ON GENERALIZED HOPF GALOIS EXTENSIONS

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INTRODUCTION

Let H be a Hopf algebra over a commutative base ring k, and A a right H-comodule algebra with comodule structure $\delta \colon A \to A \otimes H$, $\delta(a) =: a_{(0)} \otimes a_{(1)}$. Denote by $B := A^{\operatorname{co} H} := \{b \in A | \delta(b) = b \otimes 1\}$ the subalgebra of coinvariant elements. A is said to be an H-Galois extension of B if the Galois map

$$\beta \colon A \underset{B}{\otimes} A \to A \otimes H, x \otimes y \mapsto xy_{(0)} \otimes y_{(1)},$$

is a bijection.

A faithfully flat (as B-module) H-Galois extension A is a noncommutative-geometric version of a principal fiber bundle or torsor in the sense of [8]: If A and H are commutative, and represent respectively an affine scheme X and an affine group scheme G acting on X, then $B = A^{\operatorname{co} H}$ represents the quotient Y of X under the action of G. Bijectivity of the Galois map β means that

$$X \times G \to X \times_Y X, (x, g) \mapsto (xg, x),$$

is an isomorphism, which can be interpreted as the correct algebraic formulation of the condition that the G-action of X should be free, and transitive on the fibers of the map $X \to Y$.

In many applications surjectivity of the Galois map β, which, in the commutative case, means freeness of the action of G, is obvious, or at least easy to prove (it is sufficient to find $1 \otimes h$ in the image for each h in a generating set for the algebra H). It is usually much harder to decide whether β is injective.

The present paper has two main topics: When does surjectivity of β already imply bijectivity? What can we conclude about the module structure of A over B, or the comodule structure of A, or general Hopf modules, over H? Both questions will be studied for more general extensions.

The Kreimer-Takeuchi Theorem [16, Thm. 1.7] says that if β is onto and H is finite, then β is bijective and A is a projective B-module. This generalizes a Theorem of Grothendieck [8, III, §2, 6.1] on the actions of finite group schemes. Theorem 3.5 in [32] implies that if β is surjective and A is a relative injective H-comodule, then β is bijective and A is a faithfully flat B-module. This generalizes results of Oberst [26], and Cline, Parshall, and Scott [6] for the case where H represents a closed subgroup of an affine group scheme represented by A; in this situation the canonical map is trivially surjective, while injectivity of the H-comodule A means that the induction functor from the subgroup in question is exact.

A new proof for both of these results appeared in [31], where it is also shown that in the situation of [32, Thm.I] the B-module A is projective as well. The unified proof and the stronger conclusion are based on the observation that the Galois map $\beta_0 \colon A \otimes A \to A \otimes H$

(where the tensor product in the source is taken over k rather than the coinvariant subalgebra B), which is surjective by assumption, can be shown to be split as an H-comodule map in each case.

In the present paper we will show (with a further simplified proof) that having an H-colinearly split surjective map β_0 characterizes relative projective H-Galois extensions. We will in fact show this for more general extensions, and we will discuss applications of the generalized result, with appropriate additional hypotheses, to a variety of Galois-type situations.

In Section 4 we give a new proof for the criteria [32, Thm.3.5, Thm.I] mentioned above.

In Section 3 we prove a strong generalization of the Kreimer-Takeuchi Theorem. Among its corollaries are the original Kreimer-Takeuchi Theorem as well as a result of Beattie, Dăscălescu, and Raianu [1] for the co-Frobenius case; we improve on the latter by proving that the extension is projective rather than flat.

In Section 5 we will consider another condition on a Hopf Galois extension, which we call equivariant projectivity. This is a stronger requirement than projectivity of A over B; it was studied by Dąbrowski, Grosse, and Hajac [7], who showed that a Hopf Galois extension is equivariantly projective if and only if it has a so-called strong connection. This notion in turn was defined by Hajac [12] with motivations from differential geometry; see also [13]. Most notably we will show in Theorem 5.6 that if H is a Hopf algebra with bijective antipode over a field, then every faithfully flat H-Galois extension is equivariantly projective. Thus, strong connections always exist in the situation for which they were originally defined. Meanwhile, the notion of equivariant projectivity and its geometric interpretations were discussed for more general Galois-type extensions by Brzezinski and Hajac [3]. For a large class of these which is of particular interest for quantum group theory, we also prove equivariant projectivity in Theorem 5.9.

The reason why we are interested in generalizations of Hopf Galois extensions lies in the quotient theory of noncommutative Hopf algebras. The quotient Hopf algebras of a commutative Hopf algebra H correspond naturally to the closed subgroups of the affine group scheme represented by H. If H is noncommutative, however, it is not enough to consider quotient Hopf algebras. Rather, one should also take into account quotient coalgebras and right (or left) H-modules Q of H; quotient theory of Hopf algebras in this sense was studied by Takeuchi [37] and Masuoka [24]. Thus it becomes natural to consider Q-Galois extensions, that is, H-comodule algebras A for which the canonical map $A \otimes_B A \to A \otimes Q$ is bijective, for $B = A^{\mathrm{co}\,Q}$. Such extensions are already studied in [32]. It is important for the theory that the notion of a Hopf module $M \in \mathcal{M}_A^Q$ can be defined for a right H-module coalgebra quotient Q of H in the same way as a Hopf module in $\mathcal{M}_A^H$. Later Q-extensions were studied in successively more general frameworks. In the most general version a Coalgebra Galois extension [4] is simply an algebra A which is a C-comodule for a coalgebra C such that the canonical map $\beta\colon A \otimes_B A \to A \otimes C$ is a bijection; here $B = A^{\mathrm{co}\,C}$ is defined, following Takeuchi, as the largest subalgebra for which the comodule structure of A is left B-linear. The notion of a Hopf module, which is a central tool for studying Hopf Galois extensions, also underwent a series of generalizations: First, one can study an H-comodule algebra A as before, but replace Q by an H-module coalgebra C; thus, one arrives at the self-dual notion of Doi-Koppinen data (A, C, H) for which much of the theory of Hopf modules can be developed [10, 15].

The Hopf algebra's main role in this formalism is to induce a generalized switching map

$$C \otimes A \ni c \otimes a \mapsto a_{(0)} \otimes c \cdot a_{(1)} \in A \otimes C$$

between the algebra and coalgebra in consideration. A further step abstracts this switching map from the auxiliary Hopf algebra H and defines an entwining [5] to be a map $\psi \colon C \otimes A \to A \otimes C$ subject to certain axioms that, again, are sufficient to define a notion of Hopf module in $\mathcal{M}_A^C$, which is now called an entwined module. An entwining between C and A naturally arises from every C-Galois extension A, in such a way that A itself is an entwined module. We will develop many of our criteria for Galois-type extensions for entwinings between a coalgebra C and an algebra A for which A itself is an entwined module, in most cases with a comodule structure induced by a distinguished grouplike in C. We are most interested in the special case where A is an H-comodule algebra, $C = Q$ is a quotient coalgebra and right module of H, the distinguished grouplike e is the image of $1 \in H$ in Q, and the entwining is given by $\psi(\overline{h} \otimes a) = a_{(0)} \otimes \overline{h a_{(1)}}$. In fact, no example of a C-Galois extension that would not have this form seems to be known to date. Even though the theory of C-Galois extensions is potentially more general than that of Q-Galois extensions, it is perhaps more important that the formalism of entwinings is more elegant and transparent in some situations than the use of an auxiliary Hopf algebra, and it may serve to make proofs more transparent and draw attention to those instances where a Hopf algebra in the background is truly needed for more than formal reasons.

We believe that the results of the present paper show that the following two conditions on a Q-extension are of particular interest: Equivariant projectivity, and the property that all Hopf modules are relative injective as comodules. The status of the former has changed significantly by our results: At its conception, this strong, geometrically motivated condition seemed to single out a particularly well-behaved class among Hopf Galois extensions, and of course among the more general Q-Galois extensions. Now we know that it is shared by all faithfully flat H-Galois extensions when H is a Hopf algebra with bijective antipode over a field, that is, by all those Hopf Galois extensions that are candidates for a quantum group analog of a classical principal fiber bundle. We also prove in Theorem 5.9 that it is shared by all Q-Galois extensions with cosemisimple Q over, say, the complex numbers.

On the other hand, we know very little about when this property is fulfilled in general. The problem seems to be as hard for quotients of Hopf algebras (i.e. quantum analogs of homogeneous spaces) as it is for general Q-Galois extensions (i.e. quantum analogs of principal fiber bundles). We have pointed out the second interesting property, namely that all Hopf modules are relative injective comodules, as a powerful technical tool in criteria for Q-Galois situations. Again, we have collected results that show this property to be fulfilled quite often, but we do not know in what generality it can be proved. And once again, the case where the Q-extension in consideration comes from a quotient map $H \to Q$ seems to be as hard as the general case.

If H is finite-dimensional over a field, then both of the conditions on a quotient coalgebra and right module Q mentioned above are equivalent to the (in general rather stronger) condition that H be Q-cleft. This is not hard to see using the equivalent characterizations of the latter condition proved by Hoffmann, Koppinen, and Masuoka [22, 24]. Serge Skryabin kindly gave us access to his recent preprint [34], where he proves that H is in fact always Q-cleft in the finite-dimensional case.

Preliminaries and notations

Throughout the paper k denotes a commutative base ring. All maps are at least k-linear, unadorned tensor product is understood to be over k, algebras, coalgebras, and Hopf algebras are over k. We use $\nabla\colon A \otimes A \to A$, $\eta\colon k \to A$ to denote the multiplication and unit map of an algebra A, and $\mu = \mu_M\colon A \otimes M \to M$ to denote the structure map of an A-module M. The category of left (resp. right) A-modules will be denoted $_A\mathcal{M}$ (reps. $\mathcal{M}_A$). We write $\Delta\colon C \to C \otimes C; c \mapsto c_{(1)} \otimes c_{(2)}$ and $\varepsilon\colon C \to k$ for the comultiplication and counit of a coalgebra C. For a right C-comodule $M \in \mathcal{M}^C$ we write $\delta = \delta_M\colon M \to M \otimes C; m \mapsto m_{(0)} \otimes m_{(1)}$ for its comodule structure. For left comodules we use $\delta(m) = m_{(-1)} \otimes m_{(0)}$. For $M \in \mathcal{M}^C$ and a left comodule $N \in {}^C\mathcal{M}$ we denote by $M \,\square_C\, N$ the cotensor product. The antipode of a Hopf algebra H is denoted by S.

A left R-module M is called relative projective if it fulfills the following equivalent conditions: Any surjective R-module map $f\colon N \to M$ which splits as a k-module map also splits as an R-module map; The module structure map $\mu\colon R \otimes M \to M$ splits as an R-module map; $\mathrm{Hom}_R(M, f)$ is surjective if $f\colon N \to N'$ is a surjective R-module map that is k-split. Note that direct summands and direct sums of relative projective R-modules are relative projective. Also, if V is a k-module and M a relative projective R-module, then $M \otimes V$ is relative projective.

Dually, a right C-comodule M is called relative injective if it fulfills the following equivalent conditions: Any injective C-comodule map $f\colon M \to N$ which splits as a k-module map also splits as a C-comodule map; The comodule structure map $\delta\colon M \to M \otimes C$ splits as a C-comodule map; $\mathrm{Hom}^C(f, M)$ is surjective if $f\colon N \to N'$ is an injective C-comodule map which is k-split. Note again that direct summands and finite direct sums of relative injective C-comodules are relative injective, as is $V \otimes M$ whenever V is a k-module and M is a relative injective C-comodule.

The notion of relative projectivity (which one should call k-relative projectivity, but no other versions will occur in this paper) is a special case of the terminology of relative homological algebra as found in [20, Chap.IX]. The same is true for relative injectivity, provided that C is k-flat, which ensures that the category $\mathcal{M}^C$ is abelian to begin with.

1. Generalities on entwining structures

In this section we collect some general conventions and facts on entwinings and their relation to coalgebra Galois extensions. Most of these can be found in [2]; we also refer to the survey article [3].

Definition 1.1. An **entwining structure** (A, C, ψ) consists of an algebra A, a coalgebra C, and an **entwining**, that is, a map $\psi \colon C \otimes A \to A \otimes C$ satisfying

$$\psi(C \otimes \nabla) = (\nabla \otimes C)(A \otimes \psi)(\psi \otimes A) \colon C \otimes A \otimes A \to A \otimes C$$

$$\psi(c \otimes 1) = 1 \otimes c \qquad \forall c \in C$$

$$(A \otimes \Delta)\psi = (\psi \otimes C)(C \otimes \psi)(\Delta \otimes A) \colon C \otimes A \to A \otimes C \otimes C$$

$$(A \otimes \varepsilon)\psi = \varepsilon \otimes A \colon C \otimes A \to A.$$

Definition 1.2. Let (A, C, ψ) be an entwining structure. An **entwined module** $M \in \mathcal{M}_A^C$ is a right A-module and right C-comodule such that the diagram

$$
\begin{array}{ccccc}
M \otimes A & \xrightarrow{\delta \otimes A} & M \otimes C \otimes A & \xrightarrow{M \otimes \psi} & M \otimes A \otimes C \\
\mu \downarrow & & & & \downarrow \mu \otimes C \\
M & \xrightarrow{\hspace{3cm} \delta \hspace{3cm}} & & & M \otimes C
\end{array}
$$

commutes.

Lemma 1.3. *Let (A, C, ψ) be an entwining structure.*

(1) *For any $M \in \mathcal{M}_A$ we have $M \otimes C \in \mathcal{M}_A^C$ with the obvious comodule structure and the module structure*

$$\mu_{M \otimes C} = \left(M \otimes C \otimes A \xrightarrow{M \otimes \psi} M \otimes A \otimes C \xrightarrow{\mu_M \otimes C} M \otimes C \right).$$

This construction defines a right adjoint functor $\mathcal{M}_A \to \mathcal{M}_A^C$ to the underlying functor.

  If also $M \in \mathcal{M}^C$, then M is an entwined module if and only if the comodule structure $\delta \colon M \to M \otimes C$ is an A-module map.

(2) *For any $M \in \mathcal{M}^C$ we have $M \otimes A \in \mathcal{M}_A^C$ with the obvious module structure and the comodule structure*

$$\delta_{M \otimes A} = \left(M \otimes A \xrightarrow{\delta_M \otimes A} M \otimes C \otimes A \xrightarrow{M \otimes \psi} M \otimes A \otimes C \right).$$

This construction defines a left adjoint functor $\mathcal{M}^C \to \mathcal{M}_A^C$ to the underlying functor.

  If also $M \in \mathcal{M}_A$, then M is an entwined module if and only if the module structure $\mu \colon M \otimes A \to M$ is a C-comodule map.

Remark 1.4. In particular, an entwining structure (A, C, ψ) gives rise to entwined module structures on $C \otimes A$ as well as $A \otimes C$. With these structures, ψ is a morphism of entwined modules.

Note that the right A-module structure of $A \otimes C$ determines ψ uniquely through the formula $\psi(c \otimes a) = (1 \otimes c)a$. Dually, ψ is determined by the C-comodule structure of $C \otimes A$.

Definition 1.5. Let C be a coalgebra, and let A be an algebra and a C-comodule. Put $B := A^{\mathrm{co}\, C} := \{b \in A | \forall a \in A \colon \delta(ba) = ba_{(0)} \otimes a_{(1)}\}$. Define the canonical or Galois maps $\beta_0 \colon A \otimes A \to A \otimes C$ and $\beta \colon A \otimes_B A \to A \otimes C$ by $\beta_0(x \otimes y) = \beta(x \otimes y) = xy_{(0)} \otimes y_{(1)}$. A is called a C-**Galois extension** of B if β is a bijection.

Lemma 1.6. *Let A be an algebra and a C-comodule. If there is an entwining $\psi \colon C \otimes A \to A \otimes C$ for which $A \in \mathcal{M}_A^C$, then both Galois maps β_0 and β are morphisms of entwined modules.*

If A is a C-Galois extension, then there is a unique entwining $\psi\colon C \otimes A \to A \otimes C$ satisfying this condition. It is called the canonical entwining associated to the C-Galois extension A, and is given by $\psi(c \otimes a) = \beta(\beta^{-1}(1 \otimes c)a)$.

Note that $\delta(b) = b\delta(1)$ whenever $b \in A^{\mathrm{co}\,C}$. If A is C-Galois, we now know that $A \otimes C$ is a right A-module in such a way that $\delta\colon A \to A \otimes C$ is a right A-module map. Then if $\delta(b) = b\delta(1)$ for some $b \in A$, then $\delta(ba) = \delta(b)a = b\delta(1)a = b\delta(a)$ for all $a \in A$, hence $b \in A^{\mathrm{co}\,C}$.

Lemma 1.7. *Let (A, C, ψ) be an entwining structure and assume that A has a C-comodule structure making it an entwined module $A \in \mathcal{M}_A^C$ with the regular A-module structure.*

The induction functor

$$\mathcal{M}_B \ni N \mapsto N \underset{B}{\otimes} A \in \mathcal{M}_A^C$$

is left adjoint to the functor of coinvariants

$$\mathcal{M}_A^C \ni M \mapsto M^{\mathrm{co}\,C} := \{m \in M | \delta(m) = m\delta(1)\} \in \mathcal{M}_B.$$

If C is k-flat, then the following are equivalent:

(1) *The induction functor is an equivalence.*
(2) *A is C-Galois, and faithfully flat as a left B-module.*

Lemma 1.8. *Let (A, C, ψ) be an entwining structure and assume that A has a C-comodule structure making it an entwined module $A \in \mathcal{M}_A^C$ with the regular A-module structure.*

For $M \in \mathcal{M}_A$ consider $M \otimes C \in \mathcal{M}_A^C$ as in Lemma 1.3. Then $M \otimes \varepsilon$ induces a bijection $(M \otimes C)^{\mathrm{co}\,C} \to M$ with inverse $M \to (M \otimes C)^{\mathrm{co}\,C}, m \mapsto m\delta(1)$.

PROOF. This can be verified by direct computation, or by applying the adjunctions in Lemma 1.3 and Lemma 1.7 to calculate, for $B := A^{\mathrm{co}\,C}$ and $N \in \mathcal{M}_B$:

$$\begin{aligned}
\mathcal{M}_B(N, (M \otimes C)^{\mathrm{co}\,C}) &\cong \mathcal{M}_A^C(N \underset{B}{\otimes} A, M \otimes C) \\
&\cong \mathcal{M}_A(N \underset{B}{\otimes} A, M) \\
&\cong \mathcal{M}_B(N, M)
\end{aligned}$$

and appealing to the Yoneda Lemma. Let us also note that the statement is almost trivial if the comodule structure is of the form discussed in Corollary 1.10 below. $\square$

Lemma 1.9. *Let A be a C-Galois extension of $B = A^{\mathrm{co}\,C}$. Assume that there is a grouplike element $e \in C$ with $\delta(1) = 1 \otimes e$. Then $\delta(a) = \psi(e \otimes a)$ for all $a \in A$, where ψ is the canonical entwining. Moreover, $M^{\mathrm{co}\,C} = \{m \in M | \delta(m) = m \otimes e\}$ for all $M \in \mathcal{M}_A^C$.*

Corollary 1.10. *Let (A, C, ψ) be an entwining structure, and $e \in C$ a grouplike element. Then $A \in \mathcal{M}_A^C$ with the regular right A-module structure and the comodule structure $\delta\colon A \to A \otimes C$ given by $\delta(a) = \psi(e \otimes a)$.*

In fact view ke as a C-comodule, identify $A = ke \otimes A$, and apply Lemma 1.3.

Lemma 1.11. *Let (A, C, ψ) be an entwining structure and $e \in C$ a grouplike element. Endow A with the C-comodule structure as in Corollary 1.10. Then $M^{\mathrm{co}\,C} = \{m \in M | \delta(m) = m \otimes e\}$ for every $M \in \mathcal{M}_A^C$. Assume that A is C-Galois. Then the entwining associated to the C-Galois extension A of B as in Lemma 1.6 coincides with ψ.*

PROOF. For any $x, y \in A$ we have $\delta(xy) = \psi(e \otimes xy) = (\nabla \otimes C)(A \otimes \psi)(\psi \otimes A)(e \otimes x \otimes y) = (\nabla \otimes C)(A \otimes \psi)(x_{(0)} \otimes x_{(1)} \otimes y) = x_{(0)}\psi(x_{(1)} \otimes y)$.

If $b \in A^{\mathrm{co}\, C}$, then $\delta(b) = \delta(b \cdot 1) = b\delta(1) = b \otimes e$. If, on the other hand, $\delta(b) = b \otimes e$, then $\delta(ba) = b_{(0)}\psi(b_{(1)} \otimes y) = b\psi(e \otimes y) = b\delta(y)$.

That ψ coincides with the canonical entwining is a direct consequence of Lemma 1.6. □

Remark 1.12. The definition of an entwining has an obvious asymmetry (the coalgebra starts out on the left and ends up on the right). We could call an entwining as defined above a *right* entwining, and give an analogous definition of a left entwining; all the results collected above will then have analogous left-right switched versions. We will use these freely, writing $\tilde{\psi}, \tilde{\beta}, \tilde{\beta}_0, \tilde{\delta}$ for left entwinings, and the Galois maps and comodule structures associated with them.

Assume now that (A, C, ψ) is a bijective entwining structure, by which we shall mean that the map ψ is bijective. Then the inverse $\psi^{-1} \colon A \otimes C \to C \otimes A$ is a left entwining. If $e \in C$ is a grouplike element, we thus have both a right C-comodule structure δ and a left C-comodule structure δ^L on A. It turns out that the left and right C-coinvariant elements of A coincide: Writing $e \colon k \to A$ for the map that sends $1 \in k$ to $e \in A$, we see that $A^{\mathrm{co}\, C}$ is the equalizer of $A \otimes e, \psi(e \otimes A) \colon A \to A \otimes C$, while $^{\mathrm{co}\, C}A$ is the equalizer of $e \otimes A = \psi^{-1}\psi(e \otimes A)$ and $\psi^{-1}(A \otimes e)$. We also have left versions $\beta^L \colon A \otimes_B A \to C \otimes A$ and $\beta_0^L \colon A \otimes A \to C \otimes A$ of the Galois maps, mapping $x \otimes y$ to $x_{(-1)} \otimes x_{(0)}y$. Since $\psi\beta_0^L = \beta_0$ and $\psi\beta^L = \beta$ by the calculation $\psi\beta_0^L(x \otimes y) = \psi(\psi^{-1}(x \otimes e)y) = \psi(C \otimes \nabla)(\psi^{-1} \otimes A)(x \otimes e \otimes y) = (\nabla \otimes C)(A \otimes \psi)(x \otimes e \otimes y) = \beta_0(x \otimes y)$, we see that A is left C-Galois if and only if it is (right) C-Galois.

Similarly, given a bijective left entwining $\tilde{\psi}$, we get a right entwining $\tilde{\psi}^{-1}$, and right Galois maps $\tilde{\beta}^R, \tilde{\beta}_0^R$.

Remark 1.13. (1) A Doi-Koppinen datum is a triple (H, A, C) consisting of a bialgebra H, a right H-comodule algebra A, and a right H-module coalgebra C. For every Doi-Koppinen datum, we have an entwining structure (A, C, ψ) defined by $\psi(c \otimes a) = a_{(0)} \otimes c \cdot a_{(1)}$. The entwining is bijective provided that H has a skew antipode S^- (for example, H is a Hopf algebra with bijective antipode). The inverse of ψ is then given by $\psi^{-1}(a \otimes c) = c \cdot S^-(a_{(1)}) \otimes a_{(0)}$.

(2) In particular, let H be a bialgebra, A a right H-comodule algebra, and Q a quotient coalgebra and right H-module of H. Then we have an entwining (A, Q, ψ), which is bijective if H has a skew antipode. Note that the Q-comodule structure of A is the one given in Corollary 1.10 for the grouplike $e = \overline{1} \in Q$. The Galois maps β, β_0 in this case are given by $\beta(x \otimes y) = \beta_0(x \otimes y) = xy_{(0)} \otimes \overline{y_{(1)}}$.

(3) Let A be a right H-comodule algebra, and C a *left* H-module coalgebra. Then we can view C^{cop} as a right H^{cop}-module coalgebra, and A as a *left* H^{cop}-comodule algebra, and hence we have a *left* entwining structure $(A^{\mathrm{op}}, C, \tilde{\psi})$, with $\tilde{\psi}(a \otimes c) = a_{(1)} \cdot c \otimes a_{(0)}$. Entwined modules in $_A^{C^{\mathrm{cop}}}\mathcal{M}$ are Hopf modules in $_A\mathcal{M}^C$, that is, left A-modules and right C-comodules M satisfying $\delta(am) = a_{(0)}m_{(0)} \otimes a_{(1)} \cdot m_{(1)}$ for all $a \in A$ and $m \in M$. The left entwining in this situation is bijective if H has an antipode; we have $(\tilde{\psi})^{-1}(c \otimes a) = a_{(0)} \otimes S(a_{(1)}) \cdot c$.

(4) A special case arises when $C = Q'$ is a quotient coalgebra and left H-module of the bialgebra H, and A is an H-comodule algebra. Note that in this case the left Galois maps $\tilde{\beta}_0 \colon A \otimes A \to C^{\mathrm{cop}} \otimes A$ and $\tilde{\beta} \colon A \otimes_B A \to C^{\mathrm{cop}} \otimes A$ identify,

respectively, with $\beta_0' \colon A \otimes A \to A \otimes Q'$, and $\beta' \colon A \otimes_B A \to A \otimes Q'$ given by $\beta_0'(x \otimes y) = \beta'(x \otimes y) = x_{(0)} y \otimes x_{(1)}$. If H is a Hopf algebra, so the left entwining is bijective, we can also consider the right Galois map $\tilde{\beta}_0^R \colon A \otimes A \to A \otimes Q'$, given by $\tilde{\beta}_0^R = \tilde{\psi}^{-1} \tilde{\beta}_0$, or $\tilde{\beta}_0^R(x \otimes y) = \tilde{\psi}^{-1}(\overline{x_{(1)}} \otimes x_{(0)} y) = x_{(0)} y_{(0)} \otimes \overline{S(x_{(1)} y_{(1)}) x_{(2)}} = xy_{(0)} \otimes \overline{S(y_{(1)})}$.

(5) Let H be a Hopf algebra with bijective antipode. Then quotient coalgebras and right H-modules Q of H (i.e. coideal right ideals $I \subset H$) and quotient coalgebras and left H-modules Q' of H (i.e. coideal left ideals I' of H) are in bijection via $I' = S(I)$. If Q' corresponds to Q, then the antipode of H induces a coalgebra anti-isomorphism $S \colon Q \to Q'$. For a right H-comodule algebra A, the Galois maps $\beta_0 \colon A \otimes A \to A \otimes Q$ as in (2) and $\tilde{\beta}_0^R \colon A \otimes A \to A \otimes Q'$ as in (4) identify along $A \otimes S$.

Remark 1.14. Let (A, C, ψ) be an entwining structure, where A is a finite-dimensional algebra (and k is a field).

By [30] there exists a Doi-Koppinen data, that is, a bialgebra H, a right H-module coalgebra structure on C, and a right H-comodule algebra structure on A, such that the entwining ψ has the form given above, i.e. $\psi(c \otimes a) = a_{(0)} \otimes c \cdot a_{(1)}$, where $A \ni a \mapsto a_{(0)} \otimes a_{(1)} \in A \otimes H$ denotes the H-comodule structure of A. If we are given a grouplike $e \in C$, then an H-module coalgebra map $\pi \colon H \to C$ is given by $\pi(h) = e \cdot h$. The relevant Galois map for the induced right C-comodule structure on A is

$$A \otimes A \ni x \otimes y \mapsto xy_{(0)} \otimes \pi(y_{(1)}) \in A \otimes C.$$

Thus, if A is C-Galois, then π has to be surjective, and we can consider C as a quotient coalgebra and right H-module of H.

It is not known even in the situation where A is finite-dimensional whether H can be chosen to be a Hopf algebra.

There are examples of entwining structures (A, C, ψ) with infinite-dimensional A that do not come from Doi-Koppinen data [30]. It seems to be an open question, however, whether there exist C-Galois extensions whose entwining cannot be induced by a Doi-Koppinen data.

2. Projective Galois extensions

This section contains a key result of our paper, a characterization of (relative) projective Galois-type extensions as those for which a canonical map is a split surjective comodule map. The result will be applied in many ways in the subsequent sections.

The following Lemma is a combination of the adjointness in Lemma 1.3 (2) with the morphism ψ in $\mathcal{M}_A^C$, which is assumed to be an isomorphism. Its central use in the theory of comodule algebras goes back to a paper of Doi [9].

Lemma 2.1. *Let A be an algebra, C a coalgebra, and $\psi \colon C \otimes A \to A \otimes C$ a bijective entwining. Then for each $V \in \mathcal{M}_A^C$ we have an isomorphism*

$$\Phi \colon \mathcal{M}^C(C, V) \to \mathcal{M}_A^C(A \otimes C, V)$$

given by

$$\Phi(\gamma) = \left(A \otimes C \xrightarrow{\psi^{-1}} C \otimes A \xrightarrow{\gamma \otimes A} V \otimes A \xrightarrow{\mu_V} V \right).$$

Every surjective morphism $V \to A \otimes C$ that splits as a C-comodule map also splits in $\mathcal{M}_A^C$.

PROOF. It is easy to check that $\Phi(\gamma)$ is a morphism of entwined modules. The inverse of Φ is given by $\Phi^{-1}(\varphi)(c) = \varphi(1 \otimes c)$.

If $f \colon V \to A \otimes C$ is a morphism in $\mathcal{M}_A^C$ and $g \colon A \otimes C \to V$ satisfies $gf = \mathrm{id}_{A \otimes C}$, then define $g_0 \colon C \to V$ by $g_0(c) = g(1 \otimes c)$, and put $\tilde{g} = \Phi(g_0)$. Then $\tilde{g}$ still splits f, since $fg_0(c) = 1 \otimes c$, and hence

$$f\tilde{g}\psi(c \otimes a) = f(g_0(c)a) = f(g_0(c))a = (1 \otimes c)a = \psi(c \otimes a)$$

for $c \in C$ and $a \in A$. □

Theorem 2.2. *Let (A, C, ψ) be an entwining structure, and assume A has a C-comodule structure making it an entwined module $A \in \mathcal{M}_A^C$ with the regular A-module structure.*

Put $B := A^{\mathrm{co}\,C}$.

Consider the following statements:

(1) *$\beta_0 \colon A \otimes A \to A \otimes C$ is surjective, and splits as a C-comodule map.*
(2) (a) *$\beta \colon A \otimes_B A \to A \otimes C$ is bijective.*
   (b) *A is relative projective as right B-module.*

Then (2) implies (1).

If ψ is bijective, and the obvious map $A \otimes B \to (A \otimes A)^{\mathrm{co}\,C}$ is a bijection, then (1) implies (2).

Remark 2.3. The condition that the canonical map $A \otimes B \to (A \otimes A)^{\mathrm{co}\,C}$ is bijective is fulfilled in each of the following cases:

(1) A is k-flat.
(2) The induction functor $\mathcal{M}_B \to \mathcal{M}_A^C$ is an equivalence, that is (see 1.7) A is a faithfully flat C-Galois extension; indeed, the map in question is the adjunction morphism for the B-module $A \otimes B$.
(3) A is a relative injective C-comodule and ψ is bijective (see 4.1 below).

PROOF. (2) $\Longrightarrow$ (1): If A_B is relative projective, the multiplication map $A \otimes B \to A$ splits in $\mathcal{M}_B$. Apply the functor $(-) \otimes_B A \colon \mathcal{M}_B \to \mathcal{M}_A^C$ to find that

$$\beta_0 = \left(A \otimes A \cong A \otimes B \underset{B}{\otimes} A \xrightarrow{\mu \otimes_B A} A \underset{B}{\otimes} A \xrightarrow{\beta} A \otimes C \right)$$

splits in $\mathcal{M}_A^C$, and in particular as a comodule map.

Now we prove (1) $\Longrightarrow$ (2), assuming that ψ and the map $A \otimes B \to (A \otimes A)^{\mathrm{co}\,C}$ are bijective.

By assumption, there is a C-colinear splitting of β_0, and by Lemma 2.1 it follows that there is a splitting in $\mathcal{M}_A^C$. Thus $A \otimes C$ is a direct summand of $A \otimes A$ in $\mathcal{M}_A^C$. Apply the functor $(-)^{\mathrm{co}\,C}$ to deduce that A is a direct summand of $(A \otimes A)^{\mathrm{co}\,C} = A \otimes B$ in $\mathcal{M}_B$, hence relative projective. This proves (2)(b).

To prove that β is bijective, consider more generally the adjunction map $\mu_V \colon V^{\mathrm{co}\,C} \otimes_B A \to V$ for $V \in \mathcal{M}_A^C$. Since $(A \otimes C)^{\mathrm{co}\,C} \cong A$ by Lemma 1.8, we can identify $\mu_{A \otimes C}$ with β, and will verify that $\mu_{A \otimes C}$ is a bijection by using functoriality of μ. Since $A \otimes C$ is a direct summand of $A \otimes A$, we only need to check that $\mu_{A \otimes A}$ is a bijection. But this follows from the assumptions since

$$\mu_{A \otimes A} \colon (A \otimes A)^{\mathrm{co}\,C} \underset{B}{\otimes} A \cong (A \otimes B) \underset{B}{\otimes} A \to A \otimes A$$

is the canonical isomorphism. □

Remark 2.4. There is a left version of Theorem 2.2 for a bijective left entwining $\tilde{\psi}$, concerned with the condition that the left Galois map $\tilde{\beta}_0 \colon A \otimes A \to C \otimes A$ splits as a left C-comodule map. This is equivalent to the condition that the right Galois map $\tilde{\beta}_0^R \colon A \otimes A \to A \otimes C$ splits as a left C-comodule map, where the left C-comodule structure on the source is that of the left tensor factor, and the one on the right is given by $\tilde{\delta}(a \otimes c) = \tilde{\psi}(a \otimes c_{(1)}) \otimes c_{(2)}$.

It is clear how to specialize Theorem 2.2 to the situation of an H-comodule algebra A and a quotient coalgebra and right H-module Q of a Hopf algebra H with bijective antipode. By switching sides, we also get a version for quotient coalgebras and left modules of H, which we will write down explicitly to clarify the somewhat complicated identifications.

Corollary 2.5. *Let H be a Hopf algebra, and A a right H-comodule algebra. Let Q' be a quotient coalgebra and left H-module of H. Put $B := A^{\mathrm{co}\, Q'}$. Consider the following statements:*

(1) $\beta_0' \colon A \otimes A \to A \otimes Q', x \otimes y \mapsto x_{(0)} y \otimes \overline{x_{(1)}}$ *is surjective, and splits as a right Q'-comodule map.*

(2) (a) $\beta' \colon A \otimes_B A \to A \otimes Q'$ *is bijective.*
    (b) A *is relative projective as left B-module.*

Then (2) implies (1), and if the obvious map $A \otimes B \to (A \otimes A)^{\mathrm{co}\, Q'}$ is bijective, then (1) implies (2).

If the antipode of H is bijective, and Q is the quotient coalgebra and right module of H corresponding to Q', then (1) is equivalent to

(3) $\beta_0 \colon A \otimes A \to A \otimes Q, x \otimes y \mapsto xy_{(0)} \otimes \overline{y_{(1)}}$ *is surjective, and splits as a left Q-comodule map; here, the left Q-comodule structures are given by*

$$A \otimes A \ni x \otimes y \mapsto \overline{S^{-1}(x_{(1)})} \otimes x_{(0)} \otimes y \in Q \otimes A \otimes A$$

$$A \otimes Q \ni x \otimes q \mapsto q_{(1)} S^{-1}(x_{(1)}) \otimes x_{(0)} \otimes q_{(2)} \in Q \otimes A \otimes Q.$$

PROOF. As discussed in Remark 1.13 (4), we have a bijective left entwining $\tilde{\psi}$ involving $C = (Q')^{\mathrm{cop}}$. Applying the left version of Theorem 2.2 yields the stated relations between (1) and (2).

As in Remark 2.4, (1) is equivalent to the condition that $\tilde{\beta}_0^R \colon A \otimes A \to A \otimes Q'$, given by $\tilde{\beta}_0^R(x \otimes y) = xy_{(0)} \otimes \overline{S(y_{(1)})}$, is surjective and splits as a right Q'-comodule map, where the comodule structure on the source is that of the left tensor factor, and that on the target is given by

$$\delta_{A \otimes Q'} \colon A \otimes Q' \ni x \otimes q \mapsto x_{(0)} \otimes q_{(2)} \otimes x_{(1)} q_{(1)} \in A \otimes Q' \otimes Q'.$$

If H has bijective antipode, and Q corresponds to Q' as in Remark 1.13 (5), then $\tilde{\beta}_0^R$ identifies with β_0 as in (3), and the right Q'-comodule structures can be identified with the left Q-comodule structures given in (3), since $(A \otimes S^{-1})\delta_{A \otimes Q'}(x \otimes S(q)) = x_{(0)} \otimes q_{(2)} \otimes S^{-1}(x_{(1)} S(q_{(1)})) = x_{(0)} \otimes q_{(2)} \otimes q_{(1)} S^{-1}(x_{(1)})$. □

Most of our applications of Theorem 2.2 will rely on additional hypotheses on C or the comodule structure. However, we can draw one very general conclusion on the behavior of the Galois condition when we pass to quotients:

Corollary 2.6. *Let (A, C, ψ) be an entwining structure and $e \in C$ grouplike such that A is a C-Galois extension of $B := A^{\mathrm{co}\,C}$, and a relative projective right B-module.*

Let (R, D, θ) be a bijective entwining, $\pi\colon C \to D$ a surjective coalgebra map, and $f\colon A \to R$ a k-split surjective algebra and D-comodule map such that $\theta(\pi \otimes f) = (f \otimes \pi)\psi$.

Assume that π splits as a right D-comodule map.

Then R is a D-Galois extension of $S := R^{\mathrm{co}\,D}$, and if the obvious map $R \otimes S \to (R \otimes R)^{\mathrm{co}\,D}$ is bijective, then R is a relative projective right S-module.

PROOF. The commutative diagram

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{\ \beta_0^{(A)}\ } & A \otimes C \\
{\scriptstyle f \otimes f}\Big\downarrow & & \Big\downarrow{\scriptstyle f \otimes \pi} \\
R \otimes R & \xrightarrow{\ \beta_0^{(R)}\ } & R \otimes D
\end{array}
$$

shows that the canonical map $\beta_0^{(R)}$ for the D-extension R is surjective. By assumption and Theorem 2.2 the C-comodule map $\beta_0^{(A)}$ splits. Since f splits as a k-module map, we see that $\beta_0^{(R)}$ splits as a D-comodule map, and the claim follows from Theorem 2.2. $\square$

Corollary 2.7. *Let H be a k-flat Hopf algebra with bijective antipode, and Q a quotient coalgebra and right module of H. Put $K := H^{\mathrm{co}\,Q}$. The following are equivalent:*

(1) *The surjection $H \to Q$ splits as a right Q-comodule map.*

(2) *H is a Q-Galois extension of K and a relative projective right K-module.*

PROOF. (1) $\Longrightarrow$ (2): We apply Corollary 2.6 with $A = R = C = H$ and $D = Q$.

(2) $\Longrightarrow$ (1): By Theorem 2.2 the Galois map $\beta_0\colon H \otimes H \to H \otimes Q$ splits as a Q-comodule map. Looking at the diagram in the proof of Corollary 2.6, we see that $H \otimes \pi\colon H \otimes H \to H \otimes Q$ splits as a Q-comodule map by, say, $t\colon H \otimes Q \to H \otimes H$. If we define $f\colon Q \to H$ by $f(q) = (\varepsilon \otimes H)f(1 \otimes q)$, then f splits π. $\square$

To close the section, we give an equivalent characterization of projective (rather than relative projective) C-Galois extensions. The proof is very similar to that of 2.2.

Theorem 2.8. *Let (A, C, ψ) be an entwining structure, and assume A has a C-comodule structure making it an entwined module $A \in \mathcal{M}_A^C$ with the regular A-module structure.*

Put $B := A^{\mathrm{co}\,C}$.

The following are equivalent:

(1) *There is an index set I and a surjection $A^{(I)} \to A \otimes C$ in $\mathcal{M}_A^C$ that splits as a C-comodule map.*

(2) *$A \otimes C$ is a direct summand of a direct sum of copies of A as an entwined module in $\mathcal{M}_A^C$.*

(3) (a) *$\beta\colon A \otimes_B A \to A \otimes C$ is bijective.*

    (b) *A is projective as right B-module.*

PROOF. (1)$\Leftrightarrow$(2) by 2.1.

(2) $\Longrightarrow$ (3): Applying the functor $(-)^{\mathrm{co}\,C}$ we find that $A = (A \otimes C)^{\mathrm{co}\,C}$ is a direct summand of a direct sum of copies of $A^{\mathrm{co}\,C} = B$, hence projective. As in the proof of 2.2, we consider $\beta = \mu_{A \otimes C}$ as a special case of the adjunction morphism $\mu_M\colon M^{\mathrm{co}\,C} \otimes_B A \to M$ for $M \in \mathcal{M}_A^C$. Since $A \otimes C$ is a direct summand of a direct sum of copies of A, and μ_A is bijective, so is β.

(3) $\implies$ (2): A is a right B-module direct summand of $B^{(I)}$ for some index set I, so $A \otimes_B A$ is a direct summand of $A^{(I)}$ in $\mathcal{M}_A^C$. But $A \otimes_B A \cong A \otimes C$ by assumption. $\square$

3. Kreimer-Takeuchi type theorems

In this section we will discuss a generalization of the Kreimer-Takeuchi Theorem [16, Thm. 1.7], which, in turn, is a Hopf algebraic version of a result of Grothendieck on actions of finite group schemes [8, III, §2, 6.1]. Let H be a Hopf algebra, and A an H-comodule algebra such that the Galois map $\beta \colon A \otimes A \to A \otimes H$ is surjective.

The Kreimer-Takeuchi theorem says that if H is finitely generated projective, then A is an H-Galois extension of $B := A^{\mathrm{co}\, H}$, and projective as left as well as right B-module. A partial generalization was proved by Beattie, Dăscălescu, and Raianu: If k is a field, and H is co-Frobenius, it follows again that A is an H-Galois extension of B [1, Thm. 3.2, (ii)$\Rightarrow$(i)], and at least a flat B-module.

In case that k is a field, we will see that both results (and the fact that A is projective over B also in the case studied in [1]) follow directly from Theorem 2.2; if k is not a field, projectivity of A as a B-module requires a little extra work. We will prove a more general result for entwining structures, and discuss conditions under which it applies to Q-Galois extensions, with Q a quotient of a Hopf algebra H.

Theorem 3.1. *Let (A, C, ψ) be a bijective entwining and assume A has a C-comodule structure making it an entwined module $A \in \mathcal{M}_A^C$ with the regular A-module structure.*

Assume that the Galois map $\beta_0 \colon A \otimes A \to A \otimes C$ is surjective.

If C is k-flat, and projective as right (left) C-comodule, then A is a C-Galois extension of $B := A^{\mathrm{co}\, C}$ and projective as right (left) B-module.

Proof.   We only treat the version without parentheses, which implies the one in parentheses when applied to the inverse of the entwining ψ.

Note that if A is a projective k-module, then $A \otimes C$ is a projective C-comodule, and hence the surjection β_0 splits as a comodule map. The claims then follow from Theorem 2.2.

For the general case, let $M \in \mathcal{M}_A^C$. We have isomorphisms

$$\mathcal{M}_A^C(A \otimes C, M) \cong \mathcal{M}_A^C(C \otimes A, M) \cong \mathcal{M}^C(C, M)$$

induced, respectively, by ψ and the adjunction in Lemma 1.3 (2). Since C is a projective comodule by assumption, it follows that $A \otimes C$ is a projective object in $\mathcal{M}_A^C$. Now writing A as the quotient of a free k-module $k^{(I)}$ for some index set I, we get a surjection

$$A^{(I)} \cong k^{(I)} \otimes A \to A \otimes A \xrightarrow{\beta_0} A \otimes C$$

in $\mathcal{M}_A^C$. By projectivity of $A \otimes C$, this surjection splits in $\mathcal{M}_A^C$. Thus Theorem 2.8 yields the result. $\square$

Remark 3.2.   (1) Assume that C is a projective k-module. If C is projective as left C^*-module, then it is projective as right C-comodule. Now assume that C is finitely generated projective. Then the converse holds, and moreover C is projective as a left C^*-module if and only if C^* is injective as a right C^*-module, that is, C^* is a right self-injective ring. It is worth noting that if k is a field, then C^* is right self-injective if and only if it is left self-injective [17, Thm. 15.1]; in particular the hypotheses of the Theorem are the same for its left-right switched version if C is finite-dimensional over a field.

(2) Assume that k is a field. The coalgebra C is called left co-Frobenius if there is an injective left C^*-module map from C to C^*. If C is left co-Frobenius, then C is projective as left C^*-module [19, Prop. 5], hence projective as a right C-comodule.

(3) A Hopf algebra H over a field k is left co-Frobenius as a coalgebra if and only if it admits a non-zero left integral $\lambda\colon H \to k$, if and only if H is right co-Frobenius [19, Thm. 3]. In this case the antipode of H is bijective. [27, Prop. 2].

(4) Let us say that a Hopf algebra H which is a projective k-module **has enough right integrals** if the evaluation map $H^* \otimes H \to k$ induces a surjection $H \otimes I_r(H) \to k$, where $I_r(H)$ denotes the space of left integrals on H. Thus, H has enough right integrals if and only if there are right integrals $\lambda_1, \dots, \lambda_k \in H^*$ and elements $t_1, \dots, t_k \in H$ with $\sum \lambda_i(t_i) = 1$. For example, H has enough right integrals if there is a surjective right integral $\lambda\colon H \to k$.

Now H is projective as a right H^*-module if and only if H has enough right integrals: One checks that if λ_i, t_i are as above, then

$$\varphi\colon H \ni h \mapsto \sum h t_{i(2)} \otimes \lambda_i \leftharpoonup S(h_{(1)}) \in H \otimes H^*$$

is an H^*-linear splitting of the H^*-module structure of H, and conversely, if $\varphi\colon H \to H \otimes H^*$ splits the module structure, then $\varphi(1) \in H \otimes I_r(H)$ is mapped to $1 \in k$ under evaluation.

Given the well-known properties of finite Hopf algebras, and the properties of co-Frobenius Hopf algebras discussed above, Theorem 3.1 contains both the Kreimer-Takeuchi theorem and its generalization in [1] as special cases, when we apply it to the entwining coming from an H-comodule algebra A. We will be interested in the more general situation where A is an H-comodule algebra that we view as a Q-extension for a quotient coalgebra and right H-module Q of H:

Corollary 3.3. *Let H be a Hopf algebra with bijective antipode, A an H-comodule algebra, and Q a quotient coalgebra and right H-module of H.*

Assume that the Galois map $\beta_0\colon A \otimes A \to A \otimes Q$ is surjective. Then it follows that A is a Q-Galois extension of $B := A^{\mathrm{co}\,Q}$ and a projective left B-module in each of the following cases:

(1) *k is a field, and H is finite-dimensional.*

(2) *H is finitely generated projective over k, coflat as a right Q-comodule, and the surjection $H \to Q$ splits as a left Q-comodule map.*

(3) *H has enough right integrals, is coflat as a right Q-comodule, and the surjection $H \to Q$ splits as a left Q-comodule map.*

(4) *k is a field, H is co-Frobenius, and faithfully coflat both as a left and a right Q-comodule.*

(5) *H is Q-cleft and Q is finitely generated projective.*

(6) *k is a field, H has cocommutative coradical, and Q is finite dimensional and of the form $Q = H/K^+H$ for a Hopf subalgebra $K \subset H$.*

PROOF. We will verify in each case that Q is a projective left Q-comodule (or a projective right Q^*-module, which is equivalent if Q is finitely generated projective). Then the parenthesized version of Theorem 3.1 can be applied to the entwining of A and Q to prove the claim.

As for (1), Skryabin [34] has proved that Q^* is Frobenius.

Any of (2), (3), and (4) imply that H is projective as left H-comodule: In the case that H is finitely generated projective, this follows from the structure theorem for Hopf modules over a Hopf algebra, since H can be considered as a Hopf module in $\mathcal{M}_{H^*}^{H^*}$ as in [18]. If H has enough right integrals, then H is projective as right H^*-module as we discussed above. (4) is a special case of (3): If k is a field, and H is left faithfully coflat over Q, then the surjection $H \to Q$ splits as a left Q-comodule map by [32, 1.1,1.3].

Now to prove the desired results on Q under hypotheses (2), (3), or (4), we may assume more generally that H is a coalgebra that is projective as a left H-comodule, and Q is a quotient coalgebra of H so that H is a coflat right Q-comodule, and the surjection $H \to Q$ splits as a left Q-comodule map. We have an isomorphism

$$^Q\mathcal{M}(H,V) \cong {}^H\mathcal{M}(H, H \underset{Q}{\square} V)$$

for any left Q-comodule V. Thus H projective in $^H\mathcal{M}$ and H coflat as right Q-comodule implies that the functor $^Q\mathcal{M}(H,-)$ is exact, thus H is projective as left Q-comodule. Since Q is a direct summand, it also is projective as left Q-comodule.

We note that under hypothesis (2), with k a field, Q^* was proved to be self-injective by Hoffmann, Koppinen and Masuoka [24, Thm.4.2].

Under the hypotheses in (5) Fischman, Montgomery, and Schneider [11, Thm.4.8]. show that Q^* is Frobenius if k is a field. Part of their technique still applies in the general case: We can consider Q^* as a Hopf module in $\mathcal{M}_H^Q$ with the right Q-comodule structure dual to the regular left comodule structure of Q, and the right H-module structure defined by $(\theta h)(q) = \theta(qS(h))$ for $\theta \in Q^*$, $h \in H$, and $q \in Q$. Then $Q^* \in \mathcal{M}_H^Q$ by the calculation

$$(\theta_{(0)}h_{(1)})(q)\theta_{(1)}h_{(2)} = \theta_{(0)}(qS(h_{(1)}))\theta_{(1)}h_{(2)} = q_{(1)}S(h_{(2)})\theta(q_{(2)}S(h_{(1)}))h_{(3)}$$
$$= q_{(1)}\theta(q_{(2)}S(h)) = q_{(1)}(\theta h)(q_{(2)}) = (\theta h)_{(0)}(q)(\theta h)_{(1)}$$

By the results in [25] that we will review in Lemma 5.2 it follows that Q^* is injective as right Q-comodule, and hence Q is projective as right Q^*-module.

Under the hypotheses in (6) the algebra Q^* is again Frobenius, by results of Fischman, Montgomery, and Schneider [11, Cor.4.9]. $\square$

4. Injectivity conditions

From now on many of our results on general entwinings (A, C, ψ) will depend on having a distinguished grouplike $e \in C$ and considering $A \in \mathcal{M}_A^C$ as in Corollary 1.10. We will always assume this structure is taken, and allude to the situation by simply saying that there is a grouplike $e \in C$.

The following result and the remark following it characterize, in particular, those C-Galois extensions, with distinguished grouplike $e \in C$, that are relative injective comodules. For Doi-Koppinen data the results are due to Doi [10, Prop.3.2, Prop.3.3], generalizing his result [9, 1.6] for comodule algebras. The proofs for general entwinings are not essentially more difficult.

Lemma 4.1. *Let (A, C, ψ) be a bijective entwining structure, and $e \in C$ grouplike. The following are equivalent:*

(1) *A is a relative injective C-comodule.*
(2) *There is a C-colinear map $\gamma \colon C \to A$ with $\gamma(e) = 1$.*
(3) *There is a map $\varphi \colon A \otimes C \to A$ in $\mathcal{M}_A^C$ with $\varphi\delta_A = \mathrm{id}_A$.*

If these conditions are satisfied, then $B := A^{\mathrm{co}\,C}$ is a direct summand of A as a right
B-module, the unit $N \to (N \otimes_B A)^{\mathrm{co}\,C}$ of the adjunction in Lemma 1.7 is a bijection for
every right B-module N, and in particular $(V \otimes A)^{\mathrm{co}\,C} \cong V \otimes B$ for every k-module V.

PROOF. Clearly (3) implies (1). Assuming (1), there is at least a C-colinear map
$\varphi \colon A \otimes C \to A$ with $\varphi\delta = \mathrm{id}_A$. Put $\gamma(c) = \varphi(1 \otimes c)$ to prove (2). Assuming (2), put
$\varphi := \Phi(\gamma)$ as in Lemma 2.1. We have $\varphi\delta(a) = \varphi\psi(e \otimes a) = \nabla(\gamma \otimes A)(e \otimes a) = a$, proving
(3).

For a map φ as in (3), we have in particular $\varphi(b \otimes e) = \varphi\delta(b) = b$ for $b \in B$, hence the
coinvariant part $\varphi^{\mathrm{co}\,C} \colon A \to B$ splits the inclusion $B \to A$.

Also, if $\varphi \colon A \otimes C \to A$ is a colinear map that splits the comodule structure of A,
then the pair of homomorphisms $\delta, A \otimes e \colon A \to A \otimes C$ is contractible in the sense dual
to [21, VI.6], since $\varphi\delta = \mathrm{id}_A$ and $\delta\varphi(A \otimes e) = (A \otimes e)\varphi(A \otimes e)$ by the calculation
$\delta\varphi(a \otimes e) = \varphi(a \otimes e_{(1)}) \otimes e_{(2)} = \varphi(a \otimes e) \otimes e = (A \otimes e)\varphi(A \otimes e)(a)$. But the equalizer
of a contractible pair is preserved under any functor, in particular under tensor product
with a right B-module N. □

Remark 4.2. If A is a C-Galois extension of $B := A^{\mathrm{co}\,C}$, and B is a direct summand of
A as right B-module, then A is relative injective as a C-comodule.

PROOF. If B is a direct summand of A as right B-module, then $A \cong B \otimes_B A$ is a
direct summand of $A \otimes_B A$ as a C-comodule. Since $A \otimes_B A \cong A \otimes C$ is relative injective,
so is A. □

Remark 4.3. It is clear how to specialize the results to the important case where A is an
H-comodule algebra, and $C = Q$ is a quotient coalgebra and right H-module (here, the
antipode of H should be bijective to have a bijective entwining). We can also consider a
quotient coalgebra and left module of H as in Corollary 2.5. Here, the relevant entwining
of $C = (Q')^{\mathrm{cop}}$ and A is bijective if H is a Hopf algebra. As a result, if A is injective
as Q'-comodule, then there is a map $\varphi \colon A \otimes Q' \to A$ in $_A\mathcal{M}^{Q'}$ splitting the comodule
structure, and in particular the left submodule $B \subset A$ is a direct summand. Conversely,
if A is Q'-Galois and the left submodule $B \subset A$ is a direct summand, then A is injective
as Q'-comodule.

Remark 4.4. Consider a projective right B-module A. If A contains B as a direct
summand, then in particular A is a generator. If A is a generator in $\mathcal{M}_B$, then A_B
is faithfully flat. Now if $B \subset A$ is a ring extension, then conversely, A_B projective
and faithfully flat implies that the right B-submodule $B \subset A$ is a direct summand [28,
2.11.29]. The results above and in Section 2 characterize C-Galois extensions $B \subset A$
such that A_B is a projective generator (or has the equivalent properties we have just
discussed): Let (A, C, ψ) be a bijective entwining structure, and $e \in C$ grouplike. Put
$B = A^{\mathrm{co}\,C}$. Assume that A is projective as k-module. Then A is C-Galois and a right
projective generator over B if and only if the canonical map $\beta_0 \colon A \otimes A \to A \otimes C$ is
surjective and splits as a C-comodule map, and A is an injective C-comodule. On the
other hand, such extensions can also be characterized as those C-Galois extensions that
are projective and faithfully flat as right B-modules. Faithfully flat C-Galois extensions
in turn can be characterized by the structure theorem for Hopf modules Lemma 1.7.
Thus A is a C-Galois extension and a projective generator as right B-module if and only
if the induction functor $_B\mathcal{M} \to {}^C_A\mathcal{M}$ is an equivalence, and in addition A is a projective
right B-module.

If k is a field, more can be said without assuming that A_B is projective:

Proposition 4.5. *Let (A, C, ψ) be a bijective entwining structure over a field k, and $e \in C$ grouplike such that A is a C-Galois extension of $B := A^{\mathrm{co}\,C}$ and a flat right B-module. The following are equivalent:*

(1) *A is a faithfully flat right B-module.*
(2) *The right B-submodule $B \subset A$ is a direct summand.*
(3) *A is injective as C-comodule.*
(4) *A is coflat as C-comodule.*
(5) *A is faithfully coflat as C-comodule.*

PROOF. Clearly (1) follows from (2).

(1) $\Longrightarrow$ (5): Consider a left C-comodule V. We have a chain of isomorphisms

$$A \underset{B}{\otimes} (A \underset{C}{\,\square\,} V) \cong (A \underset{B}{\otimes} A) \underset{C}{\,\square\,} V \overset{\beta\,\square\,V}{\cong} (A \otimes C) \underset{C}{\,\square\,} V \cong A \otimes V,$$

the first one using that A is B-flat. By faithful flatness of A_B it follows that A is faithfully coflat, since A is faithfully flat over k.

(5) $\Longrightarrow$ (4) trivially, and (4) $\Longrightarrow$ (3) by a result of Takeuchi [36, A.2.1].

Finally (3) $\Longrightarrow$ (2) by Lemma 4.1. $\square$

Corollary 4.6. *Let (A, C, ψ) be an entwining structure, and $e \in C$ grouplike such that A is a C-Galois extension of $B := A^{\mathrm{co}\,C}$. Assume that A is a relative projective right B-module, and the right B-submodule $B \subset A$ is a direct summand.*

Let $\pi\colon C \to D$ be a surjective coalgebra map with $\psi(\operatorname{Ker}\pi \otimes A) \subset \operatorname{Im}(A \otimes \operatorname{Ker}\pi)$. Assume that the induced map $\theta\colon D \otimes A \to A \otimes D$ is bijective.

If π splits as a right D-comodule map, and C is relative injective as right D-comodule, then

A is a D-Galois extension of $S := A^{\mathrm{co}\,D}$, a relative projective right S-module, and the right S-submodule $S \subset A$ is a direct summand.

PROOF.   We already know from Corollary 2.6 that A is a D-Galois extension of S and a relative projective right S-module. In addition, since $B \subset A$ is a right module direct summand, we know from Remark 4.2 that A is a relative injective C-comodule, hence a direct summand of the C-comodule $A \otimes C$. Since C is relative injective as right D-comodule, $A \otimes C$ and hence A is a relative injective D-comodule. From Lemma 4.1 we see that $S \subset A$ is a direct summand as right S-module. $\square$

Remark 4.7.   (1) Assume that k is a field, and $\pi\colon C \to D$ is a coalgebra surjection such that C is faithfully coflat as a right D-comodule. Then by [36, A.2.1] C is injective as right D-comodule, and by [32, 1.1,1.3] the right D-comodule map π splits.

(2) The most important application of the preceding Theorem occurs when A is an H-Galois extension for a Hopf algebra $C = H$ with bijective antipode, and $D = Q$ is a right H-module coalgebra quotient of H.

Next, we will specialize our results to the case of H-comodule algebras over a k-projective Hopf algebra H with bijective antipode. Here, the condition that the Galois map β_0 is split already follows from the condition that A is a relative injective comodule. The reason is that in this case all Hopf modules are relative injective comodules. This result is due to Doi [9]. We formulate the next corollary for general C-extensions with the property that all Hopf modules are injective comodules. It shows that this is a powerful

condition. On the other hand, it is quite unclear when it is fulfilled, although we will encounter such situations in later sections.

Corollary 4.8. *Let (A, C, ψ) be a bijective entwining, and $e \in C$ grouplike. Assume that every entwined module in $\mathcal{M}_A^C$ is relative injective as a C-comodule, and C is a projective k-module.*

If $\beta_0 \colon A \otimes A \to A \otimes C$ is surjective, then A is a C-Galois extension of $B := A^{\mathrm{co}\, C}$ and a relative projective right B-module, and the right B-submodule $B \subset A$ is a direct summand.

PROOF. The canonical map β_0 is a morphism of entwined modules, and a left A-module map. Since C is projective over k, the left A-module $A \otimes C$ is projective, and β_0 splits as an A-module map. Since the kernel of β_0 is an entwined module and a k-direct summand, β_0 splits as a C-comodule map by assumption. The assertions now follow from Theorem 2.2 and Lemma 4.1. □

Except for projectivity of A as a B-module, the following two results are in [32, Thm.3.5,Thm.I], with a different proof.

Theorem 4.9. *Let H be a Hopf algebra with bijective antipode, and A an H-comodule algebra. Put $B := A^{\mathrm{co}\, H}$. Assume that H is a projective k-module. The following are equivalent:*

(1) *The canonical map $\beta \colon A \otimes_B A \to A \otimes H$ is surjective, and A is a relative injective H-comodule.*
(2) *A is an H-Galois extension of B, and the right B-submodule $B \subset A$ is a direct summand.*
(3) *A is an H-Galois extension of B, and the left B-submodule $B \subset A$ is a direct summand.*

In this case A is relative projective as left and right B-module.

PROOF. (1) implies (2) and projectivity of A_B by Corollary 4.8, since by a result of Doi [9, 1.6] every Hopf module is a relative injective comodule.

(2) $\Longrightarrow$ (1) by Remark 4.2.

The equivalence of (1) and (3) follows by applying the one of (1) and (2) to the H^{op}-comodule algebra A^{op}. □

Theorem 4.10. *Let H be a Hopf algebra with bijective antipode over a field k. Let A be an H-comodule algebra, and put $B = A^{\mathrm{co}\, H}$. The following are equivalent:*

(1) *The canonical map $\beta \colon A \otimes_B A \to A \otimes H$ is surjective, and A is injective as an H-comodule.*
(2) *A is an H-Galois extension of B and a faithfully flat left B-module.*
(3) *A is an H-Galois extension of B and a faithfully flat right B-module.*
(4) *The induction functor $\mathcal{M}_B \to \mathcal{M}_A^H$ is an equivalence.*

In this case A is projective as a left and right B-module, and B is a direct summand of A as both left and right B-module.

PROOF. This is a combination of the preceding Theorem with Proposition 4.5 and Lemma 1.7. □

5. Equivariant projectivity and injectivity

Definition 5.1. Let R be an algebra, C a coalgebra, and V an (R,C)-bimodule, that is, a left R-module and right C-comodule such that $(rv)_{(0)} \otimes (rv)_{(1)} = rv_{(0)} \otimes v_{(1)}$ for all $r \in R$ and $v \in V$.

(1) V is called C**-equivariantly** R**-projective** (or just **equivariantly projective** if no confusion is likely) if there is a C-colinear splitting of the module structure map $R \otimes V \to V$.

(2) V is called R**-equivariantly** C**-injective** (or just **equivariantly injective**) if there is an R-linear splitting of the comodule structure map $V \to V \otimes C$.

As an immediate consequence of the definition, an equivariantly projective (R,C)-bimodule is a relative projective R-module. In addition to the requirement that $R \otimes V \to V$ splits as an R-module map, the definition of an equivariantly projective bimodule requires that such a splitting can be chosen to be equivariant with respect to the coaction of C. Dually, an equivariantly injective bimodule is a relative injective C-comodule.

We will show that in many interesting cases a Q-Galois extension A of B is equivariantly projective (that is, Q-equivariantly R-projective).

The property was studied first for Hopf Galois extensions in [7], see also [13]. It was shown there that equivariant projectivity of an H-Galois extension is equivalent to the existence of a so-called strong connection. Connections and the strong connections introduced in [12] are algebraic analogs of differential-geometric notions.

A very special class of extensions that have all the desirable properties we have discussed so far is the class of cleft extensions. The following Lemma collects properties proved by Masuoka and Doi [25] for the case where $C = Q$ is a quotient coalgebra and right module of a bialgebra H, and A is an H-comodule algebra; we use techniques from [29] in the proof. The generalization to entwinings instead of comodule algebras does not present additional problems.

Lemma 5.2. *Let (A, C, ψ) be an entwining structure, and $e \in C$ a grouplike element; put $B = A^{\mathrm{co}\,C}$. Assume that A **is** C**-cleft**, that is, there is a C-colinear convolution invertible map $j \colon C \to A$, and C is a flat k-module.*

Then A is C-Galois, equivariantly projective, and equivariantly injective. The induction functor $\mathcal{M}_B \to \mathcal{M}_A^C$ is an equivalence, and every entwined module is injective as C-comodule.

PROOF. We can assume $j(e) = 1$ without loss of generality; otherwise replace j with $\tilde{\jmath}$ defined by $\tilde{\jmath}(c) = j^{-1}(e)j(c)$.

Let $M \in \mathcal{M}_A^C$. Define $\pi_0 \colon M \to M$ by $\pi_0(m) = m_{(0)}j^{-1}(m_{(1)})$. We will first show that $\pi_0(M) \subset M^{\mathrm{co}\,C}$. To verify

$$\delta_M(\pi_0(m_{(0)})) \otimes m_{(1)} = \pi_0(m_{(0)}) \otimes e \otimes m_{(1)}$$

for $m \in M$ (from which the assertion follows by applying ε to the last tensor factor), we apply the bijective map

$$T \colon M \otimes C \otimes C \ni m \otimes c \otimes d \mapsto (m \otimes c)j(d_{(1)}) \otimes j(d_{(2)}) \in M \otimes C \otimes C$$

to both sides:

$$T(\delta_M(\pi_0(m_{(0)})) \otimes m_{(1)}) = \delta_M(m_{(0)}j^{-1}(m_{(1)}))j(m_{(2)}) \otimes m_{(3)}$$
$$= \delta_M(m_{(0)}j^{-1}(m_{(1)})j(m_{(2)})) \otimes m_{(3)} = \delta_M(m_{(0)}) \otimes m_{(1)}$$

, and, using $(m \otimes e)a = ma_{(0)} \otimes a_{(1)}$ for $m \in M$ and $a \in A$,

$$\begin{aligned}
T(\pi_0(m_{(0)}) \otimes e \otimes m_{(1)}) &= (\pi_0(m_{(0)}) \otimes e)j(m_{(1)}) \otimes m_{(2)} \\
&= \pi_0(m_{(0)})j(m_{(1)})_{(0)} \otimes j(m_{(1)})_{(1)} \otimes m_{(2)} \\
&= \pi_0(m_{(0)})j(m_{(1)}) \otimes m_{(2)} \otimes m_{(3)} = m_{(0)} \otimes m_{(1)} \otimes m_{(3)}
\end{aligned}$$

Now we can define $\pi \colon M \to M^{\mathrm{co}\,C}$ by $\pi(m) = \pi_0(m)$ for all $m \in M$. It is straightforward to check that $M^{\mathrm{co}\,C} \otimes C \ni m \otimes c \mapsto mj(c)$ is C-colinear and bijective with inverse $m \mapsto \pi_0(m_{(0)}) \otimes m_{(1)}$. In particular every entwined module in $\mathcal{M}_A^C$ is a relative injective C-comodule. The Galois map $\beta \colon A \otimes_B A \to A \otimes C$ is bijective with inverse $\beta^{-1}(a \otimes c) = aj^{-1}(c_{(1)}) \otimes j(c_{(2)})$. The B-linear and C-colinear map $A \ni a \mapsto \pi(a_{(0)}) \otimes j(a_{(1)}) \in B \otimes A$ splits the left B-module structure of A, and the B-linear and C-colinear map $A \otimes C \ni a \otimes c \mapsto \pi(a)j(c)$ splits the C-comodule structure of A. $\square$

We note for later use that one property noted for cleft extensions in Lemma 5.2 holds more generally for equivariantly injective extensions:

Remark 5.3. Let A be a C-Galois extension of $B = A^{\mathrm{co}\,C}$. If A is equivariantly injective and faithfully flat as a left B-module, then every entwined module in $\mathcal{M}_A^C$ is a relative injective C-comodule.

PROOF. Let $M \in \mathcal{M}_A^C$. By assumption and Lemma 1.7 we have $M \cong M^{\mathrm{co}\,C} \otimes_B A$, and if $\varphi \colon A \otimes C \to A$ is a left B-linear and right C-colinear splitting of the comodule structure of A, then $M^{\mathrm{co}\,C} \otimes_B \varphi$ splits the comodule structure of $M^{\mathrm{co}\,C} \otimes_B A$. $\square$

We should stress that the condition that A be C-cleft is much more restrictive than the equivariant injectivity and projectivity conditions. However, the Lemma will also have applications to extensions A that are not cleft.

More precisely, let H be a Hopf algebra, and Q a quotient coalgebra and right module of H. It will turn out to be useful for the study of a Q-extension (which need not be cleft) to know that H is equivariantly Q-injective. If H is finite-dimensional over a field, a recent important result of Skryabin [34] shows that H is even Q-cleft for any quotient coalgebra and right module Q. This had long been an open question; many equivalent characterizations of cleftness in this situation had been given by Masuoka [22] and Masuoka and Doi [25], and the property had been proved in interesting special cases by Masuoka [23].

The following general Lemma links equivariant injectivity and projectivity in a general bimodule.

Lemma 5.4. *Let R be an algebra, C a coalgebra, and V an (R, C)-bimodule.*

(1) *If V is a relative projective R-module and an R-equivariantly injective C-comodule, then V is a C-equivariantly projective R-module.*
(2) *If V is a relative injective C-comodule and a C-equivariantly projective R-module, then V is an R-equivariantly injective C-comodule.*

PROOF. We only show (1), the proof of (2) is dual. Since V is relative R-projective, there is a left R-linear map $s_0 : V \to R \otimes V$ with $\mu_V s_0 = id_V$. Define s as the composition

$$V \xrightarrow{\delta_V} V \otimes C \xrightarrow{s_0 \otimes id_C} R \otimes V \otimes C \xrightarrow{id_R \otimes \varphi} R \otimes V.$$

Then s is left R-linear, and right C-colinear. Since φ is left R-linear, $\mu_V(id_R \otimes \varphi) = \varphi(\mu_V \otimes id_C)$. Hence

$$\mu_V s = \mu_V(id_R \otimes \varphi)(s_0 \otimes id_C)\delta_V = \varphi(\mu_V \otimes id_C)(s \otimes id_C)\delta_V = \varphi\delta_V = id_V.$$

□

Remark 5.5. We will always apply Lemma 5.4 to the following situation: A is an algebra and a C-comodule, and $B \subset A$ is a subalgebra such that A is a (B, C)-bimodule (mostly even a C-Galois extension of B); there is a grouplike $e \in C$ with $\delta(1) = 1 \otimes e$. From the proof of Lemma 5.4 we see that if A is equivariantly injective, and there is a B-linear splitting $s_0 \colon A \to B \otimes A$ of the left B-module structure of A that satisfies $s_0(1) = 1 \otimes 1$, then there is a (B, C)-bimodule splitting $s \colon A \to B \otimes A$ that satisfies $s(1) = 1 \otimes 1$. If $_B A$ is relative projective, and $_B B \subset {_B A}$ is a direct summand, then s_0 can in fact be chosen in this way.

As a consequence of Lemma 5.4 and our previous results, equivariant projectivity always holds for H-Galois extensions that are relative injective comodules, if H has bijective antipode. If k is a field, this means in particular that a faithfully flat H-Galois extensions for a Hopf algebra H with bijective antipode always admits a strong connection in the sense of [7].

Theorem 5.6. *Let H be a k-projective Hopf algebra, A a right H-comodule algebra and $B = A^{\mathrm{co}\, H}$. Assume that A is relative injective as a right H-comodule.*

Then A is equivariantly injective. In particular, A is equivariantly projective if and only if it is relative projective as a left B-module.

If $B \subset A$ is an H-Galois extension and the antipode of H is bijective, then $B \subset A$ is equivariantly projective.

PROOF.   Applying Lemma 4.1 to the bijective entwining of H^{op} and A^{op} coming from the H^{op}-comodule algebra A^{op} yields a right H-colinear and left B-linear map $\varphi \colon A \otimes H \to A$, so that A is equivariantly injective. If $_B A$ is relative projective, then A is equivariantly projective by Lemma 5.4 (1). If A is an H-Galois extension, and H has bijective antipode, then A is relative projective as left B-module by Theorem 4.9. □

For applications to quantum groups, the case of C-extensions where k is an algebraically closed field, and C is a cosemisimple coalgebra, is particularly important.

Remark 5.7.   (1) A coalgebra C with comultiplication $\Delta : C \to C \otimes C$ is called **coseparable** if there is a left and right C-colinear map $\varphi : C \otimes C \to C$ with $\varphi\Delta = id_C$.

(2) We will call a coalgebra C **right cosemisimple** if it is k-flat, and fulfills the following equivalent conditions:
  (a) Every right C-comodule is relative injective.
  (b) Every right C-comodule is relative projective.
  (c) If M is a right C-comodule, and $N \subset M$ a subcomodule that is a direct summand as a k-module, then N is a direct summand as a C-comodule.
  If k is a field, this coincides with the usual definition [35, Chap.XIV]; note that a coalgebra over a field is right cosemisimple if and only if it is left cosemisimple. If k is arbitrary and C is finitely generated projective, then C is right cosemisimple if and only if C^* is a left semisimple algebra over k in the sense of Hattori [14]

(3) A k-flat coseparable coalgebra is right and left cosemisimple; see 5.8.

(4) Let C be a coalgebra over a field k. Then the following are equivalent:

(a) C is coseparable.

(b) C is cosemisimple, and for any simple (hence finite-dimensional) subcoalgebra $D \subset C$, the dual algebra D^* is separable.

(5) Any cosemisimple coalgebra over an algebraically closed field is coseparable.

According to part (3) of the preceding remark, every comodule over a coseparable coalgebra is relative injective. This is generalized and strengthened by the following observation:

Proposition 5.8. *Let C be a coseparable coalgebra. Then any (R, C)-bimodule is equivariantly injective.*

In particular, any (R, C)-bimodule that is a relative projective R-module, is equivariantly projective.

PROOF. Let $\varphi \colon C \otimes C \to C$ be a left and right colinear map satisfying $\varphi\Delta = \mathrm{id}_C$. Then, for any C-comodule V,

$$\varphi_V : V \otimes C \cong V \square_C C \otimes C \xrightarrow{id_V \otimes \varphi} V \square_C C \cong V$$

is a C-colinear retraction of the comodule structure of V. If V is an (R, C)-bimodule, then φ_V is an R-module map. $\square$

Theorem 5.9. *Let (A, C, ψ) be a bijective entwining structure, with C projective as k-module, and $e \in C$ a grouplike element. Put $B := A^{\mathrm{co}\, C}$.*

Assume that the Galois map $\beta \colon A \otimes_B A \to A \otimes C$ is surjective.

If C is right cosemisimple, then A is a C-Galois extension of B, projective as right B-module, and the right B-submodule B is a direct summand.

If C is coseparable (for example, k is an algebraically closed field and C is cosemisimple), then A is also projective as left B-module, B is a direct summand as left B-module, and A is equivariantly projective.

PROOF. Since every right C-comodule is relative injective, Corollary 4.8 implies that A is a C-Galois extension of B, and a projective right B-module; Lemma 4.1 implies that B is a direct summand as right B-module.

If C is coseparable, then so is C^{cop}, and by left-right symmetry, A has the same properties as a left B-module.

Also, it follows that A is equivariantly projective by Proposition 5.8. $\square$

6. REDUCTION TO HOMOGENEOUS SPACES

Consider a Hopf algebra H, an H-comodule algebra A, and a quotient coalgebra and right module Q of H. Put $B = A^{\mathrm{co}\, Q}$ and $K = H^{\mathrm{co}\, Q}$. In this section we will collect some results that allow us to draw conclusions on the structure of the Q-extension $B \subset A$ from assumptions on the structure of the Q-extension $K \subset H$. This shows that among the Galois type extensions, which are quantum group analogs of principal fiber bundles, the analogs of homogeneous spaces play a distinguished role. We have already mentioned above the recent result of Skryabin proving that H is Q-cleft whenever H is finite-dimensional over a field. In particular, the hypotheses on H as a Q-extension of K in Theorem 6.1, Proposition 6.2, and Theorem 6.3 below are satisfied if H is finite-dimensional, since these hypotheses are satisfied in the cleft case by Lemma 5.2.

First, we will study the question when A is equivariantly projective. So far, we have settled this for H-Galois extensions, and for the case where Q is coseparable. If k is a field, H has bijective antipode, A is H-Galois and an injective H-comodule, and H is left

and right faithfully coflat over Q, we know from Corollary 4.6 and its left-right switched version that A is left and right projective over the algebra B of Q-coinvariant elements of A. But we do not know whether $B \subset A$ is equivariantly projective. As a particular case, the following result will show that $B \subset A$ is equivariantly projective if we assume that H is K-equivariantly Q-injective. At the same time we should stress that we do not know when H has these properties.

Theorem 6.1. *Let H be a Hopf algebra, Q a quotient coalgebra and right H-module of H, and A an H-comodule algebra. Put $K = H^{\mathrm{co}\,Q}$, and $B := A^{\mathrm{co}\,Q}$.*

Assume that the (K, Q)-bimodule H is equivariantly injective, and that A is relative injective as an H-comodule.

Then the (B, Q)-bimodule A is equivariantly injective. (In particular, if A is a projective left B-module, then it is equivariantly projective.)

PROOF. By Lemma 4.1 there is a left A-linear (in particular B-linear) and right H-colinear map $\varphi_A \colon A \otimes H \to A$ with $\varphi_A \delta_A = \mathrm{id}_A$, where δ_A is the H-comodule structure of A. Let $\varphi_H \colon H \otimes Q \to H$ be a left K-linear and right Q-colinear map with $\varphi_H \delta_H = \mathrm{id}_H$. Now define

$$\tilde{\varphi} := \left(A \otimes Q \xrightarrow{\delta_A \otimes Q} A \otimes H \otimes Q \xrightarrow{A \otimes \varphi_H} A \otimes H \xrightarrow{\varphi_A} A \right).$$

Then $\tilde{\varphi}$ is Q-colinear. Since $\delta_A(B) \subset B \otimes K$, $\tilde{\varphi}$ is also left B-linear. Finally $\tilde{\varphi}\overline{\delta}_A = \mathrm{id}_A$ for $\overline{\delta}_A$ the Q-comodule structure of A, since $\varphi(a_{(0)} \otimes \overline{a_{(1)}}) = \varphi_A(a_{(0)} \otimes \varphi_H(a_{(1)} \otimes \overline{a_{(2)}})) = \varphi_A(a_{(0)} \otimes a_{(1)}) = a$ for all $a \in A$. □

Next, we will return to the criterion Corollary 4.8, which says that a surjective Galois map splits, provided every Hopf module is a relative injective comodule. Again, we do not know in general when this property holds, but we will see that assuming it for H instead of A will help.

Proposition 6.2. *Let H be a Hopf algebra, A a right H-comodule algebra that is a relative injective H-comodule, and Q a quotient coalgebra and right H-module of H.*

If every Hopf module in $\mathcal{M}_H^Q$ is a relative injective Q-comodule, then every Hopf module in $\mathcal{M}_A^Q$ is a relative injective Q-comodule.

PROOF. Let $M \in \mathcal{M}_A^Q$. The Q-colinear multiplication map $\mu \colon M \otimes A \to M$ splits as a Q-comodule map by $M \ni m \mapsto m \otimes 1 \in M \otimes A$. By assumption the comodule structure $\delta \colon A \to A \otimes H$ splits as an H-comodule map. Hence, the Q-comodule M is a direct summand of $M \otimes A \otimes H$, and it is sufficient to check that the diagonal comodule $V \otimes H$ is a relative injective Q-comodule for every Q-comodule V. But $V \otimes H \in \mathcal{M}_H^Q$ with the H-module structure defined on the right tensor factor. □

Note that by Remark 5.3, the property required of H in Proposition 6.2 holds in particular if H is a faithfully flat Q-Galois extension and equivariantly projective. This is true in particular (or directly by Lemma 5.2) if H is Q-cleft. In particular, the following result gives strong conclusions on A (which need not be cleft) if H is Q-cleft. This result is stronger than Corollary 4.6 in that it does not assume that A is H-Galois.

Theorem 6.3. *Let H be a k-flat Hopf algebra with bijective antipode and Q a quotient coalgebra and right H-module of H such that H is a Q-Galois extension of $K := H^{\mathrm{co}\,Q}$, and a faithfully flat left K-module.*

Assume that H is K-equivariantly Q-injective.

Then every Hopf module in $\mathcal{M}_H^Q$ is a relative injective Q-comodule.

In particular, if A is a right H-comodule algebra which is a relative injective H-comodule, and the Galois map $A \otimes A \to A \otimes Q$ is onto, then A is a Q-Galois extension of $B := A^{\mathrm{co}\,Q}$, a projective right B-module, and $B \subset A$ is a right B-direct summand.

PROOF. Every Hopf module in $\mathcal{M}_H^Q$ is a relative injective comodule by Remark 5.3, by Proposition 6.2 it follows that every Hopf module in $\mathcal{M}_A^Q$ is a relative injective comodule, and the remaining assertions follow from Corollary 4.8. □

Remark 6.4. The hypothesis in Theorem 6.3 that H be a faithfully flat Q-Galois extension of K is fulfilled if we assume (in addition to H being relative injective as a right Q-comodule) that H is left faithfully coflat for $\mathcal{M}_H^Q$, that is, cotensor product with H over Q preserves and reflects exact sequences in the category $\mathcal{M}_H^Q$.

PROOF. We vary arguments from [33, Sec.1]: We first observe that $\tilde{\beta} \colon K \otimes H \to H \,\square_Q\, H$ given by $\tilde{\beta}(x \otimes h) = xh_{(1)} \otimes h_{(2)}$ is an isomorphism with inverse $\tilde{\beta}^{-1}(g \otimes h) = gS(h_{(1)}) \otimes h_{(2)}$. To show that $\mu \colon M^{\mathrm{co}\,Q} \otimes_K H \to M$ is an isomorphism, it is enough, by hypothesis, to show that $\mu \,\square_Q\, H$ is an isomorphism. But the composition

$$M^{\mathrm{co}\,Q} \otimes H \cong M^{\mathrm{co}\,Q} \underset{K}{\otimes} K \otimes H \xrightarrow{\mathrm{id} \otimes_K \tilde{\beta}} M^{\mathrm{co}\,Q} \underset{K}{\otimes} (H \underset{Q}{\square} H) \cong (M^{\mathrm{co}\,Q} \underset{K}{\otimes} H) \underset{Q}{\square} H \xrightarrow{\mu \square_Q H} M \underset{Q}{\square} H$$

is given by $m \otimes h \mapsto mh_{(1)} \otimes h_{(2)}$; it is a morphism of Hopf modules in $\mathcal{M}_H^H$, and thus it is sufficient to observe that its coinvariant part is the identity on $M^{\mathrm{co}\,Q}$. That the adjunction map $N \to (N \otimes_B A)^{\mathrm{co}\,C}$ is an isomorphism for every $N \in \mathcal{M}_B$ was already observed in Lemma 4.1. □

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