LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Winter term 2021

Prof. D. Kotschick
S. Gritschacher

Mathematical Gauge Theory II

Sheet 11

Exercise 1. (Seiberg-Witten equations on the flat torus) Consider $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with a flat metric g_0 induced by the scalar product of $\mathbb{R}^4$. Prove the following statements.

(a) Any solution (A, Φ) to the unperturbed Seiberg-Witten equations on (T^4, g_0) is reducible, i.e. Φ vanishes identically. For a generic flat metric $\hat{A}$ is flat.

(b) If the expected dimension of the moduli space for a Spin^c-structure $\mathfrak{s}$ on T^4 is non-negative, and the moduli space is non-empty, then the Spin^c-structure is the unique one induced by any spin structure, and the moduli space is a copy of T^4.

Exercise 2. (Small perturbations of the Seiberg-Witten equations on T^4) Consider $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with its flat Riemannian metric g_0 induced by the scalar product of $\mathbb{R}^4$. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Note that this is a parallel g_0-self-dual 2-form.

For a Spin^c-structure $\mathfrak{s} = (\gamma, V)$ on T^4 consider the perturbed Seiberg-Witten equations

$$\begin{aligned} &D_A^+ \Phi = 0 \\ &F_{\hat{A}}^+ = \sigma(\Phi, \Phi) + i\varepsilon\omega \ , \end{aligned}$$

where $0 < \varepsilon << 1$ is real and positive, and very small. Assume that the expected dimension of the moduli space of solutions is non-negative.

(a) Prove that if there is a solution to the equations, then $\langle c_1^2(L_{\mathfrak{s}}), [T^4] \rangle = 0$, equivalently the expected dimension is zero.

(b) For the unique Spin^c-structure with $c_1(L_{\mathfrak{s}}) = 0$ prove that there is precisely one solution up to gauge equivalence for every $\varepsilon \neq 0$.

Exercise 3. Consider $\mathbb{CP}^2$ endowed with the Fubini-Study metric g_{FS} with associated fundamental form ω_{FS}, and the perturbed Seiberg-Witten equations

$$\begin{aligned} &D_A^+ \Phi = 0 \\ &F_{\hat{A}}^+ = \sigma(\Phi, \Phi) + i\varepsilon\omega_{FS} \ . \end{aligned}$$

Show that for every Spin^c-structure there is a unique ε such that the equations have precisely one solution, which is reducible. What is the relation between this value of ε and the Spin^c-structure?

(please turn)

Exercise 4. (Unperturbed SW equation on $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$) Consider $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ endowed with a metric with positive scalar curvature.

(a) Classify Spin^c-structures on $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ in terms of the cohomology.

(b) Compute the expected dimension of the moduli space of solutions to the unperturbed Seiberg-Witten equation for any Spin^c-structure.

(c) Prove that for every Spin^c-structure the unperturbed Seiberg-Witten equation has no solution.

You can hand in solutions in the lecture on Thursday, 3 February 2022.