REALIZATION OF EQUIVARIANT CHAIN COMPLEXES

BERNHARD HANKE

ABSTRACT. We discuss a question appearing in a recent article by A. Sikora [4] concerning the vanishing of certain differentials in the Leray-Serre spectral sequence for a Poincaré duality space equipped with a \( \mathbb{Z}/p \)-action.

Let \( p \) be an odd prime and let \( X \) be a finite dimensional connected \( \mathbb{Z}/p \)-CW complex which fulfills Poincaré duality over \( \mathbb{F}_p \), the field with \( p \) elements. By definition, this means that \( H^*(X; \mathbb{F}_p) \) is finitely generated over \( \mathbb{F}_p \), there is a natural number \( n \geq 0 \) and an element \( \nu \in H_n(X; \mathbb{F}_p) \) such that the map

\[
H^i(X; \mathbb{F}_p) \rightarrow H_{n-i}(X; \mathbb{F}_p), \ c \mapsto c \cap \nu,
\]

is an isomorphism for all \( i \in \mathbb{Z} \). Let \( g \in \mathbb{Z}/p \) be a fixed generator. With the induced \( \mathbb{Z}/p \)-operation, \( H^*(X; \mathbb{F}_p) \) is a graded \( \mathbb{F}_p[\mathbb{Z}/p] \)-module and as such it has a direct sum decomposition

\[
H^*(X; \mathbb{F}_p) \cong V_1^* \oplus V_2^* \oplus \ldots \oplus V_p^*,
\]

where each \( V_i^* \) is a free graded module over \( \mathbb{F}_p[\xi]/(1 - \xi)^i \) and multiplication by \( \xi \) corresponds to multiplication by \( g \). In his recent article [4], A. Sikora discusses the following question:

**Question** (cf. [4], remarks following Theorem 1.4.) Let \( V_2^* = V_3^* = \ldots = V_{p-1}^* = 0 \) and let \( \mathbb{Z}/p \) act on \( X \) with nonempty fixed point set. Consider the cohomological Leray-Serre spectral sequence with coefficients \( \mathbb{F}_p \) for the Borel fibration

\[
X \hookrightarrow E\mathbb{Z}/p \times_{\mathbb{Z}/p} X \rightarrow B\mathbb{Z}/p.
\]

Do all the differentials \( d_r : E_r^{i,j} \rightarrow E_r^{i+r,j-r+1} \) in this spectral sequence vanish, if \( i \geq n \) and if \( r \) is odd and greater than 1?

We will show by an explicit example that this is false in general. In a first step, we construct our example on an algebraic level as a certain equivariant chain complex. In a second step, we realize this chain complex \( p \)-locally as...
the cellular chain complex of a $\mathbb{Z}/p$-CW complex which is thickened up and doubled in order to get a smooth $\mathbb{Z}/p$-manifold. The idea of this approach might be of independent interest for the construction of other equivariant spaces with prescribed homological properties.

Theorem 3 shows that a modified version of the above question can be answered affirmatively. Related to this observation are the results in [1].

Consider the following chain complex $C_\ast$ of $\mathbb{Z}[\mathbb{Z}/p]$-modules:

\[
\mathbb{Z}[\mathbb{Z}/p] \xrightarrow{\nu} \mathbb{Z}[\mathbb{Z}/p] \xrightarrow{\tau} \mathbb{Z}[\mathbb{Z}/p] \xrightarrow{\tau \circ \nu} \mathbb{Z}[\mathbb{Z}/p] \xrightarrow{\tau} \mathbb{Z}[\mathbb{Z}/p] \to 0 \to 0 \to 0 \to \mathbb{Z}.
\]

The map $\nu$ is multiplication by $1 + g + \ldots + g^{p-1}$ and $\tau$ is multiplication by $1 - g$. We regard this chain complex as being graded over the natural numbers, the modules occuring above sitting in degrees $10, 9, \ldots, 0$, with differentials of degree $-1$ (and $C_\ast$ being completed by zero modules in degrees larger than 10). Because $\tau \circ \nu = \nu \circ \tau = 0$, we see that $C_\ast$ is indeed a chain complex. In a first step, we realize this complex as the cellular chain complex of a $\mathbb{Z}/p$-CW complex. As we are going to work with $\mathbb{F}_p$-coefficients later on, we consider the problem $p$-locally. We denote by $\mathbb{Z}_{(p)}$ the integers localized at $p$.

**Proposition 1.** There is a 10-dimensional $\mathbb{Z}/p$-CW complex $Y$ whose equivariant cellular chain complex with coefficients $\mathbb{Z}_{(p)}$ is $\mathbb{Z}_{(p)}[\mathbb{Z}/p]$-isomorphic to $C_\ast \otimes \mathbb{Z}_{(p)}$.

**Proof.** The space $Y$ is constructed inductively, starting with the one point union of $p$ spheres of dimension 5 permuted cyclically by the action of $\mathbb{Z}/p$ (and with fixed common basepoint). Suppose that $5 \leq k \leq 9$ and that the $k$-skeleton $Y^{(k)}$ of $Y$ has been constructed. We have to show that given an element $c \in H_k(Y^{(k)}; \mathbb{Z}_{(p)})$, there is a map $S^k \to Y^{(k)}$ which, in homology, maps the fundamental class of $S^k$ to $\lambda \cdot c$, where $\lambda$ is an integer not divisible by $p$. Then a bunch of free $\mathbb{Z}/p$-cells of dimension $k + 1$ can be attached equivariantly to $Y^{(k)}$ according to the respective differential

\[
C_{k+1} \otimes \mathbb{Z}_{(p)} \to C_k \otimes \mathbb{Z}_{(p)}
\]

(up to a $\mathbb{Z}_{(p)}[\mathbb{Z}/p]$-linear automorphism of $C_{k+1} \otimes \mathbb{Z}_{(p)})$.

In order to achieve this aim, it is enough to show that the $p$-local Hurewicz map

\[
\pi_k(Y^{(k)}) \otimes \mathbb{Z}_{(p)} \to H_k(Y^{(k)}; \mathbb{Z}_{(p)})
\]

is surjective. For $k \leq 8$, we use the fact that in the Atiyah-Hirzebruch spectral sequence

\[
E_{i,j}^2 = \widetilde{H}_i(Y^{(k)}; \pi_j^p \otimes \mathbb{Z}_{(p)})) \implies \pi_{i+j}^p(Y^{(k)}) \otimes \mathbb{Z}_{(p)}
\]


converging to the $p$-local stable homotopy of $Y^{(k)}$, the terms $E^2_{i,j}$ vanish for $j = 1, 2$ as $p$ is odd. Hence, all elements in $E^2_{i,0}$ with $i \leq 8$ are permanent cocycles (recall that $Y^{(k)}$ is $4$-connected). Freudenthal’s suspension theorem shows that the canonical map

$$\pi_k(Y^{(k)}) \rightarrow \pi_k^s(Y^{(k)})$$

is surjective. This map remains surjective after tensoring with $\mathbb{Z}/p$ and the desired surjectivity of the $p$-local Hurewicz map above is established.

If $p > 3$ and $k = 9$, the same argument completes the construction of $Y$, because $E^2_{i,j} = 0$ for $j = 3$ in this case (and Freudenthal’s suspension theorem still gives a surjection from the unstable to the stable homotopy of $Y^{(9)}$ in degree 9). However, because $\pi_3 \cong \mathbb{Z}/24$, in the case that $p = 3$, we must show that $Y^{(9)}$ can be constructed in such a way that the fourth differential $d^4$ vanishes on $E^4_{9,5}(Y^{(9)})$.

The equivariant map $\sigma : Y^{(3)} = S^5 \vee S^5 \vee S^5 \rightarrow S^5$ which is the identity on each copy of $S^5$ (and with the trivial action on the target $S^5$) extends to an equivariant map

$$Y^{(6)} = (S^5 \vee S^5 \vee S^5) \cup_{\phi} (D^6 \cup D^6 \cup D^6) \rightarrow S^5$$

because $\sigma \circ \phi : S^5 \cup S^5 \cup S^5 \rightarrow S^5$ is homotopic to a constant map. We call this extended map $\sigma$, as well. The composition of $\sigma$ with the inclusion $S^5 \rightarrow S^5 \vee S^5 \vee S^5 \subseteq Y^{(6)}$ of any $S^5$ summand is the identity. Because 3-locally the homotopy groups $\pi_6(S^5) = \pi_7(S^5) = 0$, one sees that $\sigma$ extends to a 3-local equivariant map $\sigma : Y^{(8)} \rightarrow S^5$. We have $H_8(Y^{(8)}; \mathbb{Z}/3) \cong \mathbb{Z}/3$ and the $E_\infty$-term of the Atiyah-Hirzebruch spectral sequence converging to the stable homotopy of $Y^{(8)}$ (always localized at 3) leads to a short exact sequence

$$0 \rightarrow \mathbb{Z}/3 \rightarrow \pi_8^s(Y^{(8)}) \rightarrow \mathbb{Z}/3 \rightarrow 0.$$ 

Hence there is an isomorphism

$$f : \pi_8(Y^{(8)}) \cong \pi_8^s(Y^{(8)}) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3,$$

(the first isomorphism is the Freudenthal suspension theorem, again), but the choice of $f$ is not canonical. Nevertheless, with respect to any such $f$, the $\mathbb{Z}/3$-summand is mapped isomorphically to $H_8(Y^{(8)}) \cong \mathbb{Z}/3$ under the Hurewicz map - this follows from the fact that the Hurewicz map is represented by an edge homomorphism in the Atiyah Hirzebruch spectral sequence.

Let $x = f^{-1}((0, 1)) \in \pi_8(Y^{(8)})$ and $i : S^5 \rightarrow Y^{(6)} \hookrightarrow Y^{(8)}$ be chosen such that $\sigma \circ i$ is the identity. The image of $i_* : \pi_8(S^5) \rightarrow \pi_8(Y^{(8)})$ is in the kernel of the Hurewicz map, because $\text{im } i \subseteq Y^{(3)}$. In particular, we find an element $c \in \pi_8(Y^{(8)})$ which is in the kernel of the Hurewicz map and such that $\sigma_*(c) = -\sigma_*(x) \in \pi_8(S^5)$. The element $x + c \in \pi_8(Y^{(8)})$
is then in the kernel of $\sigma_*$ and is sent to a generator of $H_8(Y^{(8)})$ under the Hurewicz map. Thus, we can attach the first 9-cell to $Y^{(8)}$ in such a way that the composition of the attaching map with $\sigma$ is null homotopic. By equivariance of $\sigma$, the composition with $\sigma$ of the attaching maps of the other two 9-cells are null homotopic as well and $\sigma$ extends (3-locally) to a map $Y^{(9)} \to S^5$ which factors the identity $S^5 \to S^5$. By naturality of the Atiyah-Hirzebruch spectral sequence, this shows that for this $Y^{(9)}$, we have indeed $d^i = 0$ on $E^i_{9,0}$.

Note that $\mathbb{Z}/p$ acts on $Y$ with exactly one fixed point. Let $T$ be an oriented compact smooth $\mathbb{Z}/p$-manifold with boundary which is $\mathbb{Z}/p$-homotopy equivalent to $Y$ (for the construction of such an equivariant smooth thickening, see, for example, [3] Theorem 2.4 and Remark 2.5 with $B = Y^{\mathbb{Z}/p}$ and $U$ and $E$ product bundles). Now define $X = T \cup_{\partial T} (-T)$ as the oriented double of $T$. The space $X$ is an oriented closed smooth $\mathbb{Z}/p$-manifold and in particular satisfies Poincaré duality over $\mathbb{F}_p$. By use of Poincaré duality for $T$ and excision, the long exact cohomology sequence of the pair $(X, T)$ becomes

$$\ldots \to H^{i-1}(T; \mathbb{F}_p) \to H_{\dim T-i}(T; \mathbb{F}_p) \to H^i(X; \mathbb{F}_p) \to H^i(T; \mathbb{F}_p) \to \ldots$$

One sees (at least, if $\dim T \geq 22$ which we can assume) that $H^i(X; \mathbb{F}_p) \cong \mathbb{F}_p$, if $i = 0, 5, 7, 8, 10, n - 10, n - 8, n - 7, n - 5, n$, where $n = \dim X$, and $H^i(X; \mathbb{F}_p) = 0$ for all other values of $i$. In particular, the induced $\mathbb{Z}/p$-action on $H^*(X; \mathbb{F}_p)$ is trivial. Let $E^{*,*}_3$ be the spectral sequence for the Borel fibration $X \hookrightarrow E\mathbb{Z}/p \times_{\mathbb{Z}/p} X \to B\mathbb{Z}/p$ with coefficients in $\mathbb{F}_p$. The following theorem shows that $X$ can be used in order to answer Sikora’s question in the negative.

**Theorem 2.** The third differential $d_3 : E^{i,i}_3 \to E^{i+i,2}_3$ is different from zero for all even $i \geq 2$.

**Proof.** Because $T$ is an equivariant retract of $X$ and $T$ is $\mathbb{Z}/p$-homotopy equivalent to $Y$, we only need to show the latter statement for the spectral sequence of the Borel construction for $Y$, which we denote by the same symbol $E^{*,*}_r$. For $r \geq 2$, this is a bigraded module over

$$H^*(\mathbb{Z}/p; \mathbb{F}_p) \cong \mathbb{F}_p[t] \otimes \Lambda(s)$$

where $s$ and $t$ are considered as indeterminates of bidegree $(1, 0)$ and $(2, 0)$ respectively and where $\Lambda(s)$ is the exterior algebra on $s$. Furthermore, the differential on $E^{*,*}_r$ is $\mathbb{F}_p[t] \otimes \Lambda(s)$-linear. In the following, we abbreviate $E\mathbb{Z}/p \times_{\mathbb{Z}/p} Y$ by $Y_{\mathbb{Z}/p}$, and take coefficients in $\mathbb{F}_p$, throughout. By use of the localization theorem, we have

$$H^*(Y_{\mathbb{Z}/p})[t^{-1}] \cong H^*(Y^{\mathbb{Z}/p} \times_{\mathbb{Z}/p} E\mathbb{Z}/p)[t^{-1}] \cong \mathbb{F}_p[t, t^{-1}] \otimes \Lambda(s),$$
because we have exactly one fixed point. It is now convenient to localize the Leray-Serre spectral sequence right away: For \( r \geq 2 \), we set \( \overline{E}_r^{*,*} = E_r^{*,*}[t^{-1}] \) and denote the induced differential on this localized spectral sequence (living in the first two quadrants) by \( \overline{d}_r \).

By induction on \( r \geq 2 \), it is not difficult to show that the map
\[
E_r^{i,j} \rightarrow E_r^{i+2,j}
\]
given by multiplication with \( t \), is a surjection, if \( 0 \leq i < r - 1 \), and an isomorphism, if \( i \geq r - 1 \). In particular, the canonical map
\[
E_3^{i,j} \rightarrow \overline{E}_3^{i,j}
\]
is an isomorphism, if \( i \geq 2 \). Hence, it suffices to show that \( \overline{d}_3^{k,*} \neq 0 \) for all even \( i \).

Recall that by construction of the Leray Serre spectral sequence, there is a decreasing filtration
\[
\ldots \supset \mathcal{F}_{\gamma - 1} H^*(Y_{\mathbb{Z}/p}) \supset \mathcal{F}_{\gamma} H^*(Y_{\mathbb{Z}/p}) \supset \mathcal{F}_{\gamma + 1} H^*(Y_{\mathbb{Z}/p}) \supset \ldots
\]
such that
\[
E_\infty^{i,j} \cong \mathcal{F}_i H^{i+j}(Y_{\mathbb{Z}/p}) / \mathcal{F}_{i+1} H^{i+j}(Y_{\mathbb{Z}/p}) .
\]
As in [2], we now define an induced filtration on the localized module \( H^*(Y_{\mathbb{Z}/p})[t^{-1}] \) as follows:
\[
x \in \mathcal{F}_\gamma \left( H^*(Y_{\mathbb{Z}/p})[t^{-1}] \right) \iff t^c \cdot x \in \mathcal{F}_{\gamma + 2c} H^{*-2c}(Y_{\mathbb{Z}/p}) \text{ for } c \gg 0 .
\]
This makes sense, because, using the remarks from the preceding paragraph, multiplication by \( t \) induces isomorphisms
\[
\mathcal{F}_\gamma H^*(Y_{\mathbb{Z}/p}) \cong \mathcal{F}_{\gamma + 1} H^{*-2}(Y_{\mathbb{Z}/p})
\]
if \( \gamma \geq 11 \), because \( E_\infty^{5,*} = E_\infty^{*,*} \) for dimension reasons. It follows that
\[
\overline{E}_\infty^{i,j} \cong \mathcal{F}_i \left( H^{i+j}(Y_{\mathbb{Z}/p})[t^{-1}] \right) / \mathcal{F}_{i+1} \left( H^{i+j}(Y_{\mathbb{Z}/p})[t^{-1}] \right) .
\]
Hence, because \( H^*(Y_{\mathbb{Z}/p})[t^{-1}] \) is a module of rank 2 over the graded field \( \mathbb{F}_p[t, t^{-1}] \), the same must be true for \( \overline{E}_\infty^{*,*} \). We will show that this cannot hold, if \( \overline{d}_3^{k,*} = 0 \) for some even \( i \).

In [2], Theorem 1, we constructed operators \( \Gamma_{1,r} : E_r^{*,*} \rightarrow E_r^{*,*+1} \) for \( r \geq 1 \) that, for \( r = 1 \), can be identified with the Bockstein operator \( \beta \) on \( H^*(Y; \mathbb{F}_p) \) associated to the short exact coefficient sequence
\[
0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p \rightarrow 0 .
\]
Because these operators \( \Gamma_{1,r} \) act as derivations, they are \( \mathbb{F}_p[t] \)-linear (for \( r \geq 2 \)) and induce corresponding operators on \( \overline{E}_r^{*,*} \), \( r \geq 2 \). We also cite the fact that these operators commute with \( \overline{d}_r \) up to sign. Notice that
\[ \beta(H^i(Y; \mathbb{F}_p)) = 0 \text{ for } i \neq 7 \text{ and } \beta : H^7(Y; \mathbb{F}_p) \to H^8(Y; \mathbb{F}_p) \text{ is an isomorphism (this explains our choice of the chain complex } C_\ast). \]

Using the operator \( \Gamma_{1,2} \), we get \( \overline{d}_2 = 0 \) by a simple diagram chase and therefore

\[ \overline{E}^{i,j}_3 \cong H^j(Y; \mathbb{F}_p) \otimes (\mathbb{F}_p[t, t^{-1}] \otimes \Lambda(s))^i. \]

We now assume that \( \overline{d}_3^{i,*} = 0 \) for some even \( i \). We know that \( \overline{E}_3^{i,*} \) is two-periodic in the horizontal direction and so our assumption implies that \( \overline{d}_3^{i,*} \) vanishes for all even \( i \). Because \( \overline{d}_3 \) has odd horizontal degree, commutes with \( s \) up to sign and \( s \cdot s = 0 \), the preceding isomorphism shows that \( \overline{d}_3^{i,*} = 0 \) for all odd \( i \), as well. Hence we get \( \overline{d}_3 = 0 \). For \( j = 0, 5, 7 \) the differential \( \overline{d}_4^{i,j} \) vanishes for dimension reasons. But also \( \overline{d}_4^{8} \) vanishes by commutativity of the diagram

\[
\begin{array}{ccc}
\overline{E}_4^{i,7} & \xrightarrow{\overline{d}_4} & \overline{E}_4^{i,4,4} \\
\downarrow \Gamma_{1,4} & & \downarrow \Gamma_{1,4} \\
\overline{E}_4^{i,8} & \xrightarrow{\overline{d}_4} & \overline{E}_4^{i,4,5}
\end{array}
\]

where the first vertical arrow is an isomorphism. In a similar way, the differential \( \overline{d}_4^{10} \) is zero.

For dimension reasons, we have \( \overline{d}_5 = 0 \). Because the \( \mathbb{Z}/p \)-operation on \( Y \) has a fixed point, the projection map \( Y_{\mathbb{Z}/p} \to B\mathbb{Z}/p \) has a section and thus factors the identity \( B\mathbb{Z}/p \to B\mathbb{Z}/p \). By comparing the localized spectral sequences for \( Y_{\mathbb{Z}/p} \) and \( *_{\mathbb{Z}/p} = B\mathbb{Z}/p \), this implies that the differential \( \overline{d}_r : \overline{E}_r^{i,r-1} \to \overline{E}_r^{i+r,0} \) vanishes for all \( r \). Altogether, we get

\[ \dim_{\mathbb{F}_p[t, t^{-1}]} \overline{E}_\infty^{i,*} \geq 6 \]

because the only possibly nonzero differential is \( \overline{d}_6 : \overline{E}_6^{i,10} \to \overline{E}_6^{i+6,5} \). However, this contradicts the above calculation of this dimension. \( \square \)

**Theorem 3.** Let \( X \) be a \( \mathbb{Z}/p \)-CW complex with finitely generated cohomology over \( \mathbb{F}_p \) in every degree. Assume that in the decomposition of \( H^\ast(X; \mathbb{F}_p) \) described at the beginning of this note, we have \( V_{p-1} = 0 \) and that \( H^\ast(X; \mathbb{Z}_{(p)}) \) does not contain \( \mathbb{Z}/p \) as a direct summand. Then, in the localized spectral sequence \( \overline{E}_r^{i,*} \) with \( \mathbb{F}_p \)-coefficients for the Borel construction of \( X \), the differential \( \overline{d}_r \) vanishes, if \( r \) is odd and \( r > 1 \).

**Proof.** The Bockstein operator \( \beta \) is zero on \( H^\ast(X; \mathbb{F}_p) \), therefore, by [2] Proposition 9 (which holds for any \( \mathbb{Z}/p \)-CW complex with vanishing Bockstein), each \( \overline{E}_r^{i,*} \) is free over \( \mathbb{F}_p[t, t^{-1}] \otimes \Lambda(s) \). The universal coefficient
sequence
\[ 0 \to H^*(X; \mathbb{Z}/p) \otimes \mathbb{Z}/p^2 \to H^*(X; \mathbb{Z}/p^2) \to H^{*+1}(X; \mathbb{Z}/p) \ast \mathbb{Z}/p^2 \to 0 \]
shows that \( H^*(X; \mathbb{Z}/p^2) \) is free over \( \mathbb{Z}/p^2 \). Because \( \mathbb{Z}/p \) is not a direct summand of \( H^*(X; \mathbb{Z}/p(\mathfrak{m})) \), the modules on the left and on the right are free over \( \mathbb{Z}/p^2 \). By the vanishing of the Bockstein operator, the short exact coefficient sequence
\[ 0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0 \]
induces a short exact sequence
\[ 0 \to H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{Z}/p^2) \to H^*(X; \mathbb{F}_p) \to 0. \]
The image of the second map consists of elements that are divisible by \( p \) as \( H^*(X; \mathbb{Z}/p^2) \) is free over \( \mathbb{Z}/p^2 \). Consequently, we get an induced isomorphism \( H^*(X; \mathbb{Z}/p^2) \otimes \mathbb{F}_p \cong H^*(X; \mathbb{F}_p) \). Proposition 6 in [2] (or a little representation theory) now implies that \( V_2^* = \ldots = V_{p-2}^* = 0 \) in the decomposition of \( H^*(X; \mathbb{F}_p) \). Together with the assumption \( V_{p-1}^* = 0 \), this shows that we can take \( \overline{E}_r^{0,*} \) as an \( \mathbb{F}_p[t, t^{-1}] \otimes \Lambda(s) \)-basis of the free module \( \overline{E}_r^{*,*} \): We have
\[ \overline{E}_2^{0,i} \cong H^*(\mathbb{Z}/p; H^i(X; \mathbb{F}_p))[t^{-1}] \cong V_1^{0,i} \otimes (\mathbb{F}_p[t, t^{-1}] \otimes \Lambda(s))^*, \]
so no basis element can sit on an odd column for \( r \geq 2 \). We now assume that there is an odd \( r \) so that \( \overline{d}_r \neq 0 \). The facts that \( \overline{d}_r \) is \( \Lambda(s) \)-linear, that \( \overline{E}_r \) is free over \( \Lambda(s) \) with basis elements located on even columns, that the horizontal part of the bidegree of \( \overline{d}_r \) is odd and that \( s \cdot s = 0 \) imply that still \( \overline{d}_r \neq 0 \), if \( i \) is odd. Hence, \( \overline{d}_r \neq 0 \) and not all basis elements in \( \overline{E}_r^{0,*} \) survive to \( \overline{E}_{r+1}^{0,*} \). Therefore, multiplication by \( s \) cannot be surjective as a map \( \overline{E}_{r+1}^{0,*} \to \overline{E}_{r+1}^{1,*} \) and the bigraded module \( \overline{E}_{r+1}^{*,*} \) is not free over \( \Lambda(s) \), contrary to what we said before. 

**REFERENCES**


**Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 München, Germany**

**E-mail address:** Bernhard.Hanke@mathematik.uni-muenchen.de