SMALL QUANTUM GROUPS AND THE CLASSIFICATION OF POINTED HOPF ALGEBRAS

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Introduction

In this paper we apply the theory of the quantum groups $U_q(\mathfrak{g})$, and of the small quantum groups $u_q(\mathfrak{g})$ for q a root of unity, $\mathfrak{g}$ a semisimple complex Lie algebra, to obtain a classification result for an abstractly defined class of Hopf algebras. Since these Hopf algebras turn out to be deformations of a natural class of generalized small quantum groups, our result can be read as an axiomatic description of generalized small quantum groups.

Let k be an algebraically closed ground-field of characteristic 0. A Hopf algebra A is called *pointed*, if any simple subcoalgebra of A, or equivalently, any simple A-comodule is one-dimensional. If A is cocommutative, or if A is generated as an algebra by group-like and skew-primitive elements, then A is pointed. In particular, the quantum groups $U_q(\mathfrak{g})$ and $u_q(\mathfrak{g})$ are pointed.

Let $G(A) = \{g \in A \mid \Delta(g) = g \otimes g, \varepsilon(g) = 1\}$ be the group of group-like elements of A. We want to classify finite-dimensional pointed Hopf algebras A with abelian group $G(A)$.

We first describe the data $\mathcal{D}, \lambda, \mu$ we need to define the Hopf algebras of the class we are considering. We fix a finite abelian group Γ.

The datum $\mathcal{D}$. A *datum $\mathcal{D}$ of finite Cartan type* for Γ ,

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta}),$$

consists of elements $g_i \in \Gamma, \chi_i \in \widehat{\Gamma}, 1 \le i \le \theta$, and a Cartan matrix $(a_{ij})_{1\le i,j\le\theta}$ of finite type satisfying

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}},\ q_{ii} \neq 1,\ \text{with } q_{ij} = \chi_j(g_i) \text{ for all } 1 \le i, j \le \theta. \tag{0.1}$$

Results of this paper were obtained during a visit of H.-J. S. at the University of Córdoba, partially supported through a grant of CONICET. The work of N. A. was partially supported by CONICET, Fundación Antorchas, Agencia Córdoba Ciencia, ANPCyT and Secyt (UNC).

The Cartan condition (0.1) implies in particular,

$$q_{ii}^{a_{ij}} = q_{jj}^{a_{ji}} \text{ for all } 1 \leq i, j \leq \theta. \tag{0.2}$$

The explicit classification of all data of finite Cartan type for a given finite abelian group Γ is a computational problem. But at least it is a finite problem since the size θ of the Cartan matrix is bounded by $2(\mathrm{ord}(\Gamma))^2$ by [AS2, 8.1], if Γ is an abelian group of odd order. For groups of prime order, all possibilities for $\mathcal{D}$ are listed in [AS2].

Let Φ be the root system of the Cartan matrix $(a_{ij})_{1\leq i,j\leq\theta}$, $\alpha_1, \dots, \alpha_\theta$ a system of simple roots, and $\mathcal{X}$ the set of connected components of the Dynkin diagram of Φ. Let Φ_J, $J \in \mathcal{X}$, be the root system of the component J. We write $i \sim j$, if α_i and α_j are in the same connected component of the Dynkin diagram of Φ. For a positive root $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i, n_i \in \mathbb{N} = \{0, 1, 2, \dots\}$, for all i, we define

$$g_\alpha = \prod_{i=1}^{\theta} g_i^{n_i}, \chi_\alpha = \prod_{i=1}^{\theta} \chi_i^{n_i}.$$

We assume that the order of q_{ii} is odd for all i, and that the order of q_{ii} is prime to 3 for all i in a connected component of type G_2. Then it follows from (0.2) that the order N_i of q_{ii} is constant in each connected component J, and we define $N_J = N_i$ for all $i \in J$.

The parameter λ. Let $\lambda = (\lambda_{ij})_{1\leq i<j\leq\theta,\, i\not\sim j}$ be a family of elements in k satisfying the following condition for all $1 \leq i < j \leq \theta$, $i \not\sim j$: If $g_ig_j = 1$ or $\chi_i\chi_j \neq \varepsilon$, then $\lambda_{ij} = 0$.

The parameter μ. Let $\mu = (\mu_\alpha)_{\alpha\in\Phi^+}$ be a family of elements in k such that for all $\alpha \in \Phi_J^+$, $J \in \mathcal{X}$, if $g_\alpha^{N_J} = 1$ or $\chi_\alpha^{N_J} \neq \varepsilon$, then $\mu_\alpha = 0$.

Thus λ and μ are finite families of free parameters in k. We can normalize λ and assume that $\lambda_{ij} = 1$, if $\lambda_{ij} \neq 0$.

The Hopf algebra $u(\mathcal{D}, \lambda, \mu)$. The definition of $u(\mathcal{D}, \lambda, \mu)$ in Section 4.2 can be summarized as follows. In Definition 2.13 we associate to any μ and $\alpha \in \Phi^+$ an element $u_\alpha(\mu)$ in the group algebra $k[\Gamma]$. By construction, $u_\alpha(\mu)$ lies in the augmentation ideal of $k[g_i^{N_i} \mid 1 \leq i \leq \theta]$. The braided adjoint action $\mathrm{ad}_c(x_i)$ of x_i is defined in (1.12), and the root vectors x_α are explained in Section 2.1.

The Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ is generated as an algebra by the group Γ, that is, by generators of Γ satisfying the relations of the group, and

$x_1, \dots, x_\theta$, with the relations:

$$
\begin{array}{ll}
(\textit{Action of the group}) & gx_ig^{-1} = \chi_i(g)x_i, \text{ for all } i, \text{ and all } g \in \Gamma,\\
(\textit{Serre relations}) & \mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0, \text{ for all } i \neq j, i \sim j,\\
(\textit{Linking relations}) & \mathrm{ad}_c(x_i)(x_j) = \lambda_{ij}(1 - g_ig_j), \text{ for all } i < j, i \nsim j,\\
(\textit{Root vector relations}) & x_\alpha^{N_J} = u_\alpha(\mu), \text{ for all } \alpha \in \Phi_J^+, J \in \mathcal{X}.
\end{array}
$$

The coalgebra structure is given by

$$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1, \quad \Delta(g) = g \otimes g, \text{ for all } 1 \leq i \leq \theta, g \in \Gamma.$$

Now we can formulate our main result.

Classification Theorem 0.1. *(1) Let $\mathcal{D}, \lambda$ and μ as above. Assume that q_{ij} has odd order for all i, j, and that the order of q_{ii} is prime to 3 for all i in a connected component of type G_2. Then $u(\mathcal{D}, \lambda, \mu)$ is a pointed Hopf algebra of dimension $\prod_{J \in \mathcal{X}} N_J^{|\Phi_J^+|}|\Gamma|$, and $G(u(\mathcal{D}, \lambda, \mu)) = \Gamma$.*

(2) Let A be a finite-dimensional pointed Hopf algebra with abelian group $\Gamma = G(A)$. Assume that all prime divisors of the order of Γ are > 7. Then $A \cong u(\mathcal{D}, \lambda, \mu)$ for some $\mathcal{D}, \lambda, \mu$.

Part (1) of Theorem 0.1 is shown in Theorem 4.4, and part (2) is a special case of Theorem 6.2.

In [AS4] we proved the Classification Theorem for groups of the form $(\mathbb{Z}/(p))^s, s \geq 1$, where p is a prime number > 17. In this special case, all the elements μ and $u_\alpha(\mu)$ are zero. In [AS1] we proved part (1) of Theorem 0.1 for Dynkin diagrams whose connected components are of type A_1, and in [AS5] for Dynkin diagrams of type A_n; in [D2] our construction was extended to Dynkin diagrams whose connected components are of type A_n for various n. In [BDR] the Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ was introduced for type B_2.

Our proof of Theorem 0.1 is based on [AS1, AS2, AS3, AS4, AS5], and on previous work on quantum groups in [dCK, dCP, L1, L2, L3, M1, Ro], in particular on Lusztig's theory of the small quantum groups. Another essential ingredient of our proof are the recent results of Heckenberger on Nichols algebras of diagonal type in [H1, H2, H3] which use Kharchenko's theory [K] of PBW-bases in braided Hopf algebras of diagonal type.

In [AS2, 1.4] we conjectured that any finite-dimensional pointed Hopf algebra (over an algebraically closed field of characteristic 0) is generated by group-like and skew-primitive elements. Our Classification Theorem and Theorem 6.2 confirm this conjecture for a large class of Hopf algebras.

Finally we note that the following analog of Cauchy's Theorem from group theory holds for the Hopf algebras $A = u(\mathcal{D}, \lambda, \mu)$: If p is a prime divisor of the dimension of A, then A contains a group-like element of order p. We conjecture that Cauchy's Theorem holds for all finite-dimensional pointed Hopf algebras.

1. Braided Hopf algebras

1.1. **Yetter-Drinfeld modules over abelian groups and the tensor algebra.** Let Γ be an abelian group, and $\widehat{\Gamma}$ the character group of all group homomorphisms from Γ to the multiplicative group $k^\times$ of the field k. The braided category ${}_\Gamma^\Gamma\mathcal{YD}$ of (left) Yetter-Drinfeld modules over Γ is the category of left $k[\Gamma]$-modules which are Γ-graded vector spaces $V = \bigoplus_{g\in\Gamma} V_g$ such that each homogeneous component V_g is stable under the action of Γ. Morphisms are Γ-linear maps $f : \bigoplus_{g\in\Gamma} V_g \to \bigoplus_{g\in\Gamma} W_g$ with $f(V_g) \subset W_g$ for all $g \in \Gamma$. The Γ-grading is equivalent to a left $k[\Gamma]$-comodule structure $\delta : V \to k[\Gamma] \otimes V$, where $\delta(v) = g \otimes v$ is equivalent to $v \in V_g$. We use a Sweedler notation $\delta(v) = v_{(-1)} \otimes v_{(0)}$ for all $v \in V$.

If $V = \bigoplus_{g\in\Gamma} V_g$ and $W = \bigoplus_{g\in\Gamma} W_g$ are in ${}_\Gamma^\Gamma\mathcal{YD}$, the monoidal structure is given by the usual tensor product $V \otimes W$ with Γ-action $g(v \otimes w) = gv \otimes gw,\ v \in V, w \in W$, and Γ-grading $(V \otimes W)_g = \bigoplus_{ab=g} V_a \otimes W_b$ for all $g \in \Gamma$. The braiding in ${}_\Gamma^\Gamma\mathcal{YD}$ is the isomorphism

$$c = c_{V,W} : V \otimes W \to W \otimes V$$

defined by $c(v \otimes w) = g \cdot w \otimes v$ for all $g \in \Gamma, v \in V_g,$ and $w \in W$. Thus each Yetter-Drinfeld module V defines a braided vector space $(V, c_{V,V})$.

If χ is a character of Γ and V a left Γ-module, we define

$$V^\chi := \{v \in V \mid g \cdot v = \chi(g)v \text{ for all } g \in \Gamma\}.$$

Let $\theta \geq 1$ be a natural number, $g_1, \dots, g_\theta \in \Gamma$, and $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$. Let V be a vector space with basis $x_1, \dots, x_\theta$. V is an object in ${}_\Gamma^\Gamma\mathcal{YD}$ by defining $x_i \in V_{g_i}^{\chi_i}$ for all i. Thus each x_i has degree g_i, and the group Γ acts on x_i via the character χ_i. We define

$$q_{ij} := \chi_j(g_i) \text{ for all } 1 \leq i, j \leq \theta.$$

The braiding on V is determined by the matrix (q_{ij}) since

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i \text{ for all } 1 \leq i, j \leq \theta.$$

We will identify the tensor algebra $T(V)$ with the free associative algebra $k\langle x_1, \dots, x_\theta\rangle$. It is an algebra in ${}_\Gamma^\Gamma\mathcal{YD}$, where a monomial

$$x = x_{i_1} x_{i_1} \cdots x_{i_n}, 1 \leq i_1, \dots, i_n \leq \theta,$$

has Γ-degree $g_{i_1}g_{i_1}\cdots g_{i_n}$ and the action of $g \in \Gamma$ on x is given by $g \cdot x = \chi_{i_1}\chi_{i_1}\cdots\chi_{i_n}(g)x$. $T(V)$ is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ with comultiplication

$$\Delta_{T(V)} : T(V) \to T(V)\underline{\otimes}T(V),\ x_i \mapsto x_i \otimes 1 + 1 \otimes x_i,\ 1 \leq i \leq \theta.$$

Here we write $T(V)\underline{\otimes}T(V)$ to indicate the braided algebra structure on the vector space $T(V) \otimes T(V)$, that is

$$(x \otimes y)(x' \otimes y') = x(g \cdot x') \otimes yy',$$

for all $x, x', y, y' \in T(V)$ and $y \in T(V)_g, g \in \Gamma$.

Let $I = \{1, 2, \ldots, \theta\}$, and $\mathbb{Z}[I]$ the free abelian group of rank θ with basis $\alpha_1, \ldots, \alpha_\theta$. Given the matrix (q_{ij}), we define the bilinear map

$$(1.1)\quad \mathbb{Z}[I] \times \mathbb{Z}[I] \to k^\times,\ (\alpha, \beta) \mapsto q_{\alpha,\beta},\ \text{by } q_{\alpha_i,\alpha_j} = q_{ij}, 1 \leq i, j \leq \theta.$$

We consider V as a Yetter-Drinfeld module over $\mathbb{Z}[I]$ by defining $x_i \in V_{\alpha_i}^{\psi_i}$ for all $1 \leq i \leq \theta$, where ψ_j is the character of $\mathbb{Z}[I]$ with

$$\psi_j(\alpha_i) = q_{ij} \text{ for all } 1 \leq i, j \leq \theta.$$

Thus $T(V) = k\langle x_1, \ldots, x_\theta\rangle$ is also a braided Hopf algebra in ${}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$. The $\mathbb{Z}[I]$-degree of a monomial $x = x_{i_1}x_{i_1}\cdots x_{i_n}, 1 \leq i_1, \ldots, i_n \leq \theta$, is $\sum_{i=1}^{\theta} n_i\alpha_i$, where for all i, n_i is the number of occurences of i in the sequence $(i_1, i_2, \ldots, i_n)$. The braiding on $T(V)$ as a Yetter-Drinfeld module over Γ or $\mathbb{Z}[I]$ is in both cases given by

$$(1.2)\quad c(x \otimes y) = q_{\alpha,\beta}y \otimes x,\ \text{where } x \in T(V)_\alpha, y \in T(V)_\beta, \alpha, \beta \in \mathbb{Z}[I].$$

The comultiplication of $T(V)$ as a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ only depends on the matrix (q_{ij}), hence it coincides with the comultiplication of $T(V)$ as a coalgebra in ${}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$. In particular, the comultiplication of $T(V)$ is $\mathbb{Z}[I]$-graded.

1.2. **Bosonization and twisting.** Let R be a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. We will use a Sweedler notation for the comultiplication

$$\Delta_R : R \to R \otimes R,\ \Delta_R(r) = r^{(1)} \otimes r^{(2)}.$$

For Hopf algebras A in the usual sense, we always use the Sweedler notation

$$\Delta : A \to A \otimes A,\ \Delta(a) = a_{(1)} \otimes a_{(2)}.$$

Then the smash product $A = R\#k[\Gamma]$ is a Hopf algebra in the usual sense (the bosonization of R). As vector spaces, $R\#k[\Gamma] = R \otimes k[\Gamma]$. Multiplication and comultiplication are defined by

$$(1.3)\quad (r\#g)(s\#h) = r(g\cdot s)\#gh,\ \Delta(r\#g) = r^{(1)}\#{r^{(2)}}_{(-1)}g\otimes {r^{(2)}}_{(0)}\#g.$$

Then the maps

$$\iota : k[\Gamma] \to R\#k[\Gamma], \text{ and } \pi : R\#k[\Gamma] \to k[\Gamma]$$

with $\iota(g) = 1\#g$ and $\pi(r\#g) = r$ for all $r \in R, g \in \Gamma$ are Hopf algebra maps with $\pi\iota = \mathrm{id}$.

Conversely, if A is a Hopf algebra in the usual sense with Hopf algebra maps $\iota : k[\Gamma] \to A$ and $\pi : A \to k[\Gamma]$ such that $\pi\iota = \mathrm{id}$, then

$$R = \{a \in A \mid (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\}$$

is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ in the following way. As an algebra, R is a subalgebra of A. The $k[\Gamma]$-coaction, Γ-action and comultiplication of R are defined by

$$\delta(r) = \pi(r^{(1)}) \otimes r^{(2)}, \; g \cdot r = \iota(g) r \iota(g^{-1}) \tag{1.4}$$

and

$$\Delta_R(r) = \vartheta(r_{(1)}) \otimes r_{(2)}. \tag{1.5}$$

Here, $\Delta_A(r) = r_{(1)} \otimes r_{(2)}$, and ϑ is the map

$$\vartheta : A \to R, \; \vartheta(r) = r_{(1)}\iota(S(\pi(r_{(2)}))), \tag{1.6}$$

where S is the antipode of A. Then

$$R\#k[\Gamma] \to A, \; r\#g \mapsto r\iota(g), \; r \in R, g \in \Gamma, \tag{1.7}$$

is an isomorphism of Hopf algebras.

We recall the notion of *twisting* the algebra structure of an arbitrary Hopf algebra A, see for example [KS, 10.2.3]. Let $\sigma : A \otimes A \to k$ be a convolution invertible linear map, and a normalized 2-cocycle, that is, for all $x, y, z \in A$,

$$\sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}), \tag{1.8}$$

and $\sigma(x, 1) = \varepsilon(x) = \sigma(1, x)$. The Hopf algebra A_σ with twisted algebra structure is equal to A as a coalgebra, and has multiplication $\cdot_\sigma$ with

$$x \cdot_\sigma y = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)}, y_{(3)}) \text{ for all } x, y \in A. \tag{1.9}$$

In the situation $A = R\#k[\Gamma]$ above, let $\sigma : \Gamma \times \Gamma \to k^\times$ be a normalized 2-cocycle of the group Γ. Then σ extends to a 2-cocycle of the group algebra $k[\Gamma]$ and it defines a normalized and invertible 2-cocycle $\sigma_\pi = \sigma(\pi \otimes \pi)$ of the Hopf algebra A. Since $k[\Gamma]$ is cocommutative, ι and π are Hopf algebra maps

$$\iota : k[\Gamma] \to A_{\sigma_\pi} \text{ and } \pi : A_{\sigma_\pi} \to k[\Gamma].$$

Hence the coinvariant elements

$$R_\sigma = \{a \in A_{\sigma_\pi} \mid (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\}$$

form a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. As a vector space, R_σ coincides with R, but R_σ and R have different multiplication and comultiplication.

To simplify the formulas, we will treat ι as an inclusion map.

In any braided Hopf algebra R with multiplication m and braiding $c : R \otimes R \to R \otimes R$ we define the *braided commutator* of elements $x, y \in R$ by

$$[x, y]_c = xy - mc(x \otimes y). \tag{1.10}$$

If $x \in R$ is a primitive element, then

$$(\mathrm{ad}_c x)(y) = [x, y]_c \tag{1.11}$$

denotes the *braided adjoint action* of x on R. For example, in the situation of the free algebra in Section 1.1 with braiding (1.2), we have for all x_i and $y = x_{j_1} \cdots x_{j_n}$,

$$(\mathrm{ad}_c x_i)(y) = x_i y - q_{ij_1} \cdots q_{ij_n} y x_i. \tag{1.12}$$

In the formulation of the next lemma we need one more notation. If V is a left C-comodule over a coalgebra C, then V is a right module over the dual algebra C^* by $v \leftharpoonup p = p(v_{(-1)})v_{(0)}$ for all $v \in V, p \in C^*$. In particular, if R is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, then the $k[\Gamma]$-coaction defines a left $k[\Gamma] \otimes k[\Gamma]$-comodule structure on $R \otimes R$, hence a right $(k[\Gamma] \otimes k[\Gamma])^*$-module structure on $R \otimes R$ denoted by $\leftharpoonup$.

Lemma 1.1. *Let Γ be an abelian group, $\sigma : \Gamma \times \Gamma \to k^\times$ a normalized 2-cocycle, R a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, $g, h \in \Gamma$, and $x \in R_g, y \in R_h, r \in R$.*

(1) *$x \cdot_\sigma y = \sigma(g, h)xy$.*
(2) *$\Delta_{R_\sigma}(r) = \Delta_R(r) \leftharpoonup \sigma^{-1}$.*
(3) *If $y \in R_h^\eta$ for some character $\eta \in \widehat{\Gamma}$, and R as an algebra is generated by primitive elements, then $g \cdot_\sigma y = \sigma(g, h)\sigma^{-1}(h, g)\eta(g)y$, and hence $[x, y]_{c_\sigma} = \sigma(g, h)[x, y]_\sigma$.*

Proof. (1) and (3) are [AS5, (2-11), (2-14)]. To prove (2), using the cocommutativity of the group algebra we compute

$$\begin{aligned}\Delta_{R_\sigma}(r) &= r_{(1)} \cdot_\sigma S(\pi(r_{(2)})) \otimes r_{(3)} \\ &= \sigma(\pi(r_{(1)}), S(\pi(r_{(5)})))\vartheta(r_{(2)})\sigma^{-1}(\pi(r_{(3)}), S(\pi(r_{(4)}))) \otimes r_{(6)}.\end{aligned}$$

On the other hand, $\Delta_R(r) = r_{(1)}S\pi(r_{(2)}) \otimes r_{(3)}$, hence

${r^{(1)}}_{(-1)} \otimes {r^{(2)}}_{(-1)} \otimes {r^{(1)}}_{(0)} \otimes {r^{(2)}}_{(0)} = \pi(r_{(1)}S(r_{(3)})) \otimes \pi(r_{(4)}) \otimes \vartheta(r_{(2)}) \otimes r_{(5)}$, and

$\Delta_R(r) \leftharpoonup \sigma^{-1} = \sigma^{-1}(\pi(r_{(1)}S(r_{(3)})), \pi(r_{(4)}))\vartheta(r_{(2)}) \otimes r_{(5)}$. Hence the claim follows from the equality

$$\sigma(a, S(b_{(3)}))\sigma^{-1}(b_{(1)}, S(b_{(2)})) = \sigma^{-1}(aS(b_{(1)}), b_{(2)}))$$

for all $a, b \in k[\Gamma]$. It is enough to check this equation for elements $a, b \in \Gamma$. Then the equality follows from the group cocycle condition. □

We now apply the twisting procedure to the braided Hopf algebra $T(V) \in {}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$.

Lemma 1.2. *Let $\theta \geq 1$, and $(q_{ij})_{1\leq i,j\leq\theta}, (q'_{ij})_{1\leq i,j\leq\theta}$ matrices with coefficients in k. Let $V \in {}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$ with basis $x_1, \dots, x_\theta$ and $x_i \in V^{\psi_i}_{\alpha_i}, \psi_j(\alpha_i) = q_{ij}$ for all i, j as in Section 1.1, and $V' \in {}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$ with basis $x'_1, \dots, x'_\theta$ and $x'_i \in V^{\psi'_i}_{\alpha_i}, \psi'_j(\alpha_i) = q'_{ij}$ for all i, j. Then $T(V)$ and $T(V')$ are braided Hopf algebras in ${}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$ as in Section 1.1. Assume*

$$q_{ij}q_{ji} = q'_{ij}q'_{ji}, \text{ and } q_{ii} = q'_{ii} \text{ for all } 1 \leq i, j \leq \theta. \tag{1.13}$$

Then there is a 2-cocycle $\sigma : \mathbb{Z}[I] \times \mathbb{Z}[I] \to k^\times$ with

$$\sigma(\alpha, \beta)\sigma^{-1}(\beta, \alpha) = q_{\alpha\beta}q'^{-1}_{\alpha\beta} \text{ for all } \alpha, \beta \in \mathbb{Z}[I], \tag{1.14}$$

and a k-linear isomorphism $\varphi : T(V) \to T(V')$ with $\varphi(x_i) = x'_i$ for all i and such that for all $\alpha, \beta \in \mathbb{Z}[I], x \in T(V)_\alpha, y \in T(V)_\beta$ and $z \in T(V)$

1. *$\varphi(xy) = \sigma(\alpha, \beta)\varphi(x)\varphi(y)$.*
2. *$\Delta_{T(V')}(\varphi(z)) = (\varphi \otimes \varphi)(\Delta_{T(V)}(z)) \leftharpoonup \sigma$.*
3. *$\varphi([x, y]_c) = \sigma(\alpha, \beta)[\varphi(x), \varphi(y)]_{c'}$.*

Proof. Define σ as the bilinear map with $\sigma(\alpha_i, \alpha_j) = q_{ij}q'^{-1}_{ij}$ if $i \leq j$, and $\sigma(\alpha_i, \alpha_j) = 1$ if $i > j$ (see [AS5, Prop. 3.9]).

Let $\varphi : T(V) \to T(V')_\sigma$ be the algebra map with $\varphi(x_i) = x'_i$ for all i. Then φ is bijective since it follows from Lemma 1.1 (1) and the bilinearity of σ that for all monomials $x = x_{i_1}x_{i_2}\cdots x_{i_n}$ of length $n \geq 1$ with $x' = x'_{i_1}x'_{i_2}\cdots x'_{i_n}$,

$$\varphi(x) = \prod_{r<s} \sigma(\alpha_{i_r}, \alpha_{i_s})x'.$$

In particular, φ is $\mathbb{Z}[I]$-graded. To see that φ is $\mathbb{Z}[I]$-linear, let $\alpha, \beta \in \mathbb{Z}[I]$ and $x \in T(V)_\beta$. Then by Lemma 1.1 (3),

$$\alpha \cdot x = q_{\alpha\beta}x, \text{ and } \alpha \cdot_\sigma \varphi(x) = \sigma(\alpha, \beta)\sigma^{-1}(\beta, \alpha)q'_{\alpha\beta}\varphi(x),$$

and $\varphi(\alpha \cdot x) = \alpha \cdot_\sigma \varphi(x)$ follows by (1.14). Since the elements x_i and x'_i are primitive we now see that $\varphi : T(V) \to T(V')_\sigma$ is an isomorphism of braided Hopf algebras. Then the claim follows from Lemma 1.1. □

2. Serre relations and root vectors

2.1. **Datum of finite Cartan type and root vectors.**

Definition 2.1. A *datum of Cartan type*

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$$

consists of an abelian group Γ, elements $g_i \in \Gamma, \chi_i \in \widehat{\Gamma}, 1 \le i \le \theta$, and a Cartan matrix (a_{ij}) of size θ satisfying

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}},\ q_{ii} \neq 1,\ \text{with } q_{ij} = \chi_j(g_i) \text{ for all } 1 \le i, j \le \theta. \tag{2.1}$$

A datum $\mathcal{D}$ of Cartan type will be called of finite Cartan type if (a_{ij}) is of finite type.

Example 2.2. A Cartan datum $(I, \cdot)$ in the sense of Lusztig [L3, 1.1.1] defines a datum of Cartan type for the free abelian group ZI with $g_i = \alpha_i, \chi_i = \psi_i, 1 \le i \le \theta$, as in Section 1.1, where

$$q_{ij} = v^{d_i a_{ij}}, d_i = \frac{i \cdot i}{2}, a_{ij} = 2\frac{i \cdot j}{i \cdot i} \text{ for all } 1 \le i, j \le \theta.$$

In Example 2.2, $d_i a_{ij} = i \cdot j$ is the symmetrized Cartan matrix, and $q_{ij} = q_{ji}$ for all $1 \le i, j \le \theta$. In general, the matrix (q_{ij}) of a datum of Cartan type is not symmetric, but by Lemma 1.2 we can reduce to the symmetric case by twisting.

We fix a finite abelian group Γ and a datum

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$$

of finite Cartan type. The Weyl group $W \subset \mathrm{Aut}(\mathbb{Z}[I])$ of (a_{ij}) is generated by the reflections $s_i : \mathbb{Z}[I] \to \mathbb{Z}[I]$ with $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ for all i, j. The root system is $\Phi = \cup_{i=1}^{\theta} W(\alpha_i)$, and

$$\Phi^+ = \{\alpha \in \Phi \mid \alpha = \sum_{i=1}^{\theta} n_i\alpha_i, n_i \ge 0 \text{ for all } 1 \le i \le \theta\}$$

denotes the set of positive roots with respect to the basis of simple roots $\alpha_1, \dots, \alpha_\theta$. Let p be the number of positive roots.

For $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i \in \mathbb{Z}[I], n_i \in \mathbb{Z}$ for all i we define

$$g_\alpha = g_1^{n_1} g_2^{n_2} \cdots g_\theta^{n_\theta} \text{ and } \chi_\alpha = \chi_1^{n_1}\chi_2^{n_2} \cdots \chi_\theta^{n_\theta}. \tag{2.2}$$

In this section, we assume that the Dynkin diagram of (a_{ij}) is *connected*. In this case we say that $\mathcal{D}$ is connected.

We fix a reduced decomposition of the longest element

$$w_0 = s_{i_1}s_{i_2} \cdots s_{i_p}$$

of W in terms of the simple reflections. Then

$$\beta_l = s_{i_1} \cdots s_{i_{l-1}}(\alpha_{i_l}), 1 \le l \le p,$$

is a convex ordering of the positive roots.

Let $d_1, \dots, d_\theta \in \{1,2,3\}$ such that $d_i a_{ij} = d_j a_{ji}$ for all i, j. We assume for all $1 \le i, j \le \theta$,

$$q_{ij} \text{ has odd order, and} \tag{2.3}$$

$$\text{the order of } q_{ii} \text{ is prime to 3, if } (a_{ij}) \text{ is of type } G_2. \tag{2.4}$$

Then it follows from (2.1) ([AS2, 4.3]) that the elements q_{ii} have the same order in $k^\times$. We define

$$N = \text{ order of } q_{ii}, 1 \le i \le \theta. \tag{2.5}$$

Definition 2.3. Let $V = V(\mathcal{D})$ be a vector space with basis $x_1, \dots, x_\theta$, and let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ by $x_i \in V^{\chi_i}_{g_i}$ for all $1 \le i \le \theta$. Then $T(V)$ is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ as in Section 1.1. Let

$$R(\mathcal{D}) = T(V)/((\mathrm{ad}_c x_i)^{1-a_{ij}}(x_j) \mid 1 \le i, j \le \theta)$$

be the quotient Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

It is well-known that the elements $(\mathrm{ad}_c x_i)^{1-a_{ij}}(x_j), 1 \le i, j \le \theta$ are primitive in the free algebra $T(V)$ (see for example [AS2, A.1]), hence they generate a Hopf ideal. By abuse of language, we denote the images of the elements x_i in $R(\mathcal{D})$ again by x_i.

In the situation of Example 2.2, Lusztig [L2] defined root vectors x_α in $R(\mathcal{D}) = U^+$ for each positive root α using the convex ordering of the positive roots. As noted in [AS4], these root vectors can be seen to be iterated braided commutators of the elements $x_1, \dots, x_\theta$ with respect to the braiding given by the matrix $(v^{d_i a_{ij}})$. This follows for example from the inductive definition of the root vectors in [Ri].

In the case of our general braiding given by (q_{ij}) we define root vectors $x_\alpha \in R(\mathcal{D})$ for each $\alpha \in \Phi^+$ by the same iterated braided commutator of the elements $x_1, \dots, x_\theta$ as in Lusztig's case but with respect to the general braiding.

Definition 2.4. Let $K(\mathcal{D})$ be the subalgebra of $R(\mathcal{D})$ generated by the elements $x_\alpha^N, \alpha \in \Phi^+$.

Theorem 2.5. *Let $\mathcal{D}$ be a connected datum of finite Cartan type, and assume* (2.3), (2.4)*.*

(1) *The elements*

$$x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \cdots x_{\beta_p}^{a_p}, a_1, a_2, \dots, a_p \ge 0,$$

form a basis of $R(\mathcal{D})$.

(2) *$K(\mathcal{D})$ is a braided Hopf subalgebra of $R(\mathcal{D})$.*

(3) *For all $\alpha, \beta \in \Phi^+, x_\alpha x_\beta^N = \chi_\beta^N(g_\alpha) x_\beta^N x_\alpha$, that is, $[x_\alpha, x_\beta^N]_c = 0$.*

Proof. (a) In the situation of 2.2, the elements in (1) form Lusztig's PBW-basis of U^+ over $\mathbb{Z}[v, v^{-1}]$ by [L2, 5.7].

(b) Now we assume that the braiding has the form $(q_{ij} = q^{d_i a_{ij}})$, where $(d_i a_{ij})$ is the symmetrized Cartan matrix, and q is a non-zero element in k of odd order, and not divisible by 3 if the Dynkin diagram of (a_{ij}) is G_2. Then (1) follows from Lusztig's result by extension of scalars, and (2) is shown in [dCP, 19.1] (for another proof see [M2, 3.1]). The algebra $K(\mathcal{D})$ is commutative since it is a subalgebra of the commutative algebra Z_0 of [dCP, 19.1]. This proves (3) since $q^N = 1$, hence $\chi_\beta^N(g_\alpha) = 1$

(c) In the situation of a general braiding matrix $(q_{ij})_{1\le i,j\le\theta}$ assumed in the theorem, we define a matrix $(q'_{ij})_{1\le i,j\le\theta}$ by $q'_{ii} = q_{ii}$ for all i, and for all $i \neq j$ we define $q'_{ij} = q'_{ji}$ to be a square root of $q_{ij}q_{ji}$. By [AS2, 4.3], $q'_{ij} = q^{d_i a_{ij}}$ for all i, j, and for some $q \in k$. Thus by part (b) of the proof, (1),(2) and (3) hold for the braiding (q'_{ij}), and hence by Lemma 1.2 for (q_{ij}). $\square$

2.2. **The Hopf algebra $K(\mathcal{D})\#k[\Gamma]$.** We assume the situation of Section 2.1. By Theorem 2.5 (2), $K(\mathcal{D})$ is a braided Hopf algebra in ${}^\Gamma_\Gamma\mathcal{YD}$, and the smash product $K(\mathcal{D})\#k[\Gamma]$ is a Hopf algebra in the usual sense. We want to describe all Hopf algebra maps

$$K(\mathcal{D})\#k[\Gamma] \to k[\Gamma]$$

which are the identity on the group algebra $k[\Gamma]$.

Definition 2.6. For any $1 \le l \le p$ and $a = (a_1, a_2, \dots, a_p) \in \mathbb{N}^p$ we define

$$\begin{aligned}
h_l &= g_{\beta_l}^N,\\
\eta_l &= \chi_{\beta_l}^N,\\
z_l &= x_{\beta_l}^N,\\
z^a &= z_1^{a_1} z_2^{a_2} \cdots z_p^{a_p} \in K(\mathcal{D}),\\
h^a &= h_1^{a_1} h_2^{a_2} \cdots h_p^{a_p} \in \Gamma,\\
\eta^a &= \eta_1^{a_1} \eta_2^{a_2} \cdots \eta_p^{a_p} \in \widehat{\Gamma},\\
\underline{a} &= a_1\beta_1 + a_2\beta_2 + \cdots + a_p\beta_p \in \mathbb{Z}[I].
\end{aligned}$$

For $\alpha = \sum_{i=1}^\theta n_i\alpha_i \in \mathbb{Z}[I], n_i \in \mathbb{Z}$ for all i, we call $\mathrm{ht}(\alpha) = \sum_{i=1}^\theta n_i$ the *height* of α. Let $e_l = (\delta_{kl})_{1\le k\le p} \in \mathbb{N}^p$, where $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ if $k \neq l$.

Note that for all $a, b, c \in \mathbb{N}^p$,

$$h^a = h^b h^c,\ \eta^a = \eta^b\eta^c,\ \text{if } \underline{a} = \underline{b} + \underline{c}, \tag{2.6}$$

$$\mathrm{ht}(\underline{b}) < \mathrm{ht}(\underline{a}), \text{ if } \underline{a} = \underline{b} + \underline{c} \text{ and } c \neq 0. \tag{2.7}$$

As explained in Section 1.1, we view $T(V)$ as a braided Hopf algebra in ${}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$. Then the quotient Hopf algebra $R(\mathcal{D})$ and its Hopf subalgebra $K(\mathcal{D})$ are braided Hopf algebras in ${}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$. In particular, the comultiplication $\Delta_{K(\mathcal{D})} : K(\mathcal{D}) \to K(\mathcal{D}) \otimes K(\mathcal{D})$ is $\mathbb{Z}[I]$-graded. By construction, for any $\alpha \in \Phi^+$, the root vector x_α in $R(\mathcal{D})$ is $\mathbb{Z}[I]$-homogeneous of $\mathbb{Z}[I]$-degree α. Thus $x_\alpha \in R(\mathcal{D})^{\chi_\alpha}_{g_\alpha}$, and for all $a \in \mathbb{N}^p$, z^a has $\mathbb{Z}[I]$-degree $N\underline{a}$, and

$$z^a \in K(\mathcal{D})^{\eta^a}_{h^a}. \tag{2.8}$$

For $z \in K(\mathcal{D}), g \in \Gamma$, we will denote $z\#g \in K(\mathcal{D})\#k[\Gamma]$ by zg. By Theorem 2.5 the elements $z^a g$ with $a \in \mathbb{N}^p, g \in \Gamma$, form a basis of $K(\mathcal{D})\#k[\Gamma]$, and it follows that for all $a, b = (b_i), c = (c_i) \in \mathbb{N}^p$,

$$z^b z^c = \gamma_{b,c} z^{b+c}, \text{ where } \gamma_{b,c} = \prod_{k>l} \eta_l(h_k)^{b_k c_l}, \tag{2.9}$$

$$h^a z^b = \eta^b(h^a) z^b h^a \text{ in } R\#k[\Gamma]. \tag{2.10}$$

Lemma 2.7. *For any $0 \neq a \in \mathbb{N}^p$ there are uniquely determined scalars $t^a_{b,c} \in k, 0 \neq b, c \in \mathbb{N}^p$, such that*

$$\Delta_{K(\mathcal{D})}(z^a) = z^a \otimes 1 + 1 \otimes z^a + \sum_{b,c\neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, z^b \otimes z^c. \tag{2.11}$$

Proof. Since $\Delta_{K(\mathcal{D})}$ is $\mathbb{Z}[I]$-graded, $\Delta_{K(\mathcal{D})}(z^a)$ is a linear combination of elements $z^b \otimes z^c$ where $\underline{b} + \underline{c} = \underline{a}$. Hence

$$\Delta_{K(\mathcal{D})}(z^a) = x \otimes 1 + 1 \otimes y + \sum_{b,c\neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, z^b \otimes z^c,$$

where x, y are elements in $K(\mathcal{D})$. By applying the augmentation ε it follows that $x = y = z^a$. $\square$

We now define recursively a family of elements u^a in $k[\Gamma]$ depending on parameters μ_a which behave like the elements z^a with respect to comultiplication.

Lemma 2.8. *Let $n \geq 2$. For all $0 \neq b \in \mathbb{N}^p, \mathrm{ht}(\underline{b}) < n$, let $\mu_b \in k$ and $u^b \in k[\Gamma]$ such that*

$$u^b = \mu_b(1 - h^b) + \sum_{d,e\neq 0, \underline{d}+\underline{e}=\underline{b}} t^b_{d,e}\, \mu_d u^e, \tag{2.12}$$

$$\Delta(u^b) = h^b \otimes u^b + u^b \otimes 1 + \sum_{d,e\neq 0, \underline{d}+\underline{e}=\underline{b}} t^b_{d,e}\, u^d h^e \otimes u^e. \tag{2.13}$$

Let $a \in \mathbb{N}^p$ with $\mathrm{ht}(\underline{a}) = n$, and $u^a \in k[\Gamma]$. Then the following statements are equivalent:

$$u^a = \mu_a(1 - h^a) + \sum_{b,c\neq 0,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \mu_b u^c \text{ for some } \mu_a \in k. \tag{2.14}$$

$$\Delta(u^a) = h^a \otimes u^a + u^a \otimes 1 + \sum_{b,c\neq 0,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, u^b h^c \otimes u^c. \tag{2.15}$$

Proof. Let

$$v_a = u^a - \sum_{b,c\neq 0,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \mu_b u^c.$$

Then u^a can be written as in (2.14) if and only if $\Delta(v_a) = h^a \otimes v_a + v_a \otimes 1$. Hence it is enough to prove that

$$\Delta(v_a) - h^a \otimes v_a - v_a \otimes 1 = \Delta(u^a) - h^a \otimes u^a - u^a \otimes 1 - \sum_{b,c\neq 0,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, u^b h^c \otimes u^c.$$

We compute

$$\begin{aligned}
&\Delta(v_a) - h^a \otimes v_a - v_a \otimes 1 = \\
&\quad = \Delta(u^a) - \sum_{b,c\neq 0,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \mu_b \Delta(u^c) - h^a \otimes v_a - v_a \otimes 1 \\
&\quad = \Delta(u^a) - h^a \otimes u^a - u^a \otimes 1 + \sum_{b,c\neq 0,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \mu_b (h^a \otimes u^c - h^c \otimes u^c) \\
&\quad - \sum_{\substack{b,c,f,g\neq 0 \\ \underline{b}+\underline{c}=\underline{a},\underline{f}+\underline{g}=\underline{c}}} t^a_{b,c}\, t^c_{f,g}\, \mu_b u^f h^g \otimes u^g,
\end{aligned}$$

using the definition of v_a in the first equation, and the formula for $\Delta(u^c)$ from (2.13) in the second equation. Note that the term

$$\sum_{b,c\neq 0,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \mu_b u^c \otimes 1$$

cancels. Hence we have to show that

$$\begin{aligned}
&\sum_{\substack{b,c,f,g\neq 0 \\ \underline{b}+\underline{c}=\underline{a},\underline{f}+\underline{g}=\underline{c}}} t^a_{b,c}\, t^c_{f,g}\, \mu_b u^f h^g \otimes u^g = \\
&= \sum_{b,c\neq 0,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c} (\mu_b h^a \otimes u^c - \mu_b h^c \otimes u^c + u^b h^c \otimes u^c).
\end{aligned}$$

Since for all $b, c \neq 0, \underline{b} + \underline{c} = \underline{a}$, we have $h^a = h^b h^c$, it follows that

$$\mu_b h^a \otimes u^c - \mu_b h^c \otimes u^c + u^b h^c \otimes u^c = (\mu_b (h^b - 1) + u^b) h^c \otimes u^c.$$

Using the formula for u^b from (2.12), we finally have to prove

$$\sum_{\substack{b,c,f,g\neq 0\\ \underline{b}+\underline{c}=\underline{a},\underline{f}+\underline{g}=\underline{c}}} t^a_{b,c}\, t^c_{f,g}\, \mu_b u^f h^g \otimes u^g = \sum_{\substack{b,c,d,e\neq 0\\ \underline{b}+\underline{c}=\underline{a},\underline{d}+\underline{e}=\underline{b}}} t^a_{b,c}\, t^b_{d,e}\, \mu_d u^e h^c \otimes u^c.$$

This last equality follows from the coassociativity of $K(\mathcal{D})$. Indeed, from

$$(\mathrm{id}\otimes\Delta_{K(\mathcal{D})})\Delta_{K(\mathcal{D})}(z^a) = (\Delta_{K(\mathcal{D})}\otimes\mathrm{id})\Delta_{K(\mathcal{D})}(z^a)$$

we obtain with (2.11) after cancelling several terms

$$\sum_{\substack{b,c,f,g\neq 0\\ \underline{b}+\underline{c}=\underline{a},\underline{f}+\underline{g}=\underline{c}}} t^a_{b,c}\, t^c_{f,g}\, z^b \otimes z^f \otimes z^g = \sum_{\substack{b,c,d,e\neq 0\\ \underline{b}+\underline{c}=\underline{a},\underline{d}+\underline{e}=\underline{b}}} t^a_{b,c}\, t^b_{d,e}\, z^d \otimes z^e \otimes z^c.$$

Thus mapping $z^r \otimes z^s \otimes z^t, r,s,t \neq 0, \mathrm{ht}(\underline{r}), \mathrm{ht}(\underline{s}), \mathrm{ht}(\underline{t}) < n$, onto $\mu_r u^s h^t \otimes u^t$ proves the claim. Here we are using that the elements z^a are linearly independent by Theorem 2.5. □

Let $K(\mathcal{D})\#k[\Gamma]$ be the Hopf algebra corresponding to the braided Hopf algebra $K(\mathcal{D})$ by (1.3). Thus by definition and Lemma 2.7, for all $0 \neq a \in \mathbb{N}^p$,

$$\Delta_{K(\mathcal{D})\#k[\Gamma]}(z^a) = h^a \otimes z^a + z^a \otimes 1 + \sum_{b,c\neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, z^b h^c \otimes z^c. \tag{2.16}$$

For all $n \geq 0$, let $K(\mathcal{D})_n$ be the vector subspace spanned by all $z^a, a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n$. Then $K(\mathcal{D})_n\#k[\Gamma] \subset K(\mathcal{D})\#k[\Gamma]$ is a subcoalgebra.

In the next Lemma we describe all coalgebra maps

$$\varphi : K(\mathcal{D})_n\#k[\Gamma] \to k[\Gamma] \text{ with } \varphi|\Gamma = \mathrm{id}.$$

Note that such a coalgebra map is given by a family of elements $\varphi(z^a) =: u^a, 0 \neq a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n$, such that (2.15) holds for all $0 \neq a, \mathrm{ht}(\underline{a}) \leq n$. It follows by induction on $\mathrm{ht}(\underline{a})$ from Lemma 2.8 with (2.14) that $\varepsilon(u^a) = 0$ for all a.

Lemma 2.9. *Let $n \geq 1$.*

(1) Let $(\mu_a)_{0\neq a\in\mathbb{N}^p, \mathrm{ht}(\underline{a})\leq n}$ be a family of elements in k such that for all a, if $h^a = 1$, then $\mu_a = 0$. Define the family $(u^a)_{0\neq a\in\mathbb{N}^p, \mathrm{ht}(\underline{a})\leq n}$ by induction on $\mathrm{ht}(\underline{a})$ by (2.14). Then

$$\varphi : K(\mathcal{D})_n\#k[\Gamma] \to k[\Gamma], \varphi(z^a g) = u^a g, a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n, g \in \Gamma,$$

is a coalgebra map.

(2) The map defined in (1) from the set of all $(\mu_a)_{0\neq a\in\mathbb{N}^p, \mathrm{ht}(\underline{a})\leq n}$ such that for all a, if $h^a = 1$, then $\mu_a = 0$, to the set of all coalgebra maps φ with $\varphi|\Gamma = \mathrm{id}$ is bijective.

Proof. This follows from Lemma 2.8 by induction on $\mathrm{ht}(\underline{a})$. Note that the coefficient μ_a in (2.14) is uniquely determined if we define $\mu_a = 0$ if $h^a = 1$. □

Definition 2.10. Let $n \geq 1$. A coalgebra map $\varphi : K(\mathcal{D})_n \# k[\Gamma] \to k[\Gamma]$ with $\varphi|\Gamma = \mathrm{id}$ is called a *partial Hopf algebra map*, if for all $x, y \in K(\mathcal{D})_n \# k[\Gamma]$ with $xy \in K(\mathcal{D})_n \# k[\Gamma]$, we have $\varphi(xy) = \varphi(x)\varphi(y)$.

Lemma 2.11. *Let $n \geq 1$, and $\varphi : K(\mathcal{D})_n \# k[\Gamma] \to k[\Gamma]$ a coalgebra map, $(\mu_a)_{0 \neq a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n}$ the family of scalars corresponding to φ by Lemma 2.9, and $u^a = \varphi(a)$ for all $a \in \mathbb{N}^p$ with $\mathrm{ht}(\underline{a}) \leq n$. Then the following are equivalent:*

(1) *φ is a partial Hopf algebra map.*
(2) *For all $0 \neq a = (a_1, \dots, a_p) \in \mathbb{N}^p$ with $\mathrm{ht}(\underline{a}) \leq n$,*
 (a) *$u^a = \prod_{a_l > 0} u_l^{a_l}$, where for all $1 \leq l \leq p, u_l = u^{e_l}$, if $a_l > 0$,*
 (b) *if $\eta^a \neq \varepsilon$, then $\mu_a = 0$, and $u^a = 0$.*
(3) (a) *As (2) (a).*
 (b) *For all $1 \leq l \leq p$ with $\mathrm{ht}(\underline{e_l}) \leq n$, if $\eta_l \neq \varepsilon$, then $u^{e_l} = 0$.*

Proof. (1) ⇒ (2): If φ is a partial Hopf algebra map, then (a) follows immediately, and to prove (b), let $0 \neq a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n$, and $g \in \Gamma$, with $\eta^a \neq \varepsilon$. Then

$$\varphi(gz^a) = \eta^a(g)u^a g = u^a g,$$

since $gz^a = \eta^a(g)z^a g$ by (2.10). Thus $u^a = 0$, and it follows by induction on $\mathrm{ht}(\underline{a})$ from (2.14) that $\mu_a = 0$, since for all $0 \neq b, c \in \mathbb{N}^p$ with $\mathrm{ht}(\underline{b}) + \mathrm{ht}(\underline{c}) = \mathrm{ht}(\underline{a})$, $\eta^b \neq \varepsilon$, or $\eta^c \neq \varepsilon$.

(2) ⇒ (3) is trivial. (3) ⇒ (1): The coalgebra map φ is a partial Hopf algebra map if and only if for all $b, c \in \mathbb{N}^p$ with $\mathrm{ht}(\underline{b}) + \mathrm{ht}(\underline{c}) \leq n$, and $g, h \in \Gamma$,

$$\varphi(z^b g z^c h) = u^b g u^c h.$$

By (2.9) and (2.10), $z^b g z^c h = \eta^c(g)\gamma_{b,c} z^{b+c} gh$. Thus (1) is equivalent to

$$\eta^c(g)\gamma_{b,c} u^{b+c} = u^b u^c \text{ for all } b, c \in \mathbb{N}^p, \mathrm{ht}(\underline{b}) + \mathrm{ht}(\underline{c}) \leq n, g \in \Gamma. \tag{2.17}$$

Let $b, c \in \mathbb{N}^p, \mathrm{ht}(\underline{b}) + \mathrm{ht}(\underline{c}) \leq n, g \in \Gamma$. By (a),

$$u^{b+c} = u^b u^c = \prod_{b_l + c_l > 0} u_l^{b_l + c_l}.$$

To prove (2.17) assume that $u^b u^c \neq 0$. Then $u_l \neq 0$ for all l with $c_l > 0$. Hence by (b), $\eta_l = \varepsilon$ for all l with $c_l > 0$, and $\eta^c(g) = 1, \gamma_{b,c} = 1$. □

To formulate the main result of this section, we define $M(\mathcal{D})$ as the set of all families $(\mu_l)_{1 \leq l \leq p}$ of elements in k satisfying the following condition for all $1 \leq l \leq p$: If $h_l = 1$ or $\eta_l \neq \varepsilon$, then $\mu_l = 0$.

Theorem 2.12. *(1) Let $\mu = (\mu_l)_{1\le l\le p} \in M(\mathcal{D})$. Then there is exactly one Hopf algebra map*

$$\varphi_\mu : K(\mathcal{D})\#k[\Gamma] \to k[\Gamma],\ \varphi|\Gamma = \mathrm{id}$$

such that the family $(\mu_a)_{0\ne a\in\mathbb{N}^p}$ associated to φ_μ by Lemma 2.9 satisfies $\mu_{e_l} = \mu_l$ for all $1 \le l \le p$.

(2) The map $\mu \mapsto \varphi_\mu$ defined in (1) from $M(\mathcal{D})$ to the set of all Hopf algebra homomorphisms $\varphi : K(\mathcal{D})\#k[\Gamma] \to k[\Gamma]$ with $\varphi|\Gamma = \mathrm{id}$ is bijective.

Proof. (1) We proceed by induction on n to construct partial Hopf algebra maps on $K(\mathcal{D})_n\#k[\Gamma]$, the case $n = 0$ being trivial. We assume that we are given a partial Hopf algebra map

$$\varphi : K(\mathcal{D})_{n-1}\#k[\Gamma] \to k[\Gamma],\ n \ge 1,$$

such that $\mu_{e_l} = \mu_l$ for all $1 \le l \le p$ with $\mathrm{ht}(\underline{e_l}) \le n-1$. Here $(\mu_a)_{0\ne a\in\mathbb{N}^p,\mathrm{ht}(\underline{a})\le n-1}$ is the family of scalars associated to φ by Lemma 2.9. We define $u^b = \varphi(z^b)$ for all $0 \ne b, \mathrm{ht}(\underline{b}) \le n-1$. It is enough to show that there is exactly one partial Hopf algebra map

$$\psi : K(\mathcal{D})_n\#k[\Gamma] \to k[\Gamma]$$

extending φ, and such that $\mu_{e_l} = \mu_l$ for all l with $\mathrm{ht}(\underline{e_l}) \le n$.

Let $a \in \mathbb{N}^p$ with $\mathrm{ht}(\underline{a}) = n$. To define $\psi(z^a) =: u^a$ we distinguish two cases.

If $a = e_l$ for some $1 \le l \le p$, we define

$$u^a = \mu_l(1 - h^a) + \sum_{b,c\ne 0,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\mu_b u^c. \tag{2.18}$$

Then (2.15) holds by Lemma 2.8.

If $a = (a_1, \dots, a_l, 0, \dots, 0), a_l \ge 1, 1 \le l \le p$, and $a \ne e_l$, then $a = r + s$, where $0 \ne r, s = e_l$. We define $u^a = u^r u^s$. To see that u^a satisfies (2.15), using (2.16) we write

$$\Delta(z^c) = h^c \otimes z^c + z^c \otimes 1 + T(c),\ \text{for all } 0 \ne c \in \mathbb{N}^p.$$

Since $z^r z^s = z^a$ because of (2.9) (note that $\gamma_{r,s} = 1$ in this case) we see that $\Delta(z^r)\Delta(z^s) = h^a \otimes z^a + z^a \otimes 1 + T(r, s)$, where

$$T(r,s) = h^r z^s\otimes z^r + z^r h^s\otimes z^s + (h^r\otimes z^r + z^r\otimes 1)T(s) + T(r)(h^s\otimes z^s + z^s\otimes 1),$$

and $T(r, s) = T(a)$. Since φ on $K(\mathcal{D})_{n-1}\#k[\Gamma]$ is a coalgebra map,

$$\Delta(u^c) = h^c \otimes u^c + u^c \otimes 1 + (\varphi \otimes \varphi)(T(c)),$$

for all $0 \ne c \in \mathbb{N}^p$ with $\mathrm{ht}(\underline{c}) \le n-1$. In particular,

$$\Delta(u^r)\Delta(u^s) = h^a \otimes u^a + u^a \otimes 1 + (\varphi \otimes \varphi)(T(r, s)).$$

Thus $\Delta(u^a) = h^a \otimes u^a + u^a \otimes 1 + (\varphi \otimes \varphi)(T(a))$, that is, u^a satisfies (2.15).

Thus the extension of φ defined by $\psi(z^a g) = u^a g$ for all $g \in \Gamma, a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) = n$ is a coalgebra map.

To prove that the extension ψ is a partial Hopf algebra map, we check condition (3) in Lemma 2.11. Since the restriction of ψ to $K(\mathcal{D})_{n-1}\#k[\Gamma]$ is a partial Hopf algebra map, (3) (a) is satisfied. To prove (3)(b), let $1 \leq l \leq p$ with $\mathrm{ht}(\underline{e_l}) = n$, $a = e_l$, and assume $\eta_l \neq \varepsilon$. Then for all $0 \neq b, c \in \mathbb{N}^p$ with $\underline{b} + \underline{c} = \underline{a}$, we have $\eta^b \neq \varepsilon$ or $\eta^c \neq \varepsilon$. Since φ is a Hopf algebra map, it follows from Lemma 2.11 that $\mu_b = 0$ or $u^c = 0$. By assumption, $\mu_l = 0$. Hence by (2.18), $u^a = 0$.

This proves (1) since the uniqueness of the extension follows from Lemma 2.8 and Lemma 2.9.

(2) By Lemma 2.9, the map $\mu \mapsto \varphi_\mu$ is injective. To prove surjectivity, let $\varphi : K(\mathcal{D})\#k[\Gamma] \to k[\Gamma]$ be a Hopf algebra map with $\varphi|\Gamma = \mathrm{id}$. By Lemma 2.9, φ is defined by a family $(\mu_a)_{0 \neq a \in \mathbb{N}^p}$ of scalars. By (1), φ is determined by the values $\mu_{e_l}, 1 \leq l \leq p$. □

Definition 2.13. For any $\mu \in M(\mathcal{D})$ and $1 \leq l \leq p$, let φ_μ be the Hopf algebra map defined in Theorem 2.12, and

$$u_l(\mu) = \varphi_\mu(z_l) \in k[\Gamma].$$

If α is a positive root in Φ^+ with $\alpha = \beta_l$, we define $u_\alpha(\mu) = u_l(\mu)$.

Note that by (2.14), each $u_\alpha(\mu)$ lies in the augmentation ideal of $k[g_i^N \mid 1 \leq i \leq \theta]$.

3. Linking

3.1. **Notations.** In this Section we fix a finite abelian group Γ, and a datum $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i,j \leq \theta})$ of finite Cartan type. We follow the notations of the previous Section, in particular, $q_{ij} = \chi_j(g_i)$ for all i, j.

For all $1 \leq i, j \leq \theta$ we write $i \sim j$ if i and j are in the same connected component of the Dynkin diagram of (a_{ij}). Let $\mathcal{X} = \{I_1, \dots, I_t\}$ be the set of connected components of $I = \{1, 2, \dots, \theta\}$. We assume

(3.1) $\quad q_{ij}$ has odd order for all i, j, and

(3.2) $\quad$ the order of q_{ii} is prime to 3, if i lies in a component G_2.

For all $J \in \mathcal{X}$, let N_J be the common order of $q_{ii}, i \in J$.

As in Section 2.2, for all $J \in \mathcal{X}$, we choose a reduced decomposition of the longest element $w_{0,J}$ of the Weyl group W_J of the root system

Φ_J of $(a_{ij})_{i,j\in J}$. Then for all $J, K \in \mathcal{X}$, $w_{0,J}$ and $w_{0,K}$ commute in the Weyl group W of the root system Φ of $(a_{ij})_{1\le i,j\le\theta}$, and

$$w_0 = w_{0,I_1} w_{0,I_2} \cdots w_{0,I_t}$$

gives a reduced representation of the longest element of W. For all $J \in \mathcal{X}$, let p_J be the number of positive roots in Φ_J^+, and

$$\Phi_J^+ = \{\beta_{J,1}, \dots, \beta_{J,p_J}\}$$

the corresponding convex ordering. Then

$$\Phi^+ = \{\beta_{I_1,1}, \dots, \beta_{I_1,p_{I_1}}, \dots, \beta_{I_t,1}, \dots, \beta_{I_t,p_{I_t}}\}$$

is the convex ordering corresponding to the reduced representation of $w_0 = w_{0,I_1} w_{0,I_2} \cdots w_{0,I_t}$. We also write

$$\Phi^+ = \{\beta_1, \dots, \beta_p\},\ p = \sum_{J\in\mathcal{X}} p_J,$$

for this ordering.

In Section 2.1 we have defined root vectors x_α in the free algebra $k\langle x_1, \dots, x_\theta\rangle$ for each positive root in $\Phi_J^+ \subset \Phi$, $J \in \mathcal{X}$.

We recall a notion from [AS4].

Definition 3.1. A family $\lambda = (\lambda_{ij})_{1\le i<j\le\theta,\, i\not\sim j}$ of elements in k is called a *family of linking parameters for* $\mathcal{D}$ if the following condition is satisfied for all $1 \le i < j \le \theta$, $i \not\sim j$: If $g_i g_j = 1$ or $\chi_i\chi_j \ne \varepsilon$, then $\lambda_{ij} = 0$. Vertices $1 \le i, j \le \theta$ are called *linkable* if $i \not\sim j$, $g_i g_j \ne 1$ and $\chi_i\chi_j = \varepsilon$.

Any vertex i is linkable to at most one vertex j, and if i, j are linkable, then $q_{ii} = q_{jj}^{-1}$ [AS4, Section 5.1].

The free algebra $k\langle x_1, \dots, x_\theta\rangle$ is a braided Hopf algebra in ${}_\Gamma^\Gamma\mathcal{YD}$ as explained in Section 1.1. Then $k\langle x_1, \dots, x_\theta\rangle \# k[\Gamma]$ is a Hopf algebra as in 1.2. For simplicity we write xg instead of $x\#g$ for elements $x \in k\langle x_1, \dots, x_\theta\rangle$ and $g \in \Gamma$.

3.2. **The Hopf algebra $U(\mathcal{D}, \lambda)$.** We assume the situation of Section 3.1.

Definition 3.2. Let $\lambda = (\lambda_{ij})_{1\le i<j\le\theta,\, i\not\sim j}$ be a family of linking parameters for $\mathcal{D}$. Let $U(\mathcal{D}, \lambda)$ be the quotient Hopf algebra of $k\langle x_1, \dots, x_\theta\rangle \# k[\Gamma]$ modulo the ideal generated by

$$\operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j),\ \text{for all } 1 \le i, j \le \theta, i \sim j, i \ne j, \tag{3.3}$$

$$x_i x_j - q_{ij} x_j x_i - \lambda_{ij}(1 - g_i g_j),\ \text{for all } 1 \le i < j \le \theta,\ i \not\sim j. \tag{3.4}$$

We denote the images of x_i and $g \in \Gamma$ in $U(\mathcal{D}, \lambda)$ again by x_i and g. The elements in (3.3) and (3.4) are skew-primitive. Hence $U(\mathcal{D}, \lambda)$ is a Hopf algebra with

$$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1,\ 1 \leq i \leq \theta.$$

Theorem 3.3. *Let Γ be a finite abelian group, and $\mathcal{D}$ a datum of finite Cartan type satisfying* (3.1) *and* (3.2). *Let λ be a family of linking parameters for $\mathcal{D}$. Then*

(1) *The elements*

$$x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \cdots x_{\beta_p}^{a_p} g,\ a_1, a_2, \dots, a_p \geq 0, g \in \Gamma,$$

form a basis of the vector space $U(\mathcal{D}, \lambda)$.

(2) *Let $J \in \mathcal{X}$, and $\alpha \in \Phi^+, \beta \in \Phi^+_J$. Then $[x_\alpha, x_\beta^{N_J}]_c = 0$, that is,*

$$x_\alpha x_\beta^{N_J} = q_{\alpha,\beta}^{N_J} x_\beta^{N_J} x_\alpha.$$

Proof. We adapt the method of proof of [AS4, Section 5.3] and proceed by induction on the number t of connected components.

If I is connected, (1) and (2) follow from Theorem 2.5.

If $t > 1$, we assume that $I_1 = \{1, 2, \dots, \widetilde{\theta}\},\ 1 \leq \widetilde{\theta} < \theta$. For all $1 \leq i \leq \widetilde{\theta}$, let l_i be the least common multiple of the orders of g_i and χ_i, $1 \leq i \leq \widetilde{\theta}$. Let $\widetilde{\Gamma} = \langle h_1, \dots, h_{\widetilde{\theta}} \mid h_i h_j = h_j h_i, h_i^{l_i} = 1 \text{ for all } i, j\rangle$, and define for all $1 \leq i \leq \widetilde{\theta}$ the character η_j of $\widetilde{\Gamma}$ by $\eta_j(h_i) = \chi_j(g_i),\ 1 \leq i, j \leq \widetilde{\theta}$. Then we define

$$\mathcal{D}_1 = \mathcal{D}(\widetilde{\Gamma}, (h_i)_{1 \leq i \leq \widetilde{\theta}}, (\eta_i)_{1 \leq i \leq \widetilde{\theta}}, (a_{ij})_{1 \leq i,j \leq \widetilde{\theta}}).$$

Let $\mathcal{D}_2 = \mathcal{D}(\Gamma, (g_i)_{\widetilde{\theta} < i \leq \theta}, (\chi_i)_{\widetilde{\theta} < i \leq \theta}, (a_{ij})_{\widetilde{\theta} < i,j \leq \theta})$ be the restriction of $\mathcal{D}$ to $I_2 \cup \cdots \cup I_t$, and $\lambda_2 = (\lambda_{ij})_{\widetilde{\theta} < i < j \leq \theta, i \nsim j}$. We define $U = U(\mathcal{D}_1)$ (with empty family of linking parameters) with generators $x_1, \dots, x_{\widetilde{\theta}}$, and $h \in \widetilde{\Gamma}$, and $A = U(\mathcal{D}_2, \lambda_2)$ with generators $y_{\widetilde{\theta}+1}, \dots, y_\theta$, and $g \in \Gamma$.

It is shown in [AS4, Lemma 5.19] that there are algebra maps γ_i, $(\varepsilon, \gamma)-$derivations δ_i and a Hopf algebra map φ,

$$\gamma_i : A \to k,\ \delta_i : A \to k,\ \varphi : U \to (A^0)^{\mathrm{cop}},\ 1 \leq i \leq \widetilde{\theta},$$

such that for all $1 \leq i \leq \widetilde{\theta} < j \leq \theta$,

$$\gamma_i | \Gamma = \chi_i,\ \gamma_i(y_j) = 0,$$
$$\delta_i | \Gamma = 0,\ \delta_i(y_j) = -\chi_i(g_j)\lambda_{ij},$$
$$\varphi(h_i) = \gamma_i,\ \varphi(x_i) = \delta_i.$$

Then $\sigma : U \otimes A \otimes U \otimes A \to U \otimes A$, defined for all $u, v \in U, a, b \in A$ by

$$\sigma(u \otimes a, v \otimes b) = \varepsilon(u)\tau(v, a)\varepsilon(b),\ \tau(v, a) = \varphi(v)(a),$$

is a 2-cocycle on the tensor product Hopf algebra of U and A, and $(U \otimes A)_\sigma$ is the Hopf algebra with twisted multiplication defined in (1.9). Multiplication in $(U \otimes A)_\sigma$ is given for all $u, v \in U, a, b \in A$ by

$$(3.5) \qquad (u \otimes a) \cdot_\sigma (v \otimes b) = u\tau(v_{(1)}, a_{(1)})v_{(2)} \otimes a_{(2)}\tau^{-1}(v_{(3)}, a_{(3)})b,$$

with $\tau^{-1}(u, a) = \varphi(u)(S^{-1}(a))$.

The group-like elements $h_i \otimes g_i^{-1}$, $1 \leq i \leq \widetilde{\theta}$, are central in $(U \otimes A)_\sigma$, and as in the last part of the proof of [AS4, Theorem 5.17] it can be seen that the map

$$(U \otimes A)_\sigma \to U(\mathcal{D}, \lambda),\ x_i \otimes 1 \mapsto x_i,\ h_i \otimes 1 \mapsto g_i,\ , 1 \otimes y_j \mapsto x_j,\ 1 \otimes g \mapsto g$$

for all $1 \leq i \leq \widetilde{\theta} < j \leq \theta$, $g \in \Gamma$, induces an isomorphism of Hopf algebras

$$(3.6) \qquad (U \otimes A)_\sigma / (h_i \otimes g_i^{-1} - 1 \otimes 1 \mid 1 \leq i \leq \widetilde{\theta}) \cong U(\mathcal{D}, \lambda).$$

By induction and Theorem 2.5, the elements

$$x_{\beta_1}^{a_1} \cdots x_{\beta_{p_1}}^{a_{p_1}} h \otimes y_{\beta_{p_1+1}}^{a_{p_1+1}} \cdots y_{\beta_p}^{a_p} g,\ a_1, \dots, a_p \geq 0, h \in \widetilde{\Gamma}, g \in \Gamma,$$

are a basis of $U \otimes A$. It follows from (3.5) that for all $p_1 < l \leq p$ and $1 \leq i \leq \widetilde{\theta}$,

$$(1 \otimes y_{\beta_l}) \cdot_\sigma (h_i \otimes 1) = \chi_i(g_{\beta_l}) h_i \otimes y_{\beta_l}.$$

Hence

$$(x_{\beta_1}^{a_1} \cdots x_{\beta_{p_1}}^{a_{p_1}} \otimes y_{\beta_{p_1+1}}^{a_{p_1+1}} \cdots y_{\beta_p}^{a_p}) \cdot_\sigma (h \otimes g),\ a_1, \dots a_p \geq 0, h \in \widetilde{\Gamma}, g \in \Gamma,$$

is a basis of $(U \otimes A)_\sigma$.

Let $P = \{h \otimes g \in (U \otimes A)_\sigma \mid h \in \widetilde{\Gamma}, g \in \Gamma\}$, and let $\widetilde{P} \subset P$ be the subgroup generated by $h_i \otimes g_i^{-1}$, $1 \leq i \leq \widetilde{\theta}$. Then

$$\Gamma \to P/\widetilde{P},\ g \mapsto \overline{1 \otimes g},$$

is a group isomorphism. By (3.6), $(U \otimes A)_\sigma \otimes_{k[P]} k[P/\widetilde{P}] \cong U(\mathcal{D})$. Hence

$$x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \cdots x_{\beta_p}^{a_p} g,\ a_1, a_2, \dots, a_p \geq 0, g \in \Gamma,$$

is a basis of $U(\mathcal{D}, \lambda)$.

To prove (2), we first show that for all $\widetilde{\theta} < i \leq \theta$, and $\beta \in \Phi_{I_1}^+$, with $N = N_{I_1}$

$$(3.7) \qquad (1 \otimes y_i) \cdot_\sigma (x_\beta^N \otimes 1) = \chi_\beta^N(g_i)(x_\beta^N \otimes 1) \cdot_\sigma (1 \otimes y_i)$$

in $(U \otimes A)_\sigma$. We use the notations of Section 2.2 with $N = N_{I_1}$, $z_\beta = x_\beta^N$. By (2.16)

$$\Delta_U(z_\beta) = g_\beta^N \otimes z_\beta + z_\beta \otimes 1 + \sum_{b, c \neq o, \underline{b} + \underline{c} = \beta} t_{b,c}^a z^b h^c \otimes z^c.$$

Since $\Delta(y_i) = g_i \otimes y_i + y_i \otimes 1$, and

$$\Delta^2(y_i) = g_i \otimes g_i \otimes y_i + g_i \otimes y_i \otimes 1 + y_i \otimes 1 \otimes 1,$$

we have for all $u \in U$ by (3.5)

$$\begin{aligned}(1 \otimes y_i) \cdot_\sigma (u \otimes 1) &= \varphi(u_{(1)})(g_i)u_{(2)} \otimes g_i\varphi(u_{(3)})(S^{-1}(y_i))\\ &\quad + \varphi(u_{(1)})(g_i)u_{(2)} \otimes y_i\varphi(u_{(3)})(1)\\ &\quad + \varphi(u_{(1)})(y_i)u_{(2)} \otimes 1\varphi(u_{(3)})(1).\end{aligned}$$

It follows from the definition of φ that

$$\varphi(x_{\beta_l})(g) = 0 \text{ for all } \beta_l \in \Phi_1^+, g \in \Gamma.$$

Hence to compute $(1 \otimes y_i) \cdot_\sigma (u \otimes 1)$ with $u = z_\beta$, we only need to take into account the term $g_\beta^N \otimes z_\beta \otimes 1$ of $\Delta^2(z_\beta)$, and we obtain

$$\begin{aligned}(1 \otimes y_i) \cdot_\sigma (u \otimes 1) &= \varphi(g_\beta^N)(y_{i(1)})z_\beta \otimes y_{i(2)}\varphi(1)(S^{-1}(y_{i(3)}))\\ &= \varphi(g_\beta^N)(y_{i(1)})z_\beta \otimes y_{i(2)}\\ &= \varphi(g_\beta^N)(g_i)z_\beta \otimes y_i + \varphi(g_\beta^N)(y_i)z_\beta \otimes 1\\ &= \chi_\beta^N(g_i)(x_\beta^N \otimes 1) \cdot_\sigma (1 \otimes y_i),\end{aligned}$$

since $\varphi(g_\beta^N) = \chi_\beta^N$ and $\varphi(g_\beta^N)(y_i) = 0$ by the definition of φ.

¿From (3.6) and (3.7) we see that for all simple roots $\alpha \in \Phi_K^+, K \in \mathcal{X}, K \neq I_1$ and all roots $\beta \in \Phi_J^+$ with $J = I_1$

$$x_\alpha x_\beta^{N_J} = \chi_\beta^{N_J}(g_\alpha)x_\beta^{N_J}x_\alpha \tag{3.8}$$

in $U(\mathcal{D}, \lambda)$. Since the root vectors x_α are homogeneous, (3.8) holds for all $\alpha \in \Phi_K^+, K \neq I_1$, and $\beta \in \Phi_{I_1}^+$. Since $U(\mathcal{D}, \lambda)$ and the root vectors $x_\alpha, \alpha \in \Phi^+$, do not depend on the order of the connected components, we can reorder the connected components and obtain (3.8) for all positive roots α, β lying in different connected components. For roots in the same connected component, (3.8) follows from Theorem 2.5. □

4. Finite-dimensional quotients

4.1. **A general criterion.** We need a generalization of Theorem [AS5, 6.24].

In this section, let Γ be an abelian group, A an algebra containing the group algebra $k[\Gamma]$ as a subalgebra and $p \geq 1$. We assume

$$y_1, \dots, y_p \in A, h_1, \dots, h_p \in \Gamma, \psi_1, \dots, \psi_p \in \widehat{\Gamma}, \text{ and } N_1, \dots, N_p \geq 1,$$

such that

$$gy_l = \psi_l(g) y_l g, \text{ for all } 1 \leq l \leq p, g \in \Gamma, \tag{4.1}$$

$$y_k y_l^{N_l} = \psi_l^{N_l}(h_k) y_l^{N_l} y_k, \text{ for all } 1 \leq k, l \leq p, \tag{4.2}$$

$$y_1^{a_1} \cdots y_p^{a_p} g, \ a_1, \cdots, a_p \geq 0, g \in \Gamma, \text{ form a basis of } A. \tag{4.3}$$

For all $a = (a_1, \dots, a_p) \in \mathbb{N}^p$, we define $y^a = y_1^{a_1} \cdots y_p^{a_p}$ and

$$\mathbb{L} = \{l = (l_1, \dots, l_p) \in \mathbb{N}^p \mid 0 \leq l_i < N_i \text{ for all } 1 \leq i \leq p\}.$$

Hence any element of $y \in A$ can be written as

$$y = \sum_{l \in \mathbb{L}, a \in \mathbb{N}^p} y^l y^{aN} w_{l,a}, \ w_{l,a} \in k[\Gamma] \text{ for all } l \in \mathbb{L}, a \in \mathbb{N}^p,$$

where the coefficients $w_{l,a} \in k[\Gamma]$ are uniquely determined. In [AS5] we assumed that $A = R\#k[\Gamma]$, and the subalgebra R of A generated by $y_1, \dots, y_p$ had the basis $y_1^{a_1} \cdots y_p^{a_p}, a_1, \dots, a_p \geq 0$. Hence for $y \in R$ we could assume that the $w_{l,a}$ were scalars.

Theorem 4.1. *Assume the situation above, and let $u_l \in k[\Gamma], 1 \leq l \leq p$. Then the following are equivalent:*

(1) *The residue classes of $y_1^{a_1} \cdots y_p^{a_p} g, \ a_1, \cdots, a_p \geq 0, g \in \Gamma$, form a basis of the quotient algebra $A/(y_l^{N_l} - u_l \mid 1 \leq l \leq p)$.*
(2) *For all $1 \leq l \leq p$, u_l is central in A, and if $\psi_l^{N_l} \neq \varepsilon$, then $u_l = 0$.*

Proof. As in [AS5] this follows from Lemma [AS5, 6.23]. To extend the proof of this Lemma to the more general case considered here, we use the following rule. Assume (2), and let $u^a = u_1^{a_1} \cdots u_p^{a_p}$, for all $a = (a_1, \dots, a_p) \in \mathbb{N}^p$. For all $1 \leq l \leq p$, let $\widetilde{\psi}_l : k[\Gamma] \to k[\Gamma]$ be the algebra isomorphism with $\widetilde{\psi}_l(g) = \psi_l(g) g$ for all $g \in \Gamma$. Then

$$u^a \widetilde{\psi}^{aN}(w) = u^a w, \text{ for all } w \in k[\Gamma], a \in \mathbb{N}^p, \tag{4.4}$$

where $\widetilde{\psi}^{aN} = \widetilde{\psi}_1^{a_1 N_1} \dots \widetilde{\psi}_p^{a_p N_p}$. □

4.2. **The Hopf algebra $u(\mathcal{D}, \lambda, \mu)$.** Let Γ be a finite abelian group, and $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i,j \leq \theta})$ a datum of finite Cartan type. We assume the situation of Section 3.1.

Definition 4.2. A family $\mu = (\mu_\alpha)_{\alpha \in \Phi^+}$ of elements in k is called a *family of root vector parameters for $\mathcal{D}$* if the following condition is satisfied for all $\alpha \in \Phi_J^+, J \in \mathcal{X}$: If $g_\alpha^{N_J} = 1$ or $\chi_\alpha^{N_J} \neq \varepsilon$, then $\mu_\alpha = 0$.

Let μ be a family of root vector parameters for $\mathcal{D}$. For all $J \in \mathcal{X}$, and $\alpha \in \Phi_J^+$, we define

$$\pi_J(\mu) = (\mu_\beta)_{\beta \in \Phi_J^+}, \text{ and } u_\alpha(\mu) = u_\alpha(\pi_J(\mu)), \tag{4.5}$$

where $u_\alpha(\pi_J(\mu))$ is introduced in Definition 2.13. Let λ be a family of linking parameters for $\mathcal{D}$. Then we define

$$u(\mathcal{D},\lambda,\mu) = U(\mathcal{D},\lambda)/(x_\alpha^{N_J} - u_\alpha(\mu) \mid \alpha \in \Phi_J^+, J \in \mathcal{X}). \tag{4.6}$$

By abuse of language we still write x_i and g for the images of x_i and $g \in \Gamma$ in $u(\mathcal{D},\lambda,\mu)$. For all $1 \le l \le p$, we define $N_l = N_J$, if $\beta_l \in \Phi_J^+, J \in \mathcal{X}$.

Lemma 4.3. *Let $\mathcal{D},\lambda$ and μ as above, and $\alpha \in \Phi^+$. Then $u_\alpha(\mu)$ is central in $U(\mathcal{D},\Lambda)$.*

Proof. Let $\alpha \in \Phi_J^+$, where $J \in \mathcal{X}$, and $N = N_J$. To simplify the notation, we assume $J = I_1 = \{1,2,\dots,\widetilde{\theta}\}$, and $\Phi_J^+ = \{\beta_1,\beta_2,\dots,\beta_{\widetilde{p}}\}$. We apply the results and notations of Section 2.2 to the connected component I_1. For all $a = (a_1,\dots,a_{\widetilde{p}}) \in \mathbb{N}^{\widetilde{p}}$, and $1 \le i \le \theta$, we will show that

$$\mu_a h^a x_i = \mu_a x_i h^a. \tag{4.7}$$

We can assume that $\mu_a \neq 0$. Let $1 \le l \le \widetilde{\theta}$, and $\beta_l = \sum_{j=1}^{\widetilde{\theta}} n_j \alpha_j$, where $n_j \in \mathbb{N}$ for all $1 \le j \le \widetilde{\theta}$. Then by definition, $g_{\beta_l} = \prod_{1\le j\le\widetilde{\theta}} g_j^{n_j}$, and $\chi_{\beta_l} = \prod_{1\le j\le\widetilde{\theta}} \chi_j^{n_j}$. Hence

$$\chi_i(g_{\beta_l}^N)\chi_{\beta_l}^N(g_i) = \prod_{1\le j\le\widetilde{\theta}} q_{ii}^{a_{ij}Nn_j} = 1,$$

since $q_{ii}^N = 1$, if $i \in I_1$, and $a_{ij} = 0$, if $i \notin I_1$. By Lemma 2.11, $\chi_{\beta_l}^N = \varepsilon$ for all $1 \le l \le \widetilde{\theta}$ with $a_l > 0$. Hence $\chi_i(g_{\beta_l}^N) = 1$ for all l with $a_l > 0$. This implies (4.7) since $h^a x_i = \chi_i(h^a) x_i h^a$.

Finally we prove by induction on $\mathrm{ht}(\underline{a})$ using (4.7) and (2.14) that u^a is central in $U(\mathcal{D},\lambda)$ (and in $k\langle x_1,\dots,x_\theta\rangle\#k[\Gamma]$). $\square$

Theorem 4.4. *Let $\mathcal{D}$ be a datum of finite Cartan type satisfying* (3.1) *and* (3.2)*. Let λ and μ be families of linking and root vector parameters for $\mathcal{D}$. Then $u(\mathcal{D},\lambda,\mu)$ is a quotient Hopf algebra of $U(\mathcal{D},\lambda)$ with group-like elements $G(u(\mathcal{D},\lambda,\mu)) \cong \Gamma$, and the elements*

$$x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \cdots x_{\beta_p}^{a_p} g,\ 0 \le a_l < N_l,\ 1 \le l \le p,\ g \in \Gamma$$

form a basis of $u(\mathcal{D},\lambda,\mu)$. In particular,

$$\dim u(\mathcal{D},\lambda,\mu) = \prod_{J\in\mathcal{X}} N_J^{|\Phi_J^+|} |\Gamma|.$$

Proof. By Theorem 3.3, the elements

$$x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \cdots x_{\beta_p}^{a_p} g,\ 0 \le a_l,\ 1 \le l \le p,\ g \in \Gamma$$

are a basis of $U(\mathcal{D},\lambda)$. We want to apply Theorem 4.1 with

$$y_l = x_{\beta_l},\ \psi_l = \chi_{\beta_l},\ u_l = u_{\beta_l}(\mu),\ 1 \le l \le p.$$

For each connected component $J \in \mathcal{X}$ we apply the results of Section 2.2 with

$$\eta_l = \chi_{\beta_l}^{N_l},\ 1 \le l \le p,\ \beta_l \in \Phi_J^+.$$

If $\chi_{\beta_l}^{N_l} \neq \varepsilon$ for some $1 \le l \le p, \beta_l \in \Phi_J^+$, then by assumption, $\mu_{\beta_l} = 0$, and by Lemma 2.11, $u_{\beta_l}(\mu) = 0$. By Lemma 4.7, $u_{\beta_l}(\mu)$ is central in $U(\mathcal{D},\lambda)$. Hence the claim concerning the basis of $u(\mathcal{D},\lambda,\mu)$ follows from Theorem 3.3 and Theorem 4.1.

We now show that $u(\mathcal{D},\lambda,\mu)$ is a Hopf algebra. Let $J \in \mathcal{X}$. We denote the restriction of $\mathcal{D}$ to the connected component J by $\mathcal{D}_J$. By Theorem 2.12, the map $\varphi_\mu : K(\mathcal{D}_J)\#k[\Gamma] \to k[\Gamma]$ is a Hopf algebra homomorphism. The kernel of φ_μ is generated by all $x_\alpha^{N_J} - u_\alpha(\mu), \alpha \in \Phi_J^+$. Hence the elements $x_\alpha^{N_J} - u_\alpha(\mu), \alpha \in \Phi_J^+$, generate a Hopf ideal in $K(\mathcal{D}_J)\#k[\Gamma]$ and in $U(\mathcal{D},\lambda)$.

The Hopf algebra $u(\mathcal{D},\lambda,\mu)$ is generated by the skew-primitive elements $x_1,\dots,x_\theta$ and the image of Γ. In particular, $G(u(\mathcal{D},\lambda,\mu)) \cong \Gamma$. $\square$

For explicit examples of the Hopf algebras $u(\mathcal{D},\lambda,\mu)$ see [AS5, Section 6] for type $A_n, n \ge 1$, and [BDR] for type B_2. In these papers, and for these types, the elements $u_\alpha(\mu)$ are precisely written down. An interesting problem is to find an explicit algorithm describing the $u_\alpha(\mu)$ for any connected Dynkin diagram.

5. The associated graded Hopf algebra

5.1. **Nichols algebras.** To determine the structure of a given pointed Hopf algebra, we proceed as in [AS1] and study the associated graded Hopf algebra.

Let A be a pointed Hopf algebra with group of group-like elements $G(A) = \Gamma$. Let

$$A_0 = k[\Gamma] \subset A_1 \subset \cdots \subset A,\ A = \cup_{n\ge 0} A_n$$

be the coradical filtration of A. We define the associated graded Hopf algebra [M, 5.2.8] by

$$\operatorname{gr}(A) = \oplus_{n\ge 0} A_n/A_{n-1},\ A_{-1} = 0.$$

Then $\operatorname{gr}(A)$ is a pointed Hopf algebra with the same dimension and coradical as A. The projection map $\pi : \operatorname{gr}(A) \to k[\Gamma]$ and the inclusion

$\iota : k[\Gamma] \to \mathrm{gr}(A)$ are Hopf algebra maps with $\iota\pi = \mathrm{id}_{k[\Gamma]}$. Let

$$R = \{x \in \mathrm{gr}(A) \mid (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\} \tag{5.1}$$

be the algebra of $k[\Gamma]$-coinvariant elements. Then $R = \oplus_{n\geq 0} R(n)$ is a graded Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, and by (1.7)

$$\mathrm{gr}(A) \cong R\# k[\Gamma]. \tag{5.2}$$

Let $V = P(R) \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ be the Yetter-Drinfeld module of primitive elements in R. We call its braiding

$$c : V \otimes V \to V \otimes V$$

the *infinitesimal braiding of* A.

Let $\mathfrak{B}(V)$ be the subalgebra of R generated by V. Thus $B = \mathfrak{B}(V)$ is the *Nichols algebra* of V [AS2], that is,

(5.3) $B = \oplus_{n\geq 0} B(n)$ is a graded Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$,

(5.4) $B(0) = k1,\ B(1) = V$,

(5.5) $B(1) = P(B)$,

(5.6) B is generated as an algebra by B(1).

$\mathfrak{B}(V)$ only depends on the vector space V with its Yetter-Drinfeld structure (see the discussion in [AS5, Section 2]). As an algebra and coalgebra, $\mathfrak{B}(V)$ only depends on the braided vector space (V, c).

We assume in addition that A is finite-dimensional and Γ is abelian. Then there are $g_1, \dots, g_\theta \in \Gamma$, $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ and a basis $x_1, \dots, x_\theta$ of V such that $x_i \in V_{g_i}^{\chi_i}$ for all $1 \leq i \leq \theta$. We call

$$(q_{ij} = \chi_j(g_i))_{1\leq i,j\leq\theta}$$

the *infinitesimal braiding matrix of* A.

The first step to classify pointed Hopf algebras is the computation of the Nichols algebra.

Using results of Lusztig [L1],[L2], Rosso [Ro] and Müller [M1] and twisting we proved in [AS4, Theorem 4.5] the following description of the Nichols algebra of Yetter-Drinfeld modules of finite Cartan type.

Theorem 5.1. *Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, (a_{ij})_{1\leq i,j\leq\theta})$ be a datum of finite Cartan type with finite abelian group Γ. Assume* (3.1) *and* (3.2)*. Let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ be a vector space with basis $x_1, \dots, x_\theta$ and $x_i \in V_{g_i}^{\chi_i}$ for all $1 \leq i \leq \theta$. Then $\mathfrak{B}(V)$ is the quotient algebra of $T(V)$ modulo the ideal generated by the elements*

(5.7) $\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j)$ *for all* $1 \leq i, j \leq \theta, i \neq j$,

(5.8) $x_\alpha^{N_J}$ *for all* $\alpha \in \Phi_J^+, J \in \mathcal{X}$.

Corollary 5.2. *Assume the situation of Theorem 5.1, and let λ and μ be linking and root vector parameters for $\mathcal{D}$. Then*

$$\operatorname{gr}(u(\mathcal{D},\lambda,\mu)) \cong u(\mathcal{D},0,0) \cong \mathfrak{B}(V)\#k[\Gamma].$$

Proof. Let $A = u(\mathcal{D},\lambda,\mu)$. There is a well-defined Hopf algebra map

$$u(\mathcal{D},0,0) \to \operatorname{gr}(u(\mathcal{D},\lambda,\mu)),$$

mapping $x_i, 1 \le i \le \theta$, onto the residue class of x_i in A_1/A_0, and $g \in \Gamma$ onto g. Since $\dim(u(\mathcal{D},0,0)) = \dim(u(\mathcal{D},\lambda,\mu)) = \dim(\operatorname{gr}(u(\mathcal{D},\lambda,\mu))$ by Theorem 4.4, it follows that $u(\mathcal{D},0,0) \cong \operatorname{gr}(u(\mathcal{D},\lambda,\mu))$. By Theorem 5.1, $u(\mathcal{D},0,0) \cong \mathfrak{B}(V)\#k[\Gamma]$. □

As an application of Corollary 5.2 we derive some information about isomorphisms between Hopf algebras of the form $u(\mathcal{D},\lambda,\mu)$.

Remark 5.3. Let Γ and Γ' be finite abelian groups, and

$$\begin{aligned}\mathcal{D} &= \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta}),\\ \mathcal{D}' &= \mathcal{D}(\Gamma', (g_i')_{1\le i\le\theta'}, (\chi_i')_{1\le i\le\theta'}, (a_{ij}')_{1\le i,j\le\theta'})\end{aligned}$$

data of finite Cartan type satisfying (3.1) and (3.2). Moreover we assume

$$q_{ii} = \chi_i(g_i) > 3 \text{ for all } 1 \le i \le \theta. \tag{5.9}$$

Let λ and λ' be linking parameters, and μ and μ' root vector parameters for $\mathcal{D}$ and $\mathcal{D}'$. We assume there is a Hopf algebra isomorphism

$$F : A = u(\mathcal{D},\lambda,\mu) \to A' = u(\mathcal{D}',\lambda',\mu').$$

Then F preserves the coradical filtration and induces an isomorphism $A_0 = k[\Gamma] \cong A_0' = k[\Gamma']$, given by a group isomorphism $\varphi : \Gamma \to \Gamma'$, and by Corollary 5.2 an isomorphism

$$A_1 = k[\Gamma] \oplus \bigoplus_{\substack{g\in\Gamma,\\ 1\le i\le\theta}} kx_i g \cong A_1' \oplus \bigoplus_{\substack{g'\in\Gamma',\\ 1\le i\le\theta'}} kx_i' g'.$$

Hence (see [AS2, 6.3]) $\theta = \theta'$, and there are a permutation $\rho \in S_\theta$ and elements $0 \ne s_i \in k, 1 \le i \le \theta$ such that for all $1 \le i \le \theta$,

$$\varphi(g_i) = g'_{\rho(i)}, \tag{5.10}$$

$$\chi_i = \chi'_{\rho(i)}\varphi, \tag{5.11}$$

$$F(x_i) = s_i x'_{\rho(i)}. \tag{5.12}$$

Note that the Nichols algebras $u(\mathcal{D},0,0)$ and $u(\mathcal{D}',0,0)$ are isomorphic if and only if $\theta = \theta'$, and there are $\varphi, \rho, (s_i)$ with (5.10),(5.11).

Let $q_{ij} = \chi_j(g_i)$, and $q'_{ij} = \chi'_j(g'_i)$, for all $1 \le i, j \le \theta$. Then it follows from (5.10), (5.11) and (5.9) that for all $1 \le i, j \le \theta$,

$$q_{ij} = q'_{\rho(i)\rho(j)}, \tag{5.13}$$

$$a_{ij} = a'_{\rho(i)\rho(j)}, \tag{5.14}$$

since $q_{ii}^{a_{ij}} = q_{ii}^{a'_{\rho(i)\rho(j)}}$, and $a_{ij} - a'_{\rho(i)\rho(j)} \in \{0, \pm 1, \pm 2, \pm 3\}$. We see from (5.13) that for all $1 \le i, j \le \theta$,

$$F([x_i, x_j]_c) = s_i s_j [x'_{\rho(i)}, x'_{\rho(j)}]_{c'}, \tag{5.15}$$

hence by the linking relations for all $1 \le i < j \le \theta, i \not\sim j$,

$$\lambda_{ij} = \begin{cases} s_i s_j \lambda'_{\rho(i)\rho(j)}, & \text{if } \rho(i) < \rho(j), \\ -s_i s_j \chi_j(g_i) \lambda'_{\rho(j)\rho(i)}, & \text{if } \rho(i) > \rho(j). \end{cases} \tag{5.16}$$

To obtain more precise results we now assume as in [AS5, 6.26] that for all $1 \le i, j \le \theta, i \ne j$,

$$\operatorname{ord}(g_i) = \operatorname{ord}(g'_i) \ne \operatorname{ord}(g_j) = \operatorname{ord}(g'_j). \tag{5.17}$$

This forces ρ to be the identity, and we can identify the root systems of $\mathcal{D}$ and $\mathcal{D}'$. Then

$$F(x_\alpha) = s_\alpha x'_\alpha \text{ for all } \alpha \in \Phi^+, \tag{5.18}$$

where we define $s_\alpha = s_1^{n_1} \cdots s_\theta^{n_\theta}$, if $\alpha = \sum_{i=1}^\theta n_i \alpha_i \in \Phi^+$. The root vector relations imply

$$s_\alpha^{N_J} u'_\alpha(\mu') = F(u_\alpha(\mu)) = u'_\alpha(\mu), \text{ for all } \alpha \in \Phi_J^+, J \in \mathfrak{X}. \tag{5.19}$$

It follows from the inductive definition of the $u_\alpha(\mu)$, that (5.18) is equivalent to

$$s_\alpha^{N_J} \mu'_\alpha = \mu_\alpha, \text{ for all } \alpha \in \Phi_J^+, J \in \mathfrak{X}. \tag{5.20}$$

Conversely these data allow to define a Hopf algebra isomorphism. Assuming (5.17) and $\theta = \theta'$, we conclude that $u(\mathcal{D}, \lambda, \mu)$ is isomorphic to $u(\mathcal{D}', \lambda', \mu')$ if and only if $a_{ij} = a'_{ij}$ for all $1 \le i, j \le \theta$, and there are scalars $0 \ne s_i \in k, 1 \le i \le \theta$, and a group isomorphism $\varphi : \Gamma \to \Gamma'$ satisfying

$$\varphi(g_i) = g'_i, \text{ for all } 1 \le i \le \theta \tag{5.21}$$

$$\chi_i = \chi'_i \varphi, \text{ for all } 1 \le i \le \theta \tag{5.22}$$

$$\lambda_{ij} = s_i s_j \lambda'_{ij}, \text{ for all } 1 \le i < j \le \theta, \tag{5.23}$$

$$s_\alpha^{N_J} \mu'_\alpha = \mu_\alpha, \text{ for all } \alpha \in \Phi_J^+, J \in \mathfrak{X}. \tag{5.24}$$

In [AS2] and [AS4] we determined the structure of finite-dimensional Nichols algebras assuming that V is of Cartan type and satisfies some more assumptions in the case of small orders (≤ 17) of the diagonal elements q_{ii}. Recent results of Heckenberger [H1], [H2], [H3] together with Theorem 5.1 allow to prove the following very general structure theorem on Nichols algebras.

Theorem 5.4. *Let Γ be a finite abelian group, and $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ a Yetter-Drinfeld module such that $\mathfrak{B}(V)$ is finite-dimensional. Choose a basis $x_i \in V$ with $x_i \in V^{\chi_i}_{g_i}, g_i \in \Gamma, \chi_i \in \widehat{\Gamma}$, for all $1 \leq i \leq \theta$. For all $1 \leq i, j \leq \theta$, define $q_{ij} = \chi_j(g_i)$, and assume*

$$\operatorname{ord}(q_{ij}) \text{ is odd, and } \operatorname{ord}(q_{ii}) \text{ is not } 3, \tag{5.25}$$

$$\operatorname{ord}(q_{ii}) \text{ is prime to 3 if } q_{il}q_{li} \in \{q_{ii}^{-3}, q_{ll}^{-3}\} \text{ for some } l. \tag{5.26}$$

Then there is a datum $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, (a_{ij})_{1\leq i,j\leq\theta})$ of finite Cartan type such that

$$\mathfrak{B}(V)\#k[\Gamma] \cong u(\mathcal{D}, 0, 0).$$

Proof. For all $1 \leq i, j \leq \theta, i \neq j$, let V_{ij} be the vector subspace of V spanned by x_i, x_j. Then $\mathfrak{B}(V_{kj})$ is isomorphic to a subalgebra of $\mathfrak{B}(V)$, hence it is finite-dimensional. Heckenberger [H1], [H2] classified finite-dimensional Nichols algebras of rank 2. By (5.25) it follows from the list in [H1, Theorem 4] that V_{ij} is of finite Cartan type, that is, there are $a_{ij}, a_{ji} \in \{0, -1, -2, -3\}$ with $a_{ij}a_{ji} \in \{0, 1, 2, 3\}$, and

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}} = q_{jj}^{a_{ji}}.$$

Since $\mathfrak{B}(V)\#k[\Gamma]$ is finite-dimensional, $q_{ii} \neq 1$ for all $1 \leq i \leq \theta$ by [AS1, Lemma 3.1]. Thus $(q_{ij})_{1\leq i,j\leq\theta}$ is of Cartan type in the sense of [AS2, page 4] with (generalized) Cartan matrix (a_{ij}). In [H3, Theorem 4] Heckenberger extended part (ii) of [AS2, Theorem 1.1] (where we had to exclude some small primes) and showed that a diagonal braiding (q_{ij}) of a braided vector space V is of finite Cartan type if it is of Cartan type and $\mathfrak{B}(V)$ is finite-dimensional. Hence (a_{ij}) is a Cartan matrix of finite type, and the claim follows from Theorem 5.1. □

5.2. **Generation in degree one.** We generalize our results in [AS4, Section 7]. Let A be a finite-dimensional pointed Hopf algebra with Γ, V, and R as in Section 5.1. To prove that $\mathfrak{B}(V) = R$, we dualize. Let $S = R^*$ the dual Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ as in [AS2, Lemma 5.5]. Then $S = \oplus_{n\geq 0}S(n)$ is a graded Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, and by [AS2, Lemma 5.5], R is generated in degree one, that is, $\mathfrak{B}(V) = R$, if and only $P(S) = S(1)$. The dual vector space $S(1)$ of $V = R(1)$ has the same braiding (q_{ij}) (with respect to the dual basis) as V. Our strategy to show $P(S) = S(1)$ is to identify S as a Nichols algebra. In the next Lemma we use [H1, H2] to prove a very general version of [AS4, Lemma 7.2].

Lemma 5.5. *Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$ be a datum of finite Cartan type with finite abelian group Γ. Let $S = \oplus_{n\ge 0} S_n$ be a finite-dimensional graded Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ with $S(0) = k1$, and let $x_1, \dots, x_\theta$ be a basis of $S(1)$ with $x_i \in S(1)^{\chi_i}_{g_i}$ for all $1 \le i \le \theta$. Assume for all $1 \le i \le \theta$ that the order of $q_{ii} = \chi_i(g_i)$ is odd and > 7. Then*

$$\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0 \text{ for all } 1 \le i, j \le \theta,\ i \ne j. \tag{5.27}$$

Proof. We first note that the Nichols algebra of the primitive elements $P(S) \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ is finite-dimensional. This can be seen by looking at $\mathrm{gr}(S\#k[\Gamma])$.

Assume that there are $1 \le i, j \le \theta, i \ne j$, with $\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) \ne 0$. We define

$$y_1 = x_1,\ y_2 = \mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j).$$

By [AS2, A.1], y_2 is a primitive element. Since y_1, y_2 are non-zero elements of different degree, they are linearly independent. We know that the Nichols algebra of $W = ky_1 + ky_2$ is finite-dimensional, since $B(P(S))$ is finite-dimensional. We denote

$$h_1 = g_i,\ h_2 = g_i^{1-a_{ij}} \in \Gamma, \text{ and } \eta_1 = \chi_i,\ \eta_2 = \chi_i^{1-a_{ij}}\chi_j \in \widehat{\Gamma}.$$

Thus $y_i \in S^{\eta_i}_{h_i}, 1 \le i \le 2$. Let $(Q_{ij} = \eta_j(h_i))_{1\le i,j\le 2}$ be the braiding matrix of y_1, y_2. We compute

$$Q_{11} = q_{ii},\ Q_{22} = q_{ii}^{1-a_{ij}} q_{jj},\ Q_{12}Q_{21} = q_{ii}^{2-a_{ij}}.$$

By assumption, the order of $Q_{11} = q_{ii}$ is odd and > 3. Since $B(W)$ is finite-dimensional, $Q_{22} \ne 1$ by [AS1, Lemma 3.1]. Thus Q_{22} has odd order, since the orders of q_{ii}, q_{jj} are odd. By checking Heckenberger's list in [H1, Theorem 4], and thanks to [H2], we see that the braiding (Q_{ij}) is of finite Cartan type or that we are in case (T3) with

$$Q_{12}Q_{21} = Q_{11}^{-1}.$$

Hence there exists $A_{12} \in \{0, -1, -2, -3\}$ with

$$Q_{12}Q_{21} = Q_{11}^{A_{12}}.$$

Since $Q_{12}Q_{21} = q_{ii}^{2-a_{ij}}$, and $Q_{11} = q_{ii}$, it follows that the order of q_{ii} divides $2 - a_{ij} - A_{12} \in \{2, 3, 4, 5, 6, 7, 8\}$. This is a contradiction since the order of q_{ii} is odd and > 7. $\square$

The next theorem is one of the main results of this paper.

Theorem 5.6. *Let A be a finite-dimensional pointed Hopf algebra with abelian group $G(A) = \Gamma$ and infinitesimal braiding matrix $(q_{ij})_{1\le i,j\le\theta}$. Assume for all $1 \le i, j \le \theta$, that the order of q_{ij} is odd, the order of q_{ii} is > 7, and that (5.26) holds. Then A is generated by group-like and skew-primitive elements, that is,*

$$R = \mathfrak{B}(V),$$

where R is defined by (5.1), and $V = R(1)$.

Proof. We argue as in the proof of [AS4, Theorem 7.6]. Let $S = R^*$ be the dual Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Then $S(1) = R(1)^*$ has the same braiding (q_{ij}) as $R(1)$ with respect to the dual basis (x_i) of the corresponding basis of $R(1)$. By Theorem 5.4 (q_{ij}) is of finite Cartan type. By Lemma 5.5 the Serre relations (5.7) hold for the elements x_i. Then the root vector relations (5.8) follow by [AS4, Lemma 7.5]. Hence $S \cong \mathfrak{B}(S(1))$ by Theorem 5.1, and $S(1) = P(S)$. By duality, R is a Nichols algebra. □

6. Lifting

¿From Section 5 we know a presentation of $\mathrm{gr}(A)$ by generators and relations under the assumptions of Theorems 5.4 and 5.6. To lift this presentation to A we need the following formulation of [AS1, Lemma 5.4] which is a consequence of the theorem of Taft and Wilson [M, Theorem 5.4.1]. Here it is crucial that the group is abelian.

Lemma 6.1. *Let A be a finite-dimensional pointed Hopf algebra with abelian group $G(A) = \Gamma$. Write $\mathrm{gr}(A) \cong R\#k[\Gamma]$ as in (5.2), and let $V = R(1)$ with basis $x_i \in V_{g_i}^{\chi_i}, g_i \in \Gamma, \chi_i \in \widehat{\Gamma}, 1 \le i \le \theta$. Let $A_0 \subset A_1$ be the first two terms of the coradical filtration of A. Then*

(6.1) $\quad \oplus_{g,h\in\Gamma,\varepsilon\neq\chi\in\widehat{\Gamma}}\, P_{g,h}^{\chi}(A) \xrightarrow{\cong} A_1/A_0 \xleftarrow{\cong} V\#k[\Gamma].$

(6.2) $\quad$ *For all* $g \in \Gamma, P_{g,1}(A)^{\varepsilon} = k(1-g)$, *and if* $\varepsilon \neq \chi \in \widehat{\Gamma}$, *then*

(6.3) $\quad P_{g,1}(A)^{\chi} \neq 0 \iff g = g_i, \chi = \chi_i,$ *for some* $1 \le i \le \theta$.

We can now prove our main structure theorem.

Theorem 6.2. *Let A be a finite-dimensional pointed Hopf algebra with abelian group $G(A) = \Gamma$ and infinitesimal braiding matrix $(q_{ij})_{1\le i,j\le\theta}$. Assume for all $1 \le i, j \le \theta$, that the order of q_{ij} is odd, the order of q_{ii} is > 7, and that (5.26) holds.Then*

$$A \cong u(\mathcal{D}, \lambda, \mu),$$

where $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$ is a datum of finite Cartan type, and λ and μ are families of linking and root vector parameters for $\mathcal{D}$.

Proof. By Theorems 5.4 and 5.6, there is a datum $\mathcal{D}$ of finite Cartan type such that $\operatorname{gr}(A) \cong u(\mathcal{D}, 0, 0)$. By Lemma 6.1, for all $1 \le i \le \theta$ we can choose

$$a_i \in P(A)^{\chi_i}_{g_i,1} \text{ corresponding to } x_i \text{ in (6.1).}$$

We have shown in Theorem [AS4, 6.8] that

$$\operatorname{ad}_c(a_i)^{1-a_{ij}}(a_j) = 0, \text{ for all } 1 \le i, j \le \theta, i \sim j, i \ne j,$$
$$a_i a_j - q_{ij} a_j a_i - \lambda_{ij}(1 - g_i g_j) = 0, \text{ for all } 1 \le i < j \le \theta, \ i \not\sim j,$$

for some family λ of linking parameters. Thus there is a homomorphism of Hopf algebras

$$\varphi : U(\mathcal{D}, \lambda) \to A, \ \varphi|\Gamma = \operatorname{id}_\Gamma, \ \varphi(x_i) = a_i, \text{ for all } 1 \le i \le \theta.$$

By Theorem 5.6, φ is surjective.

We now use the notation of Section 2.2 and show that

$$\varphi(x_\alpha^{N_J}) \in k[\Gamma] \text{ for all } \alpha \in \Phi_J^+, J \in \mathcal{X}. \tag{6.4}$$

We fix $J \in \mathcal{X}$ with $p = |\Phi_J^+|$, and show by induction on $\operatorname{ht}(\underline{a})$ that

$$\varphi(z^a) \in k[\Gamma] \text{ for all } a \in \mathbb{N}^p. \tag{6.5}$$

Let $0 \ne a \in \mathbb{N}^p$. Since φ is a Hopf algebra map, we see from (2.16) that

$$\Delta(\varphi(z^a)) = h^a \otimes \varphi(z^a) + \varphi(z^a) \otimes 1 + w,$$

where by induction

$$w = \sum_{b,c \ne o, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \varphi(z^b) h^c \otimes \varphi(z^c) \in k[\Gamma] \otimes k[\Gamma].$$

In particular, $\varphi(z^a) \in A_1$ by definition of the coradical filtration. We multiply this equation with $g \otimes g, g \in \Gamma$, from the left and $g^{-1} \otimes g^{-1}$ from the right. Since $g z^a g^{-1} = \eta^a(g) z^a$, we obtain $w = \eta^a(g) w$ for all $g \in \Gamma$.

Suppose $\eta^a \ne \varepsilon$. Then $w = 0$, and $\varphi(z^a) \in P^{\eta^a}_{h^a,1}$. Then $\varphi(z^a) = 0$ by Lemma 6.1 (6.3), since $\chi_l(g_l) \ne 1$ for all $1 \le l \le \theta$, but $\eta^a(h^a) = 1$ by the Cartan condition (see the proof of [AS2, Lemma 7.5] for a similar computation).

If $\eta^a = \varepsilon$, then $\varphi(z^a) \in A_1^\varepsilon = k[\Gamma]$ by Lemma 6.1 (6.2).

This proves (6.5) and (6.4). Then we conclude for each $J \in \mathcal{X}$ from Theorem 2.12 that the map

$$K(\mathcal{D}_J) \# k[\Gamma] \to U(\mathcal{D}, \lambda) \xrightarrow{\varphi} A$$

has the form φ_{μ^J} for some family of scalars μ^J as in Theorem 2.12 for the connected component J. Define $\mu = (\mu_\alpha)_{\alpha \in \Phi^+}$ by $\mu_\alpha = \mu^J_\alpha$

for all $\alpha \in \Phi_J^+$. Then μ is a family of root vector parameters for $\mathcal{D}$, and the elements $u_\alpha(\mu) \in k[\Gamma]$ are defined in (4.5) for each $J \in \mathcal{X}$ and $\alpha \in \Phi_J^+$. It follows that $\varphi(x_\alpha^{N_J}) = u_\alpha(\mu) = \varphi(u_\alpha(\mu))$ for all $J \in \mathcal{X}, \alpha \in \Phi_J^+$. Thus φ factorizes over $u(\mathcal{D}, \lambda, \mu)$. Since $\dim(A) = \dim(\operatorname{gr}(A)) = \dim(u(\mathcal{D}, \lambda, 0, 0)) = \dim(u(\mathcal{D}, \lambda, \mu))$ by Theorem 4.4, φ induces an isomorphism $u(\mathcal{D}, \lambda, \mu) \cong A$. $\square$

Corollary 6.3. *Let A be a finite-dimensional pointed Hopf algebra with abelian group $G(A) = \Gamma$ satisfying the assumptions of Theorem 6.2. Then for each prime divisor p of the dimension of A there is a group-like element of order p in A.*

Proof. This follows from Theorems 6.2 and 4.4. $\square$

We note that the analog of Cauchy's theorem in group theory is false for arbitrary, non-pointed Hopf algebras. Let A be a finite-dimensional Hopf algebra with only trivial group-like elements, such as the dual of the group algebra of a finite group G with $G = [G, G]$. Then A does not contain any Hopf subalgebra of prime dimension, since any Hopf algebra of prime dimension is a group algebra by Zhu's theorem [Z].

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