Winter term 2021

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Mathematical Gauge Theory II

Sheet 4

Exercise 1. (Intersection form of a product of surfaces) Let Σ_g, Σ_h denote smooth closed oriented surfaces of genus $g, h \geq 0$.

1. For every $n \geq 0$ determine a basis of $H_n(\Sigma_g \times \Sigma_h; \mathbb{Z})$, for example, by using the standard basis of $H_1(\Sigma_g; \mathbb{Z})$ represented by $2g$ embedded circles.
2. Calculate the Euler characteristic $\chi(\Sigma_g \times \Sigma_h)$ and compare with the general result that the Euler characteristic is multiplicative, that is, $\chi(M \times N) = \chi(M)\chi(N)$ for compact manifolds M, N.
3. Determine the intersection form of $\Sigma_g \times \Sigma_h$.

Exercise 2. (Embedded surfaces in $\mathbb{CP}^2$) A *projective line* is a linear $\mathbb{CP}^1$ in $\mathbb{CP}^2$ (coming from a linear subspace $\mathbb{C}^2 \subset \mathbb{C}^3$). Let $d \geq 0$ be a natural number.

1. We call d projective lines in $\mathbb{CP}^2$ *in general position* if all intersections between them are transverse and if at most two projective lines intersect in a given point p for all $p \in \mathbb{CP}^2$. Prove that there exists d projective lines in $\mathbb{CP}^2$ in general position for all $d \geq 0$.
2. Determine a smooth surface representing the class $d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$. What is its genus?

Exercise 3. (Embedded surfaces in $S^2 \times S^2$) Let $M = S^2 \times S^2$ and consider the homology classes $a, b \in H_2(M; \mathbb{Z})$ defined by

$$a = \left[S^2 \times \{p\}\right], \quad b = \left[\{q\} \times S^2\right],$$

where $p, q \in S^2$ are arbitrary points.

1. Prove that the class na for every $n \in \mathbb{Z}$ can be represented by an embedded sphere.
2. Prove that the class $na + mb$ for every $n, m \in \mathbb{Z} \setminus \{0\}$ can be represented by an embedded surface Σ of genus

$$g = (|n| - 1)(|m| - 1).$$

(please turn)

Exercise 4. (Tangent bundle of $\mathbb{CP}^k$) Let $\tau \subset \mathbb{CP}^k \times \mathbb{C}^{k+1}$ denote the tautological line bundle over $\mathbb{CP}^k$ and $\tau^\perp$ its orthogonal complement using the standard Hermitian metric on $\mathbb{C}^{k+1}$.

1. Prove that $T\mathbb{CP}^k \cong \mathrm{Hom}(\tau, \tau^\perp)$ as complex vector bundles.
2. Prove that $c_1(\tau^*)$ is a generator of $H^2(\mathbb{CP}^k; \mathbb{Z})$.

You can hand in solutions in the lecture on Thursday, 18 November 2021.