Abstract

We describe an explicit chain map from the standard resolution to the minimal resolution for the finite cyclic group $\mathbb{Z}_k$ of order $k$. We then demonstrate how such a chain map induces a “$\mathbb{Z}_k$-combinatorial Stokes theorem”, which in turn implies “Dold’s theorem” that there is no equivariant map from an $n$-connected to an $n$-dimensional free $\mathbb{Z}_k$-complex.

Thus we build a combinatorial access road to problems in combinatorics and discrete geometry that have previously been treated with methods from equivariant topology. The special case $k = 2$ for this is classical; it involves Tucker’s (1949) combinatorial lemma which implies the Borsuk–Ulam theorem, its proof via chain complexes by Lefschetz (1949), the combinatorial Stokes formula of Fan (1967), and Meunier’s work (2006).

1 Introduction

The Borsuk–Ulam theorem [2] about $\mathbb{Z}_2$-equivariant maps between spheres, and its extension to $\mathbb{Z}_k$-actions formulated by Dold [5], have many interesting applications in combinatorics and geometry — see Matoušek [11]. Since these are topological theorems with purely combinatorial consequences, there is great interest in combinatorial approaches to the area.

1.1 The classical case, $k = 2$

For the case $k = 2$ such a path-way is well-established: In 1945, Tucker [20] presented a combinatorial lemma that implies the Borsuk–Ulam theorem: A centrally symmetric triangulation of $S^n$ that refines the hyperoctahedral triangulation of the $n$-sphere cannot get an antipodal labelling from the set $\{\pm 1, \ldots, \pm n\}$ such that no edge gets vertex labels $+i, -i$. In 1952, Fan [6] extended this lemma: If the labels are taken from the set $\{\pm 1, \ldots, \pm m\}$, then the number of facets of the triangulation of $S^n$ that get an “alternating labelling” by $+j_0, -j_1, \ldots, (-1)^nj_n$ with $1 \leq j_0 < j_1 < \cdots < j_n \leq m$ is odd and hence non-zero. In particular, $m$ must be larger than $n$ for such a labelling to exist.
In 1952, Fan [7] presented a rainbow coloring theorem for general pseudomanifolds (interpreted as a “combinatorial Stokes theorem” by Meunier [14]), which says that for any orientable \( n \)-dimensional pseudomanifold with boundary, equipped with a coloring by \( \{\pm 1, \ldots, \pm m\} \) without antipodal edges, the number of rainbow-colored \( n \)-simplices with positive smallest label equals the number of rainbow-colored \( (n - 1) \)-simplices in the boundary (counted with appropriate signs, depending on dimension and orientations). The resulting formula is easy to prove since by linearity it can be reduced to the case of a pseudomanifold that consists of a single \( n \)-simplex. However, a treatment in terms of chain complexes yields a simple, systematic proof that also motivates the formula in question; this was first done in Lefschetz’ 1949 treatment [9, Sect. IV§7, pp. 134–140] of Tucker’s lemma, and then for Fan’s lemma by Meunier [14]. This also leads to simple, transparent, combinatorial proofs for the Kneser conjecture (see Matoušek [12], Ziegler [21]) and for its strengthening by Schrijver [16] (see Meunier [14]).

As amply demonstrated in Matoušek [11], a variety of combinatorial hypergraph coloring problems as well as various geometric multiple-incidence problems were first proved by a result known as Dold’s theorem [5], which says that there is no equivariant map from an \( n \)-connected free \( \mathbb{Z}_k \)-complex to an \( n \)-dimensional such complex. (For \( k = 2 \) this is equivalent to the Borsuk–Ulam theorem). In view of the purely combinatorial hypergraph coloring results proved with this tool (see Alon, Frankl & Lovász [1], Matoušek [10], Ziegler [21], etc.), one is led to ask for an analogous combinatorial treatment of Dold’s theorem, for a “\( \mathbb{Z}_k \)-Tucker lemma”, etc. Steps in this direction were taken by Ziegler [21] and in particular by Meunier [13], who obtained a semi-explicit combinatorial Stokes formula for the case when \( k \) is odd.

### 1.2 The \( \mathbb{Z}_k \)-combinatorial Stokes theorem

The main objective of this paper is not only to derive a “\( \mathbb{Z}_k \)-combinatorial Stokes formula”, Theorem 4.1, that is valid for all \( k \geq 2 \), but also to explain where such a result comes from, and why it has the form it has. This question arises even in the classical case of \( k = 2 \): Why should we look for, and count, simplices with alternating labels, with signs that depend on parity of dimension and on orientation?

A hint for this is given by Meunier’s treatment in [14] of Fan’s combinatorial Stokes theorem, via chain complexes: The chain complex that plays a prominent role in his proof is the minimal free resolution (in the group homology sense) of the group \( \mathbb{Z}_2 \), and Meunier’s proof in essence builds on a \( \mathbb{Z}_2 \)-equivariant chain map from the chain complex of the universal label space to the minimal resolution.

Our combinatorial Stokes formula concerns simplicial complexes \( X \) whose vertices get labels in the set \( \mathbb{Z}_k \times \mathbb{N} \), where we interpret the elements of \( \mathbb{N} \) as “colors”, while the elements of \( \mathbb{Z}_k \) play the role of “signs”. The main requirement is that adjacent vertices of \( X \) may not have the same color and different signs. Such an admissible labelling \( \ell : V(X) \to \mathbb{Z}_k \times \mathbb{N} \) amounts to a simplicial map from \( X \) to a “universal label space” \((\mathbb{Z}_k)^{\times \mathbb{N}}\) and this establishes a chain map \( \ell_\# : C_\bullet(X) \to C_\bullet((\mathbb{Z}_k)^{\times \mathbb{N}}) \) of simplicial chain complexes (with coefficient in some commutative ring \( R \)).

The label space is equipped with a canonical free simplicial \( \mathbb{Z}_k \)-action, corresponding to the natural symmetry of admissible labellings given by cyclically permuting the signs in \( \mathbb{Z}_k \). Thus there is a \( \mathbb{Z}_k \)-equivariant chain map \( C_\bullet((\mathbb{Z}_k)^{\times \mathbb{N}}) \to M_\bullet \) to the minimal resolution of the ring \( R \) over the group ring \( R[\mathbb{Z}_k] \) which commutes with the canonical augmentations on both complexes, unique up to \( \Lambda \)-linear chain homotopy. This statement relies on the fact that \( M_\bullet \) is a free resolution of \( R \) over \( R[\mathbb{Z}_k] \). The chain complex \( M_\bullet \) consists of free modules of rank one over \( R[\mathbb{Z}_k] \) in every degree, hence only label patterns of a very specific form survive to the minimal
The combinatorial Stokes formula results from an explicit description of the chain map to the minimal resolution (and in particular of the surviving label patterns) combined with the simple fact that this chain map commutes with boundary operators.

The following diagram of chain complexes and chain maps illustrates the homological-algebraic content of this mechanism.

\[
\begin{array}{cccccc}
\mathbb{C}_i(X) & \xrightarrow{\partial_i} & \mathbb{C}_{i-1}(X) & \rightarrow & \text{simplicial chain complex} \\
\downarrow \ell_\# & & \downarrow \ell_\# & & (labelling) \\
C_i((\mathbb{Z}_k)^{\infty}) & \xrightarrow{\partial_i} & C_{i-1}((\mathbb{Z}_k)^{\infty}) & \rightarrow & \text{chain complex of label space} \\
\downarrow h_i & & \downarrow h_{i-1} & & (map to standard resolution) \\
S_i & \xrightarrow{\partial_i} & S_{i-1} & \rightarrow & \text{standard resolution} \\
\downarrow f_i & & \downarrow f_{i-1} & & (map to minimal resolution, Prop. 3.8) \\
M_i = R[\mathbb{Z}_k] & \xrightarrow{\partial_i} & M_{i-1} = R[\mathbb{Z}_k] & \rightarrow & \text{minimal resolution} \\
\downarrow u & & \downarrow u & & (evaluate at the neutral element) \\
\mathbb{R} & & & & & \\
\end{array}
\]

The composite chain map \( h_\ell := h_\bullet \circ \ell_\# \) (see Section 4) sends simplices to “patterns” of label sequences (counted with multiplicities and according to orientation). Thus, \( h_\ell^i(x) \) is the formal sum of the patterns that arises from an \( i \)-chain \( x \in \mathbb{C}_i(X) \), while \( h_{\ell-1}^i(\partial_i x) \) is the corresponding sum of patterns on the \((i-1)\)-simplices in the boundary of \( x \).

Given a pattern for \( i \)-simplices, the map \( f_i \) to \( M_i \) followed by the boundary map of the minimal resolution and then by the evaluation map \( u \) (which maps an element of the group ring to the coefficient of the neutral element) tells us how to count \( i \)-patterns. Similarly, we count \((i-1)\)-patterns according to \( u \circ f_{i-1} \).

In this notation, the combinatorial Stokes formula simply reads

\[
\widehat{u}((\partial \circ f \circ h_\ell^i)(x)) = \widehat{u}((f \circ h_\ell^i)(\partial_i x)) \in \mathbb{R}
\]

for \( x \in \mathbb{C}_i(x) \). Our Theorem 4.1 combines this fact with the explicit description of the chain map \( f_\bullet \) presented in Section 3.

The formula obtained in this way depends on some choices. Indeed, the map \( \ell_\# \) is determined by the given labelling on \( X \) and there is a canonical choice for \( h_\bullet \). Furthermore, replacing \( u \) by the evaluation at another group element in \( \mathbb{Z}_k \) induces a Stokes formula which is given by shifting the signs involved in the old one cyclically by the inverse of this element. However, the map \( f_\bullet \) is determined only up to chain homotopy and different choices lead to different Stokes formulas, in general. It is easy to see (cf. Lemma 3.4) that the chain map from the standard to the minimal resolution is uniquely determined upon choosing \( R \)-linear complements of the kernels of the boundary operator in each degree of the minimal resolution. We will propose a particular choice, uniform for all \( k \) (see the remarks following Lemma 3.4), and analyze the corresponding label patterns surviving to the minimal resolution in terms of strongly alternating...
labellings (see Definition 3.5). This notion and the resulting Stokes formula restrict to the notion of alternating labellings and to the classical Fan formula if $k = 2$.

The boundary operator $\partial_i$ in the minimal resolution depends on the parity of $i$. Consequently, as in the classical case $k = 2$, we actually get two combinatorial Stokes formulas depending on whether the dimension of the given simplicial chain on $X$ is even or odd.

1.3 Plan

In Section 2 we review the combinatorial Stokes formula and the Tucker lemma in the classical case when $k = 2$. In this case $X$ is required to be a $d$-pseudomanifold, and $x = o_d \in C_d(X)$ is an orientation chain for it. However, a key example for our discussions for arbitrary $k$ is the universal label space $(\mathbb{Z}_k)^{\infty}$, and this is a pseudomanifold for $k = 2$ (at least for finite $m$), but not for $k > 2$. Thus we admit for greater generality below.

Section 3 is the technical heart of our paper: We explicitly construct the chain maps that lead to the combinatorial Stokes formula in Section 4 and we give a combinatorial interpretation of the relevant label patterns in terms of strongly alternating elements. We remark that this construction is much more difficult for $k > 2$ than in the classical case $k = 2$.

From this, in Section 5, we derive “$\mathbb{Z}_k$-Tucker lemmas”. What should such a result achieve, if we follow the model for $k = 2$? It should refer to a labeled simplicial complex $X$ with a free $\mathbb{Z}_k$ action, and predict the existence of simplices with a specified type of label pattern. Topologically, it should imply that for some $d$-connected free $\mathbb{Z}_k$-space with arbitrarily fine triangulation (for $k = 2$: antipodal triangulations of the $d + 1$-sphere) there is no equivariant map to a specific $d$-dimensional free $\mathbb{Z}_k$-space which serves as a “label space”. The Tucker lemmas should be derived from the combinatorial Stokes theorem by induction on the dimension, once we can identify suitable chains (generalized spheres, cf. Definition 5.1) in the complex $X$. In Section 5, we derive a generalized $\mathbb{Z}_k$-Tucker lemma, Theorem 5.4, which in the case $k = 2$ specializes to Fan’s and Tucker’s lemma, and which also yields the “$\mathbb{Z}_k$-Tucker–Fan lemma” of Meunier [13, Thm. 2] as an example, without Meunier’s restriction to the case of odd $k$. We also derive a (homological) version of Dold’s theorem from this setup.

Finally, in Section 6 we determine the homotopy type of the target space $(\mathbb{Z}_k)^{\infty}_{alt \leq d}$ that appears implicitly in Meunier’s and explicitly in our version of the $\mathbb{Z}_k$-Tucker lemma. In the special case $k = 2$ this yields the natural target space for rainbow colorings — which appears in Fan’s classical work [8] and its current extensions by Tardos & Simonyi [17, 18].

Acknowledgements. We are grateful to Frédéric Meunier for his unpublished paper [13], which was an important stepping-stone for our work. We also thank Mark de Longueville for important discussions.

This work grew on the hospitality of KTH Stockholm and the Mittag-Leffler Institute on occasion of the “Topological Combinatorics” workshop in May 2005, and the MSRI special semester “Computational Applications of Algebraic Topology” in the fall of 2007. We gratefully acknowledge support by a DFG Leibniz grant (GMZ, BH) and from the DFG Research Training Group “Methods for Discrete Structures” in Berlin (CS, RS). BH thanks TU Berlin for its hospitality.

2 Fan and Tucker revisited

A $d$-pseudomanifold is a finite, pure $d$-dimensional, simplicial complex $X$ such that any $(d - 1)$-face (ridge) is contained in at most two $d$-faces (facets) of the complex. The ridges that lie in exactly one facet generate the boundary $\partial X$, which is thus a pure $(d - 1)$-dimensional simplicial
complex (or empty). The vertex set of a complex \(X\) will be denoted by \(V(X)\), the edge set by \(E(X)\). A \(d\)-pseudomanifold is orientable if the facets can be oriented consistently so that they induce opposite orientations on the interior ridges, that is, if there is an orientation \(d\)-chain \(o_d\) in the chain group \(C_d(X;\mathbb{Z})\) such that the boundary \(\partial o_d\) is supported only on the boundary complex \(\partial X\).

We refer to Munkres [15] for basics about chain complexes, chain maps, and orientability.

**Definition 2.1.** An admissible vertex labelling of a pure \(d\)-dimensional simplicial complex \(X\) is a map

\[
\ell : V(X) \rightarrow \mathbb{Z} \setminus \{0\}
\]

such that no two adjacent vertices obtain opposite labels, that is, such that \(\ell(v) \neq -\ell(w)\) for \(\{v, w\} \in E(X)\).

Under such a labelling, a +alternating facet is one that obtains labels \(+j_0, -j_1, +j_2, \ldots, (-1)^{d-1}j_d\) with \(0 < j_0 < j_2 < \cdots < j_d\) (that is, all labels have different absolute values, and if we order them by absolute value, then the signs alternate, starting with a positive sign). Similarly, a −alternating facet obtains labels \(-j_0, +j_1, -j_2, \ldots, (-1)^{d+1}j_d\) with \(0 < j_0 < j_2 < \cdots < j_d\).

The main result of Fan’s 1967 paper [7] was that for any admissible vertex labelling on an oriented \(d\)-pseudomanifold, \((-1)^d\) times the number of +alternating facets (counted according to orientation and with an additional minus sign if \(d\) is odd) plus the number of −alternating facets (counted according to orientation) yields the number of +alternating facets in the boundary complex. Here “counted according to orientation” means that a facet is counted as \(-1\) if the ordering of the vertices according to the label ordering \(j_0, j_1, j_2, \ldots, j_d\) yields a negative orientation of the facet (and similarly for −alternating facets). If the \(d\)-pseudomanifold is not orientable, then all of this is still true modulo 2.

With or without explicit notation for this (Fan writes “\(\alpha(+j_0, -j_1, +j_2, \ldots, (-1)^{d-1}j_d)\)” for the number of \(d\)-simplices with the given set of labels, counted according to orientation), the precise count is a bit tricky to digest. However, it clearly relates a sum over a pseudomanifold to a sum over the boundary. This explains why Meunier [14] calls this a discrete “Stokes theorem”.

From Fan’s lemma, it is easy to derive the Tucker lemma, by induction on the dimension, using the decomposition of \(\Sigma^d\) into upper and lower hemisphere.

**Proposition 2.2 (Tucker lemma [20], Lefschetz [9, Sect. IV§7], Fan [6]).** Let \(\Sigma^d\) be a centrally symmetric triangulation of the \(d\)-sphere \(S^d\) that refines the hyperoctahedral triangulation. Then there is no admissible vertex labelling \(\ell : V(\Sigma^d) \rightarrow \{\pm 1, \ldots, \pm d\}\) that is antipodal, i.e. \(\ell(-v) = -\ell(v)\) for all vertices \(v\).

Indeed, for any antipodal vertex labelling \(\ell : V(\Sigma^d) \rightarrow \{\pm 1, \ldots, \pm m\}\), the number of +alternating facets (with labels \(+j_0, -j_1, +j_2, \ldots, (-1)^{d-1}j_d\), where \(0 < j_0 < \cdots < j_d\)) is odd and hence nonzero.

To match this with the following, and to pave the way for the transition to a more algebraic treatment, we first re-interpret the set of labels as

\[
\mathbb{Z} \setminus \{0\} = \mathbb{Z}_2 \times \mathbb{N},
\]

where \(\mathbb{N}\) are the (non-zero) natural numbers, and \(\mathbb{Z}_2 \equiv \{1, -1\}\) (which will later be identified with the multiplicative group of order 2).

Thus any admissible labelling induces a simplicial map

\[
\ell : X \rightarrow (\mathbb{Z}_2)^{\mathbb{N}}.
\]
Here $\mathbb{Z}_2 = \{1, -1\}$ is seen as a discrete two element set, $(\mathbb{Z}_2)^{*m}$ is a simplicial sphere of dimension $m-1$ (which may be identified with the boundary complex of the $m$-dimensional cross polytope), and thus the target space

$$(\mathbb{Z}_2)^{*N} = \bigcup_{m\geq 1} (\mathbb{Z}_2)^{*m}$$

is the infinite-dimensional sphere. The simplicial map $\ell$ induces a map of simplicial chain complexes

$$\ell_\#: C_\bullet(X) \to C_\bullet((\mathbb{Z}_2)^{*N})$$

with coefficients in some chosen ring $R$ (when talking about orientation classes, this is usually specified to be $\mathbb{Z}$ if the pseudomanifold is orientable, and $\mathbb{Z}/2$ otherwise).

Here the natural symmetry of admissible label patterns, given by reversing the signs, comes into play. This amounts to the usual free simplicial $\mathbb{Z}_2$-action on $(\mathbb{Z}_2)^{*N}$ and the induced action on its simplicial chain complex. We now re-interpret this: Taking into account that $(\mathbb{Z}_2)^{*N}$ is a contractible space, the chain complex $C_\bullet((\mathbb{Z}_2)^{*N})$ is a free resolution of $R$ over the group ring $R[\mathbb{Z}_2]$ (see Section 3). It is, however, a huge free resolution, of infinite rank, in each dimension: The standard basis for $C_i((\mathbb{Z}_2)^{*N})$ consists of all infinite sequences of type $(\ast, +, -, \ast, -, \ast, \ldots)$ with exactly $i+1$ non-$\ast$ elements. By [4, Lemma 7.4], there is up to homotopy a unique chain map to the minimal resolution which induces the identity of zero dimensional homology groups (which can be canonically identified with $R$). For $R = \mathbb{Z}$ the minimal resolution is given by

$$\cdots \xrightarrow{(+1,+1)} \mathbb{Z}^2 \xrightarrow{(-1,+1)} \mathbb{Z}^2 \xrightarrow{(+1,+1)} \mathbb{Z}^2 \xrightarrow{(+1,+1)} \mathbb{Z}^2 \xrightarrow{(-1,+1)} \cdots \xrightarrow{0}$$

with the rightmost $\mathbb{Z}^2$ sitting in degree 0. The identification of its zeroth dimensional homology with $\mathbb{Z}$ is induced by the map (augmentation) $\mathbb{Z}^2 \to \mathbb{Z}$ represented by the matrix $(+1\: +1)$.

Each such chain map to the minimal resolution can be factored (up to homotopy) through the canonical map from $C_\bullet((\mathbb{Z}_2)^{*N})$ to the so-called standard resolution [4, Sect. I.5], by simply deleting the $\ast$s, and further through the canonical map from the standard resolution to the so-called normalized standard resolution, by throwing away those label patterns that contain two $+$signs or two $-$signs at consecutive places.

For $k = 2$ (but not for larger $k$), the normalized standard resolution is isomorphic to the minimal resolution. One possible chain map is given by mapping the alternating sequences $(+1,-1,+1,\ldots) \in (\mathbb{Z}_2)^m$ and $(-1,+1,-1,\ldots) \in (\mathbb{Z}_2)^m$ into the first and second copy of $\mathbb{Z}$ in $\mathbb{Z}^2$, respectively.

In view of the later generalization to $\mathbb{Z}_k$, we write $\mathbb{Z}_2 = \{e, g\}$ with generator $g$, take $R := \mathbb{Z}$, identify $M_i = \mathbb{Z}^2$ with the group ring $\mathbb{Z}[\mathbb{Z}_2] = \mathbb{Z} \cdot e \oplus \mathbb{Z} \cdot g$ for $i \geq 0$ and identify the boundary maps $\partial_i : M_i \to M_{i-1}$ in the minimal resolution with the multiplication with $\tau := g - e$ for odd $i$, and with the multiplication with $\sigma := e + g$ for even $i > 0$. The augmentation map $M_0 \to \mathbb{Z}$ is defined as $\alpha e + \beta g \mapsto \alpha + \beta$. We finally define the evaluation at $e \in \mathbb{Z}_2$ by

$$u : \mathbb{Z}[\mathbb{Z}_2] \to \mathbb{Z}, \quad \alpha e + \beta g \mapsto \alpha.$$ 

In summary we get the $\mathbb{Z}_2$-Stokes formula by interpreting the labelling as a simplicial map, then constructing the chain map from the chain complex of the color sphere to the minimal resolution, and then evaluating by $u$.

It is easily checked that this Stokes formula is identical to the Fan theorem described after Definition 2.1.

Replacing $u$ by the evaluation at $g$ yields a second Stokes formula obtained from the previous one by reversing all signs.
However, there are many other isomorphisms from the normalized standard resolution to the minimal resolution. These are in one-to-one correspondence with $\mathbb{Z}$-linear complements (viewed as graded modules) of the boundary operator in the minimal resolution. Consequently, there is no “canonical” discrete Stokes formula, even not in the classical case $k = 2$.

3 Resolutions and a chain map

Let $k \geq 2$. We denote the cyclic group with $k$ elements by $\mathbb{Z}_k$ and write it multiplicatively as $\mathbb{Z}_k = \{e, g, \ldots, g^{k-1}\}$, where $g$ is a generator of $\mathbb{Z}_k$. We work over a commutative ring $R$ with 1. We set $\Lambda = R[\mathbb{Z}_k]$, the group ring of $\mathbb{Z}_k$ over $R$.

As usual we consider $R$ as a $\Lambda$-module with $g$ acting trivially. Questions about $\mathbb{Z}_k$-equivariant maps can often be related to the homology of the group $\mathbb{Z}_k$, which is by definition the homology of a chain complex obtained from a free resolution of $R$. A free resolution of $R$ is an acyclic chain complex of free $\Lambda$-modules that is augmented with the (non-free) $\Lambda$-module $R$ in dimension $-1$:

$$\cdots \rightarrow F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} R \rightarrow 0,$$

or, equivalently, a free chain complex

$$F_* : \cdots \rightarrow F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \rightarrow 0$$

such that $H_i(F) = 0$ for $i > 0$ together with a $\Lambda$-linear isomorphism (augmentation) $H_0(F) \xrightarrow{\cong} R$. In the following we use the latter convention.

For our approach, it is important to describe such resolutions explicitly.

**Definition 3.1 (Standard resolution).** The *standard resolution* of $R$ is given by

$$S_* : \cdots \rightarrow S_3 \xrightarrow{\partial_3} S_2 \xrightarrow{\partial_2} S_1 \xrightarrow{\partial_1} S_0 \rightarrow 0$$

with modules

$$S_r := \Lambda \otimes_R \cdots \otimes_R \Lambda$$

and boundary maps

$$\partial_r(h_0 \otimes \cdots \otimes h_r) := \sum_{i=0}^{r} (-1)^i h_0 \otimes \cdots \otimes \hat{h}_i \otimes \cdots \otimes h_r.$$

with the (usual) convention that $\hat{h}_i$ denotes omission from the tensor product. The boundary maps are defined on the basis elements $h_0 \otimes \cdots \otimes h_r$ with $h_1, h_2, \ldots, h_r \in \mathbb{Z}_k$ and extended to $R$-linear maps.

The diagonal action $g \cdot (h_0 \otimes h_1 \otimes \cdots \otimes h_r) := gh_0 \otimes gh_2 \cdots \otimes gh_r$ turns the modules $S_r$ into $\Lambda$-modules. It is easily seen that the boundary maps $\partial_r$ are $\Lambda$-linear.

**Definition 3.2 (Bar resolution).** A choice of a special basis of the $S_r$ as $\Lambda$-modules gives rise to the so called *bar resolution*. This particular basis is given by

$$[h_1|h_2|\cdots|h_r] := e \otimes h_1 \otimes h_2 \otimes \cdots \otimes h_1h_2 \cdots h_r$$

with $h_1, h_2, \ldots, h_r \in \mathbb{Z}_k$. We allow for $r = 0$, i.e. $[\ ] = e \in S_0$. 7
This is clearly a basis of \( S_r \) as a \( \Lambda \)-module and, for example, the elements of the standard \( R \)-basis are rewritten as
\[
h_0 \otimes \cdots \otimes h_r = h_0^{-1}h_1^{-1}h_2\cdots|h_{r-1}^{-1}h_r].
\]
In this basis, the boundary is given by
\[
\partial_r[h_1|h_2|\cdots|h_r] = h_1[h_2|\cdots|h_r] + \sum_{i=1}^{r-1} (-1)^i[h_1|\cdots|h_i-1|h_i+1|h_i+2|\cdots|h_r] + (-1)^r[h_1|h_2|\cdots|h_{r-1}].
\]

**Definition 3.3 (Minimal resolution).** The **minimal resolution** is given by
\[
M_\bullet: \cdots \longrightarrow M_3 \xrightarrow{\partial_3=m_r} M_2 \xrightarrow{\partial_2=m_\sigma} M_1 \xrightarrow{\partial_1=m_\tau} M_0 \longrightarrow 0
\]
with \( M_i := \Lambda \) for all \( i \geq 0 \). The boundary maps are defined by
\[
\partial_r := \begin{cases} 
m_\sigma, & \text{if } r \text{ is even}, \\
m_\tau, & \text{if } r \text{ is odd}, 
\end{cases}
\]
where \( m_x \) denotes multiplication by \( x \in \Lambda \) and
\[
\tau = g - e, \quad \sigma = e + g + \cdots + g^{k-1}.
\]
More generally, we define elements
\[
\tau_r := g^r - e, \quad \sigma_r := e + g + \cdots + g^{r-1}
\]
for \( 0 \leq r \leq k \). In particular, \( \sigma_0 = 0, \sigma_k = \sigma, \tau_1 = \tau \), and \( \tau_0 = \tau_k = 0 \). The sets
\[
\Sigma := \{\sigma_1, \sigma_2, \ldots, \sigma_k\}, \quad T := \{e, \tau_1, \tau_2, \ldots, \tau_{k-1}\}
\]
are both bases of \( \Lambda \) as an \( R \)-module and we have the identities
\[
\tau \sigma_i = \tau_i, \quad \sigma \tau_i = 0
\]
for \( 1 \leq i \leq k \). It will therefore be useful to represent \( M_i \) in the basis \( T \) for even \( i \) and in the basis \( \Sigma \) for odd \( i \). This choice is justified by the identities
\[
\ker m_\sigma = R\tau_1 \oplus R\tau_2 \oplus \cdots \oplus R\tau_{k-1} = \im m_\tau
\]
\[
\im m_\sigma = R\sigma_k = \ker m_\tau
\]
which prove that \( M_\bullet \) is exact in positive dimensions and, indeed, a free resolution of \( R \) with an augmentation \( M_0 \rightarrow R \) defined by
\[
\sum_{i=0}^{k-1} \alpha_i g^i \mapsto \sum_{i=0}^{k-1} \alpha_i.
\]
Because \( S_\bullet \) and \( M_\bullet \) are free resolutions, there is a \( \Lambda \)-linear chain map \( S_\bullet \rightarrow M_\bullet \) which is augmentation preserving (and indeed identifies \( S_0 \) and \( M_0 \) canonically). This chain map is unique up to chain homotopy, see [4, Lemma 7.4]. The following lemma, which is proved by an easy inductive argument, shows how we can achieve uniqueness in this situation.
Lemma 3.4. Let $K_r \subset M_r$, $r \geq 0$, be a collection of $R$-submodules so that the module $K_r$ is an $R$-complement of $\ker \partial_r$ for all $r \geq 0$ (here, $\partial_0 : M_0 \to R$ is the augmentation). Then there is a unique augmentation preserving $\Lambda$-linear chain map

$$S_\bullet \to M_\bullet$$

which sends the basis elements $[h_1|h_2|\cdots|h_s]$ from the bar resolution into $K_r$.

The $R$-bases $T$ and $\Sigma$ of $\Lambda$ introduced above motivate a feasible choice for such a complementary graded submodule $K_\bullet \subset M_\bullet$. For $s \geq 0$ we set

$$K_{2s} := Re, \quad K_{2s+1} := R\sigma_1 \oplus \ldots \oplus R\sigma_{k-1}.$$

Note that for $k = 2$, this specializes to $K_i := Re$ for all $i \geq 0$.

Our aim is to give an explicit description of the resulting chain map $S_\bullet \to M_\bullet$. It relies on the following notion.

Definition 3.5 (Strongly alternating elements). Let $h_0, h_1, \ldots, h_{2s} \in \mathbb{Z}_k$. We call the element $h_0 \otimes \cdots \otimes h_{2s}$ of $S_{2s}$ strongly alternating if its bar representative

$$h_0 \otimes \cdots \otimes h_{2s} = g^{a_0}[g^{a_1}| \cdots |g^{a_{2s}}],$$

with $0 \leq a_i < k$ for all $i = 0, \ldots, 2s$, satisfies

$$a_{2i+1} + a_{2i+2} \geq k \quad \text{for all } 0 \leq i \leq s - 1.$$

(In other words: passing from $h_{2i}$ to $h_{2i+1}$ and from $h_{2i+1}$ to $h_{2i+2}$ amounts to multiplications with elements $g^a$ and $g^\beta$, $0 < \alpha, \beta < k$, so that $\alpha + \beta \geq k$.) Let $h_0, h_1, \ldots, h_{2s+1} \in \mathbb{Z}_k$. We call the element $h_0 \otimes \cdots \otimes h_{2s+1}$ of $S_{2s+1}$ strongly alternating if there is an $a \in \mathbb{Z}_k$ such that $a \otimes h_0 \otimes \cdots \otimes h_{2s+1}$ is strongly alternating, i.e. if $h_1 \otimes \cdots \otimes h_{2s+1}$ is strongly alternating and $h_0 \neq h_1$.

Definition 3.6 (Alternating elements). The element $h_0 \otimes \cdots \otimes h_r$ of $S_r$ is alternating if $h_{r+1} \neq h_i$ for all $0 \leq i < r$.

Remark 3.7. In general, strongly alternating elements are alternating. The two notions coincide if and only if $k = 2$. In this case we get back the alternating label patterns introduced in Definition 2.1.

The strongly alternating elements are $\mathbb{Z}_k$-invariant in the sense that $x = h_0 \otimes h_1 \otimes \cdots \otimes h_{2s}$ is strongly alternating if and only if $gx$ is.

After these preparations, we can write down the chain map $f_\bullet : S_\bullet \to M_\bullet$ corresponding to the above choice of $K_\bullet \subset M_\bullet$.

The $\Lambda$-linear maps $f_r : S_r \to \Lambda$ are given by

$$f_{2s}([h_1|\cdots|h_{2s}]) := \begin{cases} e, & \text{if } [h_1|\cdots|h_{2s}] \text{ is strongly alternating, and} \\ 0, & \text{otherwise,} \end{cases}$$

$$f_{2s+1}([h_1|\cdots|h_{2s+1}]) := \sigma_i f_{2s}([h_2|\cdots|h_{2s+1}]) \quad \text{for } h_1 = g^i, 0 \leq i < k.$$
Proposition 3.8. The collection of the maps \( f_r \) is a chain map from the standard resolution to the minimal resolution, i.e. for all \( s \geq 0 \) the diagrams

\[
\begin{array}{ccc}
S_{2s+1} & \overset{\partial}{\longrightarrow} & S_{2s} \\
\downarrow f_{2s+1} & & \downarrow f_{2s} \\
M_{2s+1} & \overset{\partial}{\longrightarrow} & M_{2s}
\end{array}
\quad
\begin{array}{ccc}
S_{2s+2} & \overset{\partial}{\longrightarrow} & S_{2s+1} \\
\downarrow f_{2s+2} & & \downarrow f_{2s+1} \\
M_{2s+2} & \overset{\partial}{\longrightarrow} & M_{2s+1}
\end{array}
\]

commute:

\[
\begin{align*}
f_{2s}(\partial c) &= \tau f_{2s+1}(c) \quad & \text{for } c \in S_{2s+1} \\
f_{2s+1}(\partial c) &= \sigma f_{2s+2}(c) \quad & \text{for } c \in S_{2s+2}.
\end{align*}
\]

Proof. We proceed by induction on \( s \). Let \( c = [g^r|h_2|\cdots|h_{2s+1}] \), \( 0 \leq r < k \). If \( s = 0 \) then \( f_0(\partial c) = f_0(\partial[g^r]) = f_0(g^r) = g^r-e = \tau r = \tau f_1([g^r]) = \tau f_1(c) \). If \( s > 0 \) then by induction \( \sigma f_{2s}(\partial c) = f_{2s-1}(\partial \partial c) = 0 \), so \( f_{2s}(\partial c) \in \ker m_r \in \im m_r \), and to prove \( f_{2s}(\partial c) = \tau f_{2s+1}(c) \) it suffices to show that for \( 1 \leq i \leq k-1 \) the coefficient of \( \tau_i \) in \( f_{2s}(\partial c) \) with respect to the basis \( T \) equals the coefficient of \( \sigma_{i} \) in \( f_{2s+1}(c) \) with respect to the basis \( \Sigma \). Now \( f_{2s}(\partial[g^r|h_2|\cdots|h_{2s+1}]) = g^r f_{2s}([h_2|\cdots|h_{2s+1}]) \) plus a multiple of \( c \), so the coefficient of \( g^r \) is 1 if \( [h_2|\cdots|h_{2s+1}] \) is strongly alternating and \( i = r \), and it is 0 otherwise. Comparison with the definition of \( f_{2s+1} \) proves the first equation.

Let \( c = [g^t|g^r|h_3|\cdots|h_{2s+2}] \), \( 0 \leq t, r < k \). From the first equation we know that \( \tau f_{2s+1}(\partial c) = f_{2s}(\partial \partial c) = 0 \), so \( f_{2s+1}(\partial c) \in \ker m_r = \im m_s \), and to prove \( f_{2s+1}(\partial c) = \sigma f_{2s+2}(c) \) it suffices to show that the coefficient of \( \sigma_k \) in \( f_{2s+1}(\partial c) \) with respect to the basis \( \Sigma \) equals the coefficient of \( e \) in \( f_{2s+2}(c) \) with respect to the basis \( T \). Now \( f_{2s+1}(\partial c) = g^t f_{2s+1}([g^r|h_3|\cdots|h_{2s+2}]) \) plus a linear combination of the \( \sigma_i \) with \( 1 \leq i < k \), so the coefficient of \( \sigma_k \), which equals the coefficient of \( g^{k-1} \) with respect to the basis \( \{e,g,\ldots,g^{k-1}\} \), equals 1 if \( t+r \geq k \) and \( [h_3|\cdots|h_{2s+2}] \) is strongly alternating and 0 otherwise. So it equals 1 if \( [g^t|g^r|h_3|\cdots|h_{2s+2}] \) is strongly alternating and 0 otherwise. This proves the second equation.

\[\square\]

Remark 3.9. The maps \( f_r \) are zero on all non-alternating, or degenerate, basis elements. These generate a subcomplex of the standard resolution and \( f_* \) factors through the quotient by this subcomplex. This quotient is the so-called normalized standard resolution. The induced map from the normalized standard resolution to the minimal resolution is an isomorphism if and only if \( k = 2 \). In this case we recover exactly the chain map described in Section 2.

4 Labellings and the combinatorial \( \mathbb{Z}_k \)-Stokes theorem

Fix an integer \( k \geq 2 \) and consider an (ordered) simplicial complex \( X \) with vertices labelled with elements of \( \mathbb{Z}_k \times \mathbb{N} \). This labelling is a map

\[\ell : V \rightarrow \mathbb{Z}_k \times \mathbb{N}\]

defined on the vertex set \( V = V(X) \). For a vertex \( v \in V \) and \( \ell(v) = (s, c) \in \mathbb{Z}_k \times \mathbb{N} \) we will call \( c \) the color and \( s \) the sign of \( v \). A labelling is called admissible if the two vertices of an edge always carry different colors or the same sign (compare Definition 2.1).

Let \( X \) be a simplicial complex with an admissible \( \mathbb{Z}_k \times \mathbb{N} \)-labelling \( \ell \) and let \( C_*(X) = C_*(X;R) \) denote its simplicial chain complex with coefficients in \( R \). We define maps

\[h_{\ell}^*: C_*(X) \rightarrow S_r\]
Recall the chain map \( c \) from Section 3. The combinatorial Stokes theorem is now a consequence of the fact that the evaluation at \( e \) the coefficient sign \( c \) amounts to counting patterns “according to orientation”.

In order to obtain a counting formula, we compose the maps occurring in these equations with \( g \) and — together with the explicit description of \( f \) — obtain

\[
\langle \sum_{i=0}^{k-1} \alpha_i \cdot g^i \rangle \mapsto \alpha_0,
\]

and

**Theorem 4.1 (Combinatorial Stokes formula).** Let \( X \) be a simplicial complex with an admissible \( \mathbb{Z}_k \times \mathbb{N} \)-labelling \( \ell \) and let \( x \in C_r(X) \) be an \( r \)-chain. Then depending on the parity of \( r \), we have the following identities:

- \( (r = 2s) \). The number of label patterns \( h_0 \otimes \ldots \otimes h_{2s} \) in \( \partial x \) so that \( g \otimes h_0 \otimes \ldots \otimes h_{2s} \) is strongly alternating equals the sum of all strongly alternating label patterns occurring in \( x \).

- \( (r = 2s + 1) \). The number of label patterns \( h_0 \otimes \ldots \otimes h_{2s} \) occurring in \( \partial x \) that are strongly alternating and satisfy \( h_0 = e \) is equal to the number of label patterns \( h_0 \otimes \ldots \otimes h_{2s+1} \) occurring in \( x \) so that \( e \otimes h_0 \otimes \ldots \otimes h_{2s+1} \) is strongly alternating minus the number of label patterns \( h_0 \otimes \ldots \otimes h_{2s+1} \) occurring in \( x \) so that \( g \otimes h_0 \otimes \ldots \otimes h_{2s+1} \) is strongly alternating.
Here all label patterns are counted with multiplicities and according to orientation.

It is remarkable, and not clear a priori, that our approach via chain complexes and chain maps leads to a counting formula of the stated form, where — apart from possible multiplicities imposed by the chain $x$ itself — all relevant label patterns are counted with multiplicities $\pm 1$.

For $k = 2$, we recover the classical Fan theorem mentioned in the introduction after Definition 2.1. If we replace the evaluation map $u$ by evaluation at another group element, we obtain the above identities with all labels shifted cyclically.

5 Equivariant labellings and $\mathbb{Z}_k$-Tucker lemmas

Even though the group $\mathbb{Z}_k$ has played an important role in the definition of the objects of Section 3, group actions did not occur in the results of Section 4. We will now consider a $\mathbb{Z}_k$-equivariant labelling, and for each $k$-sphere $x$ that has played an important role in the definition of the objects of $\mathbb{Z}_k$-complexes.

As before, we consider the action of $\mathbb{Z}_k$ on the set of labels $\mathbb{Z}_k \times \mathbb{N}$ by cyclically shifting the signs, i.e. $g(s, c) := (gs, c)$. With this action we say that a labelling $\ell$ on a $\mathbb{Z}_k$-complex $X$ is equivariant if $\ell(gv) = g\ell(v)$ for all $g \in \mathbb{Z}_k$ and all vertices $v$ of $X$.

An equivariant labelling on $X$ can only exist if $X$ is a free $\mathbb{Z}_k$-space.

If $X$ a $\mathbb{Z}_k$-complex with an admissible equivariant labelling $\ell$, then the chain map $h^\ell$ considered in the last section is obviously $\Lambda$-linear.

**Definition 5.1.** Let $X$ be a free $\mathbb{Z}_k$-complex and let $r \geq 0$. A generalized $r$-sphere in $C_*(X)$ is a sequence $(x_i)_{0 \leq i \leq r}$ of chains $x_i \in C_i(X)$ satisfying

$$\partial x_i = \begin{cases} \sigma x_{i-1}, & \text{if } i \text{ is even,} \\ \tau x_{i-1}, & \text{if } i \text{ is odd} \end{cases}$$

for all $0 < i \leq r$.

The terminology is motivated by the following example.

**Example 5.2.** Let $k > 2$ and $X$ be the triangulation of $S^{2m+1} = S^1 \ast \ldots \ast S^1$ obtained by triangulating each of the $m + 1$ copies of $S^1$ as a $k$-gon. We number the copies starting with 0 and for each $i$, $0 \leq i \leq m$, choose a vertex $u^i$ in the $(m - i)$-th copy. Let $\mathbb{Z}_k$ act on $X$ in such a way that each of the 1-spheres is invariant under the action and $gu^i$ is a neighbor of $u^i$. We denote the oriented edge from $u^i$ to $gu^i$ by $w^i$. We define several chains in $C_*(X)$, starting with

$$o^i_0 := \tau w^i, \quad o^i_1 := \sigma w^i.$$

So $o^i_0 = \partial w^i$ is an orientation chain of a 0-sphere in the $(m - i)$-th copy of $S^1$, and $o^i_1$ an orientation chain of this 1-sphere. Setting

$$x_{2s} := u^s * o^s_1 * o^{s-2}_1 * \ldots * o^0_1,$$

$$x_{2s+1} := u^s * o^{s-1}_1 * o^{s-2}_1 * \ldots * o^0_1,$$

each $x_j$ is the orientation chain of a $j$-disk, and

$$\tau x_{2s} = o^s_0 * o^{s-1}_1 * o^{s-2}_1 * \ldots * o^0_1,$$

$$\sigma x_{2s+1} = o^1_1 * o^{s-1}_1 * \ldots * o^0_1$$

for all $0 < i \leq r$. 

\[ 12 \]
are orientation chains of spheres. We obtain
\[ \partial x_{2s+1} = \tau x_s, \quad \partial x_{2s+2} = \sigma x_{2s+1}. \]

**Example 5.3.** Let \( k \geq 2, \ d \geq 0. \) The construction of the preceding example translates to \((\mathbb{Z}_k)^{s(d+1)}\), since \( \mathbb{Z}_k \ast \mathbb{Z}_k \) contains the barycentric subdivision of a \( k \)-gon with the natural \( \mathbb{Z}_k \)-action. We set
\[
    u^i := \langle (e, d - 2i) \rangle,
    w^i := \langle (e, d - 2i - 1), (g, d - 2i) \rangle - \langle (e, d - 2i - 1), (e, d - 2i) \rangle
\]
and continue as in 5.2 to obtain chains \( x_i \in C_i((\mathbb{Z}_k)^{s(d+1)}) \) for \( 0 \leq i \leq d \) satisfying the conditions of Definition 5.1. Again, each \( x_i \) is the orientation chain of an \( i \)-disk, while \( \sigma x_i \) is the orientation chain of an \( i \)-sphere for odd \( i \) and \( \tau x_i \) is the orientation chain of an \( i \)-sphere for even \( i \).

Now the generalized Tucker lemma has the following form. As before, the map \( u : \Lambda \to R \) is the evaluation at \( e \in \mathbb{Z}_k \).

**Theorem 5.4 (Generalized \( \mathbb{Z}_k \)-Tucker lemma).** Let \( X \) be a \( \mathbb{Z}_k \)-complex which is equipped with an equivariant admissible \( \mathbb{Z}_k \times \mathbb{N} \)-labelling \( \ell \). Let \((x_i)_{0 \leq i \leq r} \) be a generalized \( r \)-sphere in \( C_\bullet(X) \) for some \( r \geq 0 \). We set
\[
    \alpha_i := u(\sigma \cdot (f_\bullet \circ h_\ell^i)(x_i)).
\]
(For even \( i \), this just counts the number of strongly alternating label patterns in \( x_i \).) Then

- the number \( \alpha_0 \) equals the sum of the coefficients of the 0-simplices in \( x_0 \) (and hence does not depend on \( \ell \));
- we have \( \alpha_i \equiv \alpha_0 \pmod{k} \) for all \( 0 \leq i \leq r \).

**Remark 5.5.** For \( k = 2 \) it is convenient to work over \( R = \mathbb{Z}/2 \). In this case \( \sigma = \tau \) and \( \alpha_i \) is the parity of the number of alternating in \( x_i \), which equals the parity of the number of +alternating simplices in \( \sigma x_i \).

If \( X \) is a centrally symmetric triangulation of the \( r \)-sphere \( S^r \) that refines the hyperoctahedral triangulation, we obtain the Tucker lemma (Proposition 2.2) by choosing \( x_1 \) to be the orientation chain of an \( i \)-dimensional hemisphere. Then \( x_0 \) is a chain consisting of a single point, hence \( \alpha_i = 1 \in \mathbb{Z}/2 \) for all \( i \), and we obtain that the number of +alternating simplices in \( X \) is odd.

**Proof of Theorem 5.4.** The first assertion on the value of \( \alpha_0 \) is immediate. We now show that \( \alpha_{i+1} \equiv \alpha_i \pmod{k} \) for all \( 0 \leq i < r \). For \( 0 \leq 2s + 1 < r \) this assertion follows by composing the equation
\[
    \sigma(fh^\ell(x_{2s+2})) = fh^\ell(\partial x_{2s+2}) = fh^\ell(\sigma x_{2s+1}) = \sigma(fh^\ell(x_{2s+1}))
\]
with the map \( u \). The first of these equation uses the fact that \( f_\bullet \) and \( h_\ell^i \) are chain maps, the second one the definition of a generalized sphere and the last one the equivariance of \( f_\bullet \) and \( h_\ell^i \).

Now let \( 0 \leq 2s < r \). In order to show \( \alpha_{2s+1} \equiv \alpha_{2s} \pmod{k} \), it suffices to establish
\[
    \sigma(f_{2s+1}(h_\ell^i(x_{2s+1})) - f_{2s}(h_\ell^i(x_{2s}))) \in k\Lambda
\]
and because \( \sigma^2 = k\sigma \), this will be a consequence of
\[
    f_{2s+1}(h_\ell^i(x_{2s+1})) - f_{2s}(h_\ell^i(x_{2s})) \in \text{im } m_\sigma = \ker m_\tau.
\]

13
But indeed,

$$\tau_f 2s+1(h^\ell(x_{2s+1})) = f_{2s}(h^\ell(\partial x_{2s+1})) = f_{2s}(h^\ell(\tau x_{2s})) = \tau f_{2s}(h^\ell(x_{2s}))$$

finishing the proof of Theorem 5.4.

**Remark 5.6.** In order to put Definition 5.1 and Theorem 5.4 into a more general perspective, we observe that the chains $x_i$ of a generalized $r$-sphere define a $\Lambda$-chain map $x : M^r_{\leq r} \to C_\bullet(X)$, where $M^r_{\leq r}$ denotes the truncation of the minimal resolution in degree $r$. Theorem 5.4 follows from the fact that the chain map $f \circ h^\ell \circ x : M^r_{\leq r} \to M_\bullet$ is determined up to homotopy by the induced map $R \cong H_0(M^r_{\leq r}) \to H_0(M) \cong R$, which is multiplication by $\alpha_0$. In essence, the inductive and more explicit procedure presented above is based on a systematic study of the connecting homomorphisms in cohomology resulting from the exact short exact sequences

$$\sigma C_\bullet(X) \xrightarrow{\text{incl.}} C_\bullet(X) \xrightarrow{m_r} \tau C_\bullet(X)$$

and

$$\tau C_\bullet(X) \xrightarrow{\text{incl.}} C_\bullet(X) \xrightarrow{m_\sigma} \sigma C_\bullet(X).$$

From Theorem 5.4 we can derive the following invariance property of $\alpha_i$ under a change of labellings.

**Corollary 5.7.** Let $X$ be a free $\mathbb{Z}_k$-complex, let $r \geq 0$ and $x \in C_r(X)$. If $r$ is even, assume that $\partial(\tau x) = 0$, if $r$ is odd, assume that $\partial(\sigma x) = 0$. For an arbitrary admissible $\mathbb{Z}_k \times \mathbb{N}$-labelling $\ell$, set

$$\alpha := u(\sigma \cdot (f_\bullet \circ h^\ell)(x)).$$

Then the congruence class of $\alpha$ modulo $k$ does not depend on the choice of the labelling $\ell$.

**Proof.** We will see that there exists a generalized sphere $(x_i)_{0 \leq i \leq r}$ with $x_r = x$. Consequently $\alpha = \alpha_r \equiv \alpha_0 \pmod{k}$, and $\alpha_0$ does not depend on $\ell$.

The chains $x_i$ can be constructed recursively starting with $x_r = x$. To see this, assume that for a chain $y$ the condition $\partial(\sigma y) = 0$ holds. Then $\sigma \delta y = 0$, and since $C_\bullet(X)$ is a free $\Lambda$-complex, this implies the existence of $\bar{y}$ with $\partial y = \tau \bar{y}$. It further follows that $\partial(\tau \bar{y}) = \partial(\delta y) = 0$. Analogously the condition $\partial(\tau y) = 0$ implies the existence of $\bar{y}$ with $\partial y = \sigma \bar{y}$ and $\partial(\sigma \bar{y}) = 0$.

**Corollary 5.8.** Let $X$ be any $\mathbb{Z}_k$-equivariant subdivision of the simplicial complex $(\mathbb{Z}_k)^{d+2}$. There is a subcomplex $Y$ of $X$, homeomorphic to a $(d+1)$-sphere, such that for every admissible equivariant $\mathbb{Z}_k \times \mathbb{N}$-labelling $\ell$ of $X$, the number of $(d+1)$-simplices of $Y$ to which $\ell$ assigns strongly alternating patterns, counted as in the definition of $\alpha_{d+1}$ in Theorem 5.4, is congruent to 1 modulo $k$.

**Proof.** Let $\text{sd} : C_\bullet((\mathbb{Z}_k)^{d+1}) \to C_\bullet(X)$ be the equivariant subdivision chain map and $x_i \in C_i((\mathbb{Z}_k)^{d+2})$ the chains constructed in Example 5.3. The chains $\text{sd}(x_i)$ satisfy the conditions of Theorem 5.4 with $\alpha_0 = 1$.

We formulate a consequence of this as a non-existence result for certain equivariant maps.

**Definition 5.9.** For $d \geq 0$ and $m \geq d + 1$, we denote by $(\mathbb{Z}_k)^{\leftarrow m}_{\text{alt} \leq d}$ the subcomplex of the join $(\mathbb{Z}_k)^{\leftarrow m}$ whose facets consist of all simplices $\langle i_1, \ldots, i_m \rangle$ ($i_j \in \mathbb{Z}_k$) with at most $d$ jumps, that is, such that $\# \{ j \in [m-1] : i_j \neq i_{j+1} \} \leq d$. 

14
The following is also implied by the Tucker–Fan lemma that Meunier [13, Thm. 4] obtained for odd \( k \).

**Corollary 5.10.** Let \( k \geq 2 \), and let \( X \) be any \( \mathbb{Z}_k \)-equivariant subdivision of the simplicial complex \( (\mathbb{Z}_k)^n \times (d+2) \), then there is no equivariant simplicial \( \mathbb{Z}_k \)-map

\[
\ell : X \rightarrow (\mathbb{Z}_k)^m_{\text{alt} \leq d}.
\]

**Proof.** Since all strongly alternating patterns are alternating, an equivariant map \( \ell : X \rightarrow (\mathbb{Z}_k)^m_{\text{alt} \leq d} \) would establish an admissible \( \mathbb{Z}_k \times \mathbb{N} \)-labelling of \( X \) in which no \((d+1)\)-simplex gets a strongly alternating pattern, contradicting Corollary 5.8. \( \qed \)

**Remark 5.11.** The spaces \( (\mathbb{Z}_k)^m_{\text{alt} \leq d} \) will be reconsidered in Section 6. In Corollary 6.3 we prove the existence of a \( \mathbb{Z}_k \)-equivariant map from \( (\mathbb{Z}_k)^m_{\text{alt} \leq d} \) to the \( d \)-dimensional space \( (\mathbb{Z}_k)^n \times (d+1) \). Thus Corollary 5.10 also follows directly from Dold’s theorem 5.13 below.

Instead of constructing the chains in Theorem 5.4 explicitly as in Example 5.3, we can also give a homological condition that ensures their existence. We illustrate this by giving a proof of Dold’s theorem.

**Proposition 5.12.** Let \( X \) be a simplicial complex with a free \( \mathbb{Z}_k \)-action, and \( R \) be a commutative ring with 1 such that \( kR \neq R \). Let \( r \geq 0 \). If \( \tilde{H}_i(X; R) \cong 0 \) for all \( i \leq r \) then for every equivariant admissible \( \mathbb{Z}_k \times \mathbb{N} \)-labelling there is an \((r+1)\)-simplex of \( X \) which is labelled with \( r+2 \) distinct colors and a strongly alternating pattern.

**Proof.** It will suffice to construct a generalized \((r+1)\)-sphere \((x_i)_{0 \leq i \leq r+1} \) with \( \alpha_0 = 1 \), because the conclusion \( \alpha_{r+1} \neq 0 \) of Theorem 5.4 shows the existence of the desired \((r+1)\)-simplex. Since \( \tilde{H}_{r+1}(X) \cong 0 \), \( X \) is nonempty and we can set \( x_0 = \langle v \rangle \) for a simplex \( v \), so \( \alpha_0 = 1 \). Then \( \tau x_0 \) is a reduced 0-cycle. Further, because \( \tilde{H}_0(X) \cong 0 \), we can choose \( x_1 \) with \( \partial x_1 = \tau x_0 \).

Now assume that for some \( i \) with \( 1 \leq i \leq r \), the \( x_j \) for \( j \leq i \) are already chosen. In case of odd \( i \), we have \( \partial(\tau x_i) = \sigma \partial x_i = \sigma \tau x_{i-1} = 0 \), and since \( H_i(X) \cong 0 \), there is an \( x_{i+1} \) such that \( \partial x_{i+1} = x_i \). In case of even \( i \), we get \( \partial(\tau x_i) = \tau \partial x_i = \tau \sigma x_{i+1} = 0 \), and since \( H_i(X) \cong 0 \), there is an \( x_{i+1} \) such that \( \partial x_{i+1} = x_i \). In both cases \( x_{i+1} \) with the desired property can be found. \( \qed \)

**Theorem 5.13 (Dold [5]).** Let \( X \) and \( Y \) be simplicial complexes with free \( \mathbb{Z}_k \)-actions. Let \( r \geq 0 \) and \( R \) be a commutative ring with 1 such that \( kR \neq R \). If \( \tilde{H}_i(X; R) \cong 0 \) for all \( i \leq r \) and \( \dim Y \leq r \) then there is no equivariant simplicial map from \( X \) to \( Y \).

**Proof.** The complex \( Y \) admits an equivariant admissible \( \mathbb{Z}_k \times \mathbb{N} \)-labelling. No simplex of \( Y \) is labelled with more than \( r+1 \) colors, since no simplex has more then \( r+1 \) vertices. An equivariant map from \( X \) to \( Y \) would induce a labelling with these same properties on \( X \), contradicting Proposition 5.12. \( \qed \)

**Remark 5.14.** The “\( \mathbb{Z}_p \)-Tucker lemma” from Ziegler [21, Lemma 5.3] corresponds to a different type of labelling. Namely, call a \( \mathbb{Z}_k \times \mathbb{N} \)-labelling for the vertices of a simplicial complex \( X \) a *weakly admissible* labelling if there are no \( k \) vertices of a \((k-1)\)-simplex \( \sigma^{k-1} \) that under the labelling get all the same color (second component), but all different signs (first component). Such a labelling corresponds to a simplicial map to a label space \( (\partial \sigma^{k-1})^\infty \mathbb{N} \), an infinite join of boundaries of \((k-1)\)-simplices. The action of \( \mathbb{Z}_k \) by cyclically permuting the vertices of \( \sigma^k \) is free on the boundary \( \partial \sigma^{k-1} \) only if \( k = p \) is a prime.
F. Meunier (personal communication, Oct. 2008) has shown how to construct a chain map from the chain complex of the corresponding simplicial complex to the minimal resolution for $\mathbb{Z}_p$, and to derive a combinatorial/algebraic proof for [21, Lemma 5.3] from this.

6 The $\mathbb{Z}_k$-target space for rainbow colorings

It was Fan’s basic insight from his 1952 paper [6] that one gets meaningful Tucker lemmas also for labellings of the vertices of antipodal $d$-spheres with labels from $\{\pm 1, \pm 2, \ldots, \pm m\}$ for $m > d + 1$. With subsequent generalizations from $\mathbb{Z}_2$ to $\mathbb{Z}_k$, and from $d$-spheres to arbitrary pseudomanifolds (Fan [7]), it now appears that the space $(\mathbb{Z}_k)^{\ast m}_{alt \leq d}$ introduced in Definition 5.9 is a natural target space for $\mathbb{Z}_k$-Fan theorems. Here we determine its homotopy type.

**Theorem 6.1.** All the inclusions of $\mathbb{Z}_k$-spaces

$$(\mathbb{Z}_k)^{(d+1)} = (\mathbb{Z}_k)^{(d+1)}_{alt \leq d} \subset (\mathbb{Z}_k)^{(d+2)}_{alt \leq d} \subset \cdots \subset (\mathbb{Z}_k)^{m}_{alt \leq d} \subset \cdots \subset (\mathbb{Z}_k)^{\ast m}_{alt \leq d} = \bigcup_{m \geq d+1} (\mathbb{Z}_k)^{\ast m}_{alt \leq d}.$$ are strong deformation retracts.

The proof of Theorem 6.1 is based on the following elementary homotopy theory lemma.

**Lemma 6.2.** Let $X$ be a topological space and let $A \subset X$ be a subspace which is contractible (as a topological space). Then $X$ is a strong deformation retract of the space $X \cup_A CA$, the union along $A$ of $X$ and the cone over $A$.

**Proof.** Because $A$ is contractible, we have a homotopy equivalence

$$X \cup_A CA \cong X \cup_{\{a\}} CA$$

where $a \in A$ is some point and $CA$ is glued to $X$ along a constant map $A \to \{a\}$. Furthermore, a pair of homotopy inverse maps can be chosen in such a way that their restrictions to $X$ are identity maps and that the homotopies of their compositions to the respective identity maps are constant on $X$. Because the south tip of the unreduced suspension $\Sigma A = CA/A$ is a strong deformation retract of $\Sigma A$ ($A$ being contractible), the result follows.

**Proof of Theorem 6.1.** We fix $k \geq 2$ and start with some general observations. For simplicity, we write $\mathbb{Z}_k$ as $\{0, 1, \ldots, k - 1\}$ instead of $\{e, g, \ldots, g^{k-1}\}$ in this section.

Let $d \geq 0$ and $m \geq d + 1$. We define $C_{d,m+1,i} \subset (\mathbb{Z}_k)^{m+1}_{alt \leq d}$ as the (closed) star of the vertex $i \in \mathbb{Z}_k$, where we identify $\mathbb{Z}_k$ with its $(m + 1)$st copy in $(\mathbb{Z}_k)^{m+1}_{alt \leq d}$. By the definition of joins, we can think of $C_{d,m+1,i}$ as the cone over $C_{d,m+1,i} \cap (\mathbb{Z}_k)^{m}_{alt \leq d}$ and furthermore know that the intersections $C_{d,m+1,i} \cap C_{d,m+1,j}$ are contained in $(\mathbb{Z}_k)^{m}_{alt \leq d}$ for $i \neq j$.

By induction on $d > 0$, we will now prove that each of the intersections $C_{d,m+1,i} \cap (\mathbb{Z}_k)^{m}_{alt \leq d}$ is a contractible space (for all $m \geq d + 1$). Together with Lemma 6.2 this implies in particular that the inclusion

$$(\mathbb{Z}_k)^{m}_{alt \leq d} \hookrightarrow (\mathbb{Z}_k)^{m+1}_{alt \leq d}$$

is a strong deformation retract, thus proving the theorem.

For $d = 0$ and $m \geq 1$, each intersection $C_{d,m+1,i} \cap (\mathbb{Z}_k)^{m}$ is a full $(m - 1)$-dimensional simplex $\langle i, i, i, \ldots, i \rangle$ and hence contractible.
Now let $d > 0$ and $m \geq d + 1$. For symmetry reasons it is enough to show that the intersection

$$P := C_{d,m+1,0} \cap (\mathbb{Z}_k)^{s_m}_{alt \leq d}$$

is contractible. We can write the polyhedron $P$ as the union of two subpolyhedra $P_0$ and $P_{>0}$ defined as follows: The facets of $P_0$ are all the facets of $(\mathbb{Z}_k)^{s_m}_{alt \leq d}$ whose $m$th vertex (with respect to the join construction) is equal to 0. It can be identified with the closed star in $(\mathbb{Z}_k)^{s_m}_{alt \leq d}$ over this vertex and is therefore contractible. The facets of $P_{>0}$ are all the facets of $(\mathbb{Z}_k)^{s_m}_{alt \leq (d-1)}$ whose $m$th vertex is contained in the set $\{1, 2, \ldots, k-1\} \subset \mathbb{Z}_k$. We will show that

$$P_0 \hookrightarrow P$$

is a strong deformation retract. Because $P_0$ is contractible, this finally implies contractibility of $P$.

We can write

$$P_{>0} = (\mathbb{Z}_k)^{s_{(m-1)}}_{alt \leq (d-1)} \cup (C_{d-1,m,1} \cup C_{d-1,m,2} \cup \ldots \cup C_{d-1,m,k-1})$$

By our induction hypothesis, for each $1 \leq i \leq k-1$ the intersection

$$C_{d-1,m,i} \cap (\mathbb{Z}_k)^{s_{(m-1)}}_{alt \leq (d-1)}$$

is contractible. Hence, using Lemma 6.2 again, the inclusion

$$(\mathbb{Z}_k)^{s_{(m-1)}}_{alt \leq (d-1)} \hookrightarrow P_{>0}$$

is a strong deformation retract. Because on the other hand

$$(\mathbb{Z}_k)^{s_{(m-1)}}_{alt \leq (d-1)} \subset P_0,$$

this shows that $P_0 \hookrightarrow P_0 \cup P_{>0} = P$ is a strong deformation retract.

**Corollary 6.3.** For $m \geq d+1$ the spaces $(\mathbb{Z}_k)^{s_{(d+1)}}$, $(\mathbb{Z}_k)^{s_m}_{alt \leq d}$ and $(\mathbb{Z}_k)^{s_N}_{alt \leq d}$ are all $\mathbb{Z}_k$-homotopy equivalent, in particular there exists a $\mathbb{Z}_k$-map $(\mathbb{Z}_k)^{s_N}_{alt \leq d} \to (\mathbb{Z}_k)^{s_{(d+1)}}$.

**Proof.** The inclusion map from $(\mathbb{Z}_k)^{s_{(d+1)}}$ into any of the other spaces is $\mathbb{Z}_k$-equivariant and a homotopy equivalence by Theorem 6.1. Since all of the spaces are free $\mathbb{Z}_k$-spaces, a theorem of Bredon [3, Ch. II] [19, Sec. II.2] implies that these maps are $\mathbb{Z}_k$-homotopy equivalences (i.e. they have equivariant homotopy inverses, and the homotopies can also be chosen as equivariant maps).

**References**


