Let t be an algebra automorphism of a finite-dimensional algebra A over a field k of characteristic zero. Suppose that t has finite order r and satisfies the strong vanishing trace condition. If e is an idempotent of A and $t(e) = e$ then r divides* $\mathrm{Dim}\, Ae$.

PROOF: We may assume that k is algebraically closed and that $e \neq 0$. Since t is an algebra endomorphism of A and $t(e) = e$, the idempotent endomorphism $r(e)$ of A and t commute. In particular $t(Ae) \subseteq Ae$.

Now $\mathsf{tr}(t^m|_{Ae}) = \mathsf{tr}(t^m \circ r(e))$ for all $m \geq 0$. Since t satisfies the strong vanishing trace condition it thus follows that $t|_{Ae}$ satisfies the vanishing trace condition. Since $Ae \neq (0)$ it follows that $t|_{Ae}$ has finite order r. Our conclusion that r divides $\mathrm{Dim}\, Ae$ now follows by Lemma 1. □

Let A be a finite-dimensional Hopf algebra over k and let $n \geq 0$. Then the sum of all subcoalgebras C of A which are invariant under s^2 and satisfy $s^{2n}|_C = 1_C$ is in fact a sub-Hopf algebra of A. Thus A has a unique sub-Hopf algebra B maximal with respect to the property that $s^{2n}|_B = 1_B$.

Now suppose that the characteristic of k is zero. Since A is semisimple if and only if $s^2|_A = 1_A$ by [6], we conclude that A contains a unique maximal semisimple sub-Hopf algebra which we denote by A_s. As an immediate consequence of Lemma 11:

Corollary 5 *Let A be a finite-dimensional Hopf algebra with antipode s over a field k of characteristic zero, let A_s be the unique maximal semisimple sub-Hopf algebra of A, suppose that s^2 has order r and that s^2 satisfies the strong vanishing trace condition. Then r divides* $\mathrm{Dim}\, Ae$ *for all idempotents $e \in A_s$.*
□

The dimension of the unique maximal semisimple Hopf-subalgebra of a finite-dimensional Hopf algebra A over a field of characteristic zero is related to the order of a Hopf algebra automorphism of A which satisfies the strong vanishing trace condition.

Theorem 6 *Let A be a finite-dimensional Hopf algebra over a field of characteristic zero, let $B = A_s$ be the unique maximal semisimple sub-Hopf algebra of A, suppose that t is a Hopf algebra automorphism of A of finite order r and t also satisfies the strong vanishing trace condition. Then*

a) r *divides* $\mathrm{Dim}\, A/\mathrm{Dim}\, B$, *and the minimal polynomial of* t *is* $X^r - 1$.

b) *The eigenvalues of* t *are the* r^{th} *roots of unity in* k.

c) *The eigenspaces for* t *all have the same dimension which is* $(\mathrm{Dim}\, A)/r$.

d) *If* e *is an idempotent of* A *which satisfies* $t(e) = e$ *then* r *divides* $\mathrm{Dim}\, Ae$.

PROOF: Let e be a left integral for B which satisfies $\epsilon(e) = 1$. The calculation $e^2 = \epsilon(e)e = e$ shows that e is an idempotent of A. By assumption t is a Hopf algebra automorphisms of A. Thus $t(B)$ is a semisimple sub-Hopf algebra of A which means that $t(B) \subseteq B$ by the maximalilty and uniqueness of B. As a consequence $t|_B$ is a Hopf algebra automorphism of B from which $t(e) = e$ follows. At this point we conclude that r divides $\mathrm{Dim}\, Ae$ by Lemma 11.

To calculate $\mathrm{Dim}\, Ae$ we use the fact that A is a free right B-module. Let $\{m_1, \ldots, m_s\}$ be a basis for A as a free right B-module. The calculation

$$Ae = (m_1 B \oplus \cdots \oplus m_s B)e = m_1 Be \oplus \cdots \oplus m_s Be = km_1 \oplus \cdots \oplus km_s$$

shows that $\mathrm{Dim}\, Ae = s = \mathrm{Dim}\, A/\mathrm{Dim}\, B$.

Thus we have shown the first half of part a). The second half of a) and parts b) and c) are consequences of Lemma 1. Part d) follows by Lemma 11. □

Corollary 6 *Let* A *be a finite-dimensional pointed Hopf algebra over a field of characteristic zero with coradical* $B = A_0$, *and suppose that* s^2 *has order* r. *Then a) – d) in Theorem 6 hold for* $t = s^2$. *In particular,*

$$r \textit{ divides } \mathrm{Dim}\, A/\mathrm{Dim}\, A_0,$$

and $r = \mathrm{Dim}\, A$ *if and only if* $A = k1$.

PROOF: Since A is pointed, $A_s = A_0$ and Theorem 6 applies by Theorem 1.

To see that $r = \mathrm{Dim}\, A$ if and only if $A = k1$, suppose that $r = \mathrm{Dim}\, A$. Then each eigenspace for s^2 is one-dimensional by part b). In particular 1 is the only grouplike element in A. Let Λ be a non-zero left integral for A. Then $s^2(\Lambda) = \alpha(g)\Lambda$, where $\alpha \in G(A^*)$ and $g \in G(A)$. See [16] for example.

Therefore $g = 1$ and consequently $\alpha(g) = 1$. This means $\Lambda \in k1$, which is equivalent to $A = k1$.

Another way to see this is to note that if $\mathrm{Dim}\, A_0 = 1$ then $A_1 = A_0$, for otherwise A would contain a non-zero primitive element which is impossible in characteristic zero since A is finite-dimensional. In this case $A_0 = A_1 = \cdots = A$. $\Box$

For the Taft algebras $r = \mathrm{Dim}\, A/\mathrm{Dim}\, A_0$. Thus the estimate of Corollary 6 on the order of s^2 is sharp in the case of pointed Hopf algebras.

In the general case much less is known. By (2) it follows that

$$\mathrm{order}(s^2) \text{ divides } 2(\mathrm{Dim}\, A).$$

We assume that the characteristic of k is zero and introduce the element x_A of Section 5 into our discussion. Let B be the sub-Hopf algebra of A generated by the coradical of A. Since $\epsilon(x_A) = (\mathrm{Dim}\, A)1 \neq 0$ it follows that $x_A \neq 0$. Now $x_A = (\mathrm{Dim}\, A/\mathrm{Dim}\, B)x_B$ by Proposition 5. Thus the idempotent $x_A/\mathrm{Dim}\, A = x_B/\mathrm{Dim}\, B \in B$. Mimicking the proof of Theorem 6 one can easily show that:

Proposition 6 *Let A be a finite-dimensional Hopf algebra over a field k of characteristic zero. Suppose that t is a Hopf algebra automorphism of A of finite order r and that t satisfies the strong vanishing trace condition. Let B be the sub-Hopf algebra of A generated by the coradical of A. Then r divides* $(\mathrm{Dim}\, A\, \mathrm{Dim}\, Bx_B)/\mathrm{Dim}\, B$. $\Box$

Let A_s be the unique maximal semisimple sub-Hopf algebra of A. Since semisimple Hopf algebras in characteristic zero are also cosemisimple [5, Theorem 3.3], we conclude that $A_s \subseteq A_0 \subseteq B$, where B is the sub-Hopf algebra of A generated by the coradical of A. Thus $A_s = B$ if and only if A_0 is a sub-Hopf algebra of A. In this case $x_A = (\mathrm{Dim}\, A/\mathrm{Dim}\, B)x_B$ is a non-zero left integral of B by [7, Proposition 9] since $s^2 = 1_A$; thus $\mathrm{Dim}\, Bx_A = 1$ and the conclusions of Theorem 6 and Proposition 6 are the same.

There are interesting connections between the strong vanishing trace condition and the classification problem for finite-dimensional Hopf algebras A of dimension p^2, where p is an odd prime, over an algebraically closed field of characteristic zero. Let s be the antipode of A. By the results of [1] the Hopf algebra A is the group algebra of Z_{p^2} or $Z_p \times Z_p$ over k, or is a Taft algebra $\mathrm{T}_{n,k}(q)$, or s^2 has order $2p$. If s^2 satisfies the strong vanishing trace condition then the last possibility is ruled out.

Let A be any Hopf algebra over k. The subspace of fixed points of s^2 is a subalgebra of B of A. If A is finite-dimensional and pointed, and k is an algebraically closed field of characteristic 0, then $\mathrm{Dim}\, B$ divides $\mathrm{Dim}\, A$ by part b) of Theorem 6. Generally A is not a free (left) B-module.

Example 1 *Let A be a finite-dimensional unimodular Hopf algebra with antipode s over k, and suppose that $s^2 \neq 1_A$. Then A is not a free left module over the subalgebra B of fixed points of s^2.*

To see this, let Λ be a non-zero two-sided integral for A. Then $s(\Lambda) = \Lambda$. Therefore $\Lambda \in B$. Now $s^2 \neq 1_A$ implies that $B \neq A$. Suppose $\{m_1, \ldots, m_r\}$ is a basis for A as a left B-module. Then $r > 1$ and $\epsilon(m_\imath) \neq 0$ for some $1 \leq \imath \leq r$. We may assume that $\epsilon(m_1) \neq 0$. Since $\epsilon(m_2)\Lambda m_1 - \epsilon(m_1)\Lambda m_2 = 0$ determines a non-trivial dependency relation among $m_1, \ldots, m_r$, it follows that A can not be a free left B-module.

There are an abundance of examples which satisfy the hypothesis of Example 1. Let A be any finite-dimensional Hopf algebra over k whose antipode s satisfies $s^2 \neq 1_A$. Then the quantum double $D(A)$ of A satisfies the hypothesis of Example 1.

Let A be finite-dimensional Hopf algebra over any field k. Recall that $D(A) = A^{*\,cop} \otimes A$ as coalgebras. One way of describing the multiplication in $D(A)$ is by the formula

$$(p \otimes a)(q \otimes b) = p(a_{(1)} \rightharpoonup q \leftharpoonup s^{-1}(a_{(3)})) \otimes a_{(2)} b$$

for all $p, q \in A^*$ and $a, b \in A$. The antipode s of $D(A)$ is given by

$$\mathsf{s}(p \otimes a) = \mathsf{s}((p \otimes 1)(\epsilon \otimes a)) = (\epsilon \otimes s(a))(S^{-1}(p) \otimes 1),$$

where s is the antipode of A and $S = s^*$. Observe that $\mathsf{s}^2 = S^{-2} \otimes s^2$. Thus s^2 is a semisimple endomorphism of A if and only if s is a semisimple endomorphism of $D(A)$.

Let λ be a *left* integral for A^* and Λ be a *right* integral for A. Then $\lambda \otimes \Lambda$ is a two-sided integral for $D(A)$ by part a) of [12, Theorem 4]. Let t be Hopf algebra automorphism of A and set $T = t^*$. Then $T^{-1} \otimes t$ is a Hopf algebra automorphism of $D(A)$. We leave the reader with the short exercise of establishing

$$\nu_{D(A)}(T^{-1}\otimes t) = \Big(s(\Lambda_{(4)}) \rightharpoonup (S^{-1}\circ T^{-1}(\nu_{A^*}(T))) \leftharpoonup \Lambda_{(2)}\Big) \otimes s(\Lambda_{(3)})t(\Lambda_{(1)}), \quad (17)$$

where $\nu_{A^*}(T)$ is defined with respect to the left integral λ for A^*.

Regard A and $A^{*\,cop}$ as sub-Hopf algebras of $D(A)$ by the identifications $a = \epsilon\otimes a$ and $p = p\otimes 1$ for all $a \in A$ and $p \in A^*$. Let t be a Hopf algebra automorphism of A and consider the Hopf algebra automorphism $T^{-1}\otimes t$ of $D(A)$, where $T = t^*$. Observe that $(T^{-1}\otimes t)|_A = t$ and $(T^{-1}\otimes t)|_{A^{*cop}} = T^{-1}$. Now $\nu_{A^*}(T) = 0$ if and only if $\nu_{A^{*\,cop}}(T^{-1}) = 0$ by (4). Thus either $\nu_A(t) = 0$ or $\nu_{A^*}(T) = 0$ implies that $\nu_{D(A)}(T^{-1}\otimes t) = 0$ by part c) of Proposition 2.

Proposition 7 *Let A be a finite-dimensional Hopf algebra over any field k, suppose that t is a Hopf algebra automorphism of A and let $T = t^*$. Then the Hopf algebra automorphism $T^{-1}\otimes t$ of $D(A)$ satisfies the following:*

a) *If t satisfies the vanishing trace condition then so does $T^{-1}\otimes t$.*

b) *If either t or T satisfies the strong vanishing trace condition then so does $T^{-1}\otimes t$.*

PROOF: In light of the comments above we need only note that $\mathsf{tr}((T^{-1}\otimes t)^m) = \mathsf{tr}(T^{-m})\mathsf{tr}(t^m)$ for all $m \geq 0$. $\square$

Introducing (17) into the discussion at this point we deduce some information about $\nu_{A^*}(T)$ when $\nu_A(t) = 0$.

Proposition 8 *Let A be a finite-dimensional Hopf algebra with antipode s over the field k and suppose that t is a Hopf algebra automorphism of A. If $\nu_A(t) = 0$ then*

$$\Big(s(\Lambda_{(4)}) \rightharpoonup (S^{-1}\circ T^{-1}(\nu_{A^*}(T))) \leftharpoonup \Lambda_{(2)}\Big) \otimes s(\Lambda_{(3)})t(\Lambda_{(1)}) = 0,$$

where Λ is a right integral for A, $S = s^*$ and $T = t^*$. $\square$

For a Hopf algebra automorphism t of A it may very well be the case that $\nu_A(t) = 0$ and $\nu_{A^*}(t^*) \neq 0$. For example, suppose that A is a finite-dimensional Hopf algebra which is not semisimple whose dual A^* is semisimple. Case in point, the group algebra $k[Z_p]$ of the cyclic group of order p over k when the characteristic of k is $p \geq 2$. Then $\nu_A(s^2) = 0$ and $\nu_{A^*}(s^{2\,*}) \neq 0$ by part a) of Proposition 2.

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