Winter term 2021

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Mathematical Gauge Theory II

Sheet 12

Exercise 1. (Small perturbations of the Seiberg-Witten equations on T^4 II) As in Exercise 2 from Sheet 11 consider $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with its flat Riemannian metric g_0 induced by the scalar product of $\mathbb{R}^4$. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. For a Spin^c-structure $\mathfrak{s}$ on T^4 consider the perturbed Seiberg-Witten equations

$$D_A^+\Phi = 0$$
$$F_{\hat{A}}^+ = \sigma(\Phi,\Phi) + i\varepsilon\omega \ ,$$

where $0 < \varepsilon << 1$ is real and positive, and very small. Assume that the expected dimension of the moduli space of solutions is non-negative. Prove that $c_1(L_\mathfrak{s}) = 0$, as soon as there is a solution.

Exercise 2. (Even intersection forms and spin manifolds) Let X be a closed, oriented, connected, smooth 4-manifold without 2-torsion in $H^2(X;\mathbb{Z})$. Recall that a **characteristic** element for Q_X is an element $c \in H^2(X;\mathbb{Z})$ which satisfies

$$Q_X(c,a) \equiv Q_X(a,a) \mod 2$$

for all $a \in H^2(X;\mathbb{Z})$.

(a) Let $L_\mathfrak{s}$ be the characteristic line bundle of a Spin^c-structure $\mathfrak{s}$. Use the Atiyah Index Theorem to show that $c_1(L_\mathfrak{s})$ is a characteristic element for Q_X.

(b) Show that any characteristic element $c \in H^2(X;\mathbb{Z})$ satisfies $c = c_1(L_\mathfrak{s})$ for some Spin^c-structure $\mathfrak{s}$ on X.

(c) Conclude that a closed, oriented, connected, smooth 4-manifold X without 2-torsion in $H^2(X;\mathbb{Z})$ is spin if and only if its intersection form Q_X is even.

Exercise 3. (Complex conjugation on $\mathbb{CP}^2$) We want to show that there exists an orientation preserving diffeomorphism $d\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ which is the identity on some ball $D^4 \subset \mathbb{CP}^2$ and induces $-\mathrm{Id}$ on $H_2(\mathbb{CP}^2;\mathbb{Z})$.

1. Consider the map $c\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ given by complex conjugation of the homogeneous coordinates. Prove that c is orientation preserving and induces $-\mathrm{Id}$ on $H_2(\mathbb{CP}^2;\mathbb{Z})$.
2. Show that c preserves $\mathbb{C}^2 = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid z_0 = 1\}$ and find an explicit isotopy f_t on $\mathbb{C}^2$ with $f_0 = \mathrm{Id}_{\mathbb{C}^2}$ and $f_1 = c^{-1}|_{\mathbb{C}^2}$.
3. Let $D^4 \subset \mathbb{C}^2$ be a closed ball. Prove that c is isotopic to an orientation preserving diffeomorphism $d\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ with $d|_{D^4} = \mathrm{Id}_{D^4}$.

(please turn)

Exercise 4. (Reflection in (± 1)-sphere)

1. Let N be a smooth oriented 4-manifold and $M = N \# \mathbb{CP}^2$ or $M = N \# \overline{\mathbb{CP}}^2$. Let $E \in H_2(M;\mathbb{Z})$ be the homology class of the sphere $\mathbb{CP}^1 \subset \mathbb{CP}^2 \setminus D^4 \subset M$ with self-intersection $E^2 = \pm 1$. Use the diffeomorphism d from Exercise 3 to show that there exists an orientation preserving diffeomorphism $f\colon M \to M$ which induces on integer homology the map f_* given by

$$\begin{aligned} f_*\colon H_2(M;\mathbb{Z}) &\longrightarrow H_2(M;\mathbb{Z}) \\ A &\longmapsto A \mp 2(A \cdot E)E. \end{aligned}$$

2. For an arbitrary smoothly embedded sphere S^2 of self-intersection ± 1 in a 4-manifold X, show that a tubular neighbourhood is diffeomorphic to a punctured $\mathbb{CP}^2$, and conclude that X is diffeomorphic to $Y \# \mathbb{CP}^2$ or $Y \# \overline{\mathbb{CP}}^2$, so that the above result is applicable.

You can hand in solutions in the lecture on Thursday, 10 February 2022.